A study in derived algebraic geometry
Volume II: Deformations, Lie theory and formal geometry

Dennis Gaitsgory

Nick Rozenblyum

Contents

Preface

1. What is the object of study in this book?

The main unifying theme of the two volumes of this book is the notion of *ind-coherent sheaf*, or rather, categories of such on various geometric objects. In this section we will try to explain what ind-coherent sheaves are and why we need this notion.

1.1. Who are we? Let us start with a disclosure: this book is *not* really about algebraic geometry.

Or, rather, in writing this book, its authors do not act as real algebraic geometers. This is because the latter are ultimately interested in geometric objects that are constrained/enriched by the algebraicity requirement.

We, however, use algebraic geometry as a tool: this book is written with a view toward applications to *representation theory*.

It just so happens that algebraic geometry is a very (perhaps, even the most) convenient way to formulate representation-theoretic problems of categorical nature. This is not surprising, since, after all, algebraic groups are themselves objects of algebraic geometry.

The most basic example of how one embeds representation theory into algebraic geometry is this: take the category $\text{Rep}(G)$ of algebraic representations of a linear algebraic group G. Algebraic geometry allows us to define/interpret $\text{Rep}(G)$ as the category of quasi-coherent sheaves on the classifying stack BG.

The advantage of this point of view is that many natural constructions associated with the category of representations are already contained in the package of 'quasi-coherent sheaves on stacks'. For example, the functors of restriction and

coinduction[1] along a group homomorphism $G' \to G$ are interpreted as the functors of inverse and direct image along the map of stacks

$$BG' \to BG.$$

But what is the advantage of this point of view? Why not stick to the explicit constructions of all the required functors within representation theory?

The reason is that 'explicit constructions' involve 'explicit formulas', and once we move to the world of higher categories (which we inevitably will, in order to meet the needs of modern representation theory), we will find ourselves in trouble: constructions in higher category theory are intolerant of explicit formulas (for an example of a construction that uses formulas see point (III) in Sect. 1.5 below). Rather, when dealing with higher categories, there is a fairly limited package of constructions that we are allowed to perform (see Volume I, Chapter 1, Sects. 1 and 2 where some of these constructions are listed), and algebraic geometry seems to contain a large chunk (if not all) of this package.

1.2. A stab in the back. Jumping ahead slightly, suppose for example that we want to interpret algebro-geometrically the category $\mathfrak{g}$-mod of modules over a Lie algebra $\mathfrak{g}$.

The first question is: why would one want to do that? Namely, take the universal enveloping algebra $U(\mathfrak{g})$ and interpret $\mathfrak{g}$-mod as modules over $U(\mathfrak{g})$. Why should one mess with algebraic geometry if all we want is the category of modules over an associative algebra?

But let us say that we have already accepted the fact that we want to interpret $\mathrm{Rep}(G)$ as $\mathrm{QCoh}(BG)$. If we now want to consider restriction functor

(1.1) $\mathrm{Rep}(G) \to \mathfrak{g}\text{-mod},$

(where $\mathfrak{g}$ is the Lie algebra of G), we will need to give an algebro-geometric interpretation of $\mathfrak{g}$-mod as well.

If $\mathfrak{g}$ is a usual (=classical) Lie algebra, one can consider the associated formal group, denoted in the book $\exp(\mathfrak{g})$, and one can show (see Chapter 7, Sect. 5) that the category $\mathfrak{g}$-mod is canonically equivalent to $\mathrm{QCoh}(B(\exp(\mathfrak{g})))$, the category of quasi-coherent sheaves on the classifying stack[2] of $\exp(\mathfrak{g})$. With this interpretation of $\mathfrak{g}$-mod, the functor (1.1) is simply the pullback functor along the map

$$B(\exp(\mathfrak{g})) \to BG,$$

induced by the (obvious) map $\exp(\mathfrak{g}) \to G$.

Let us now be given a homomorphism of Lie algebras $\alpha : \mathfrak{g}' \to \mathfrak{g}$. The functor of restriction $\mathfrak{g}$-mod $\to \mathfrak{g}'$-mod still corresponds to the pullback functor along the corresponding morphism

(1.2) $B(\exp(\mathfrak{g}')) \xrightarrow{f_\alpha} B(\exp(\mathfrak{g})).$

---

[1]What we call 'coinduction' is the functor right adjoint to restriction, i.e., it is the usual representation-theoretic operation.

[2]One can (reasonably) get somewhat uneasy from the suggestion to consider the category of quasi-coherent sheaves on the *classifying stack of a formal group*, but, in fact, this is a legitimate operation.

Note, however, that when we talk about representations of Lie algebras, the natural functor in the opposite direction is *induction*, i.e., the *left* adjoint to restriction. And being a left adjoint, it *cannot* correspond to the direct image along (1.2) (whatever the functor of direct image is, it is the *right* adjoint of pullback).

This inconsistency leads to the appearance of *ind-coherent sheaves*.

1.3. The birth of IndCoh.

What happens is that, although we can interpret $\mathfrak{g}$-mod as $\mathrm{QCoh}(B(\exp(\mathfrak{g})))$, a more natural interpretation is as $\mathrm{IndCoh}(B(\exp(\mathfrak{g})))$. The symbol 'IndCoh' will of course be explained in the sequel. It just so happens that for a classical Lie algebra, the categories $\mathrm{QCoh}(B(\exp(\mathfrak{g})))$ and $\mathrm{IndCoh}(B(\exp(\mathfrak{g})))$ are equivalent (as $\mathrm{QCoh}(BG)$ is equivalent to $\mathrm{IndCoh}(BG)$).

Now, the functor of restriction along the homomorphism α will be given by the functor
$$(f_\alpha)^! : \mathrm{IndCoh}(B(\exp(\mathfrak{g}'))) \to \mathrm{IndCoh}(B(\exp(\mathfrak{g})));$$
this is the !-pullback functor, which is the *raison d'être* for the theory of IndCoh.

However, the functor of induction $\mathfrak{g}'$-mod $\to \mathfrak{g}$-mod will be the functor of *IndCoh direct image*

$$(1.3) \qquad (f_\alpha)_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(B(\exp(\mathfrak{g}'))) \to \mathrm{IndCoh}(B(\exp(\mathfrak{g}))),$$

which is the *left* adjoint of $(f_\alpha)^!$. This adjunction is due to the fact that the morphism f_α is, in an appropriate sense, proper.

Now, even though, as was mentioned above, for a usual Lie algebra $\mathfrak{g}$, the categories
$$\mathrm{QCoh}(B(\exp(\mathfrak{g}))) \text{ and } \mathrm{IndCoh}(B(\exp(\mathfrak{g})))$$
are equivalent, the functor $(f_\alpha)_*^{\mathrm{IndCoh}}$ of (1.3) is as different as can be from the functor
$$(f_\alpha)_* : \mathrm{QCoh}(B(\exp(\mathfrak{g}'))) \to \mathrm{QCoh}(B(\exp(\mathfrak{g})))$$
(the latter is quite ill-behaved).

For an analytically minded reader let us also offer the following (albeit somewhat loose) analogy: $\mathrm{QCoh}(-)$ behaves more like functions on a space, while $\mathrm{IndCoh}(-)$ behaves more like measures on the same space.

1.4. What can we do with ind-coherent sheaves? As we saw in the example of Lie algebras, the kind of geometric objects on which we will want to consider IndCoh (e.g., $B(\exp(\mathfrak{g}))$) are quite a bit more general than the usual objects on which we consider quasi-coherent sheaves, the latter being schemes (or algebraic stacks).

A natural class of algebro-geometric objects for which IndCoh is defined is that of *inf-schemes*, introduced and studied in Volume II, Part I of the book. This class includes all schemes, but also formal schemes, as well as classifying spaces of formal groups, etc. In addition, if X is a scheme, its de Rham prestack[3] X_{dR} is an inf-scheme, and ind-coherent sheaves on X_{dR} will be the same as *crystals* (a.k.a. D-modules) on X.

---

[3]The de Rham prestack of a given scheme X is obtained by 'modding' out X by the groupoid of its infinitesimal symmetries, see Chapter 4, Sect. 1.1.1 for a precise definition.

Thus, for any inf-scheme $\mathcal{X}$ we have a well-defined category $\mathrm{IndCoh}(\mathcal{X})$. For any map of inf-schemes $f : \mathcal{X}' \to \mathcal{X}$ we have functors

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}') \to \mathrm{IndCoh}(\mathcal{X})$$

and

$$f^! : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X}').$$

Moreover, if f is proper[4], then the functors $(f_*^{\mathrm{IndCoh}}, f^!)$ form an adjoint pair.

Why should we be happy to have this? The reason is that this is exactly the kind of operations one needs in geometric representation theory.

1.5. Some examples of what we can do.

(I) Take $\mathcal{X}'$ to be a scheme X and $\mathcal{X} = X_{\mathrm{dR}}$, with f being the canonical projection $X \to X_{\mathrm{dR}}$. Then the adjoint pair

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \rightleftarrows \mathrm{IndCoh}(X_{\mathrm{dR}}) : f^!$$

identifies with the pair

$$\mathbf{ind}_{\mathrm{D\text{-}mod}} : \mathrm{IndCoh}(X) \rightleftarrows \mathrm{D\text{-}mod}(X) : \mathbf{ind}_{\mathrm{D\text{-}mod}},$$

corresponding to forgetting and inducing the (right) D-module structure (as we shall see shortly in Sect. 2.3, for a scheme X, the category $\mathrm{IndCoh}(X)$ is only slightly different from the usual category of quasi-coherent sheaves $\mathrm{QCoh}(X)$).

(II) Suppose we have a morphism of schemes $g : Y \to X$ and set

$$Y_{\mathrm{dR}} \xrightarrow{f := g_{\mathrm{dR}}} X_{\mathrm{dR}}.$$

The corresponding functors

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Y_{\mathrm{dR}}) \to \mathrm{IndCoh}(X_{\mathrm{dR}}) \text{ and } f^! : \mathrm{IndCoh}(X_{\mathrm{dR}}) \to \mathrm{IndCoh}(Y_{\mathrm{dR}})$$

identify with the functors

$$g_{*,\mathrm{Dmod}} : \mathrm{Dmod}(Y) \to \mathrm{Dmod}(X) \text{ and } g_{\mathrm{Dmod}}^! : \mathrm{Dmod}(X) \to \mathrm{Dmod}(Y)$$

of D-module (a.k.a. de Rham) push-forward and pullback, respectively.

Note that while the operation of pullback of (right) D-modules corresponds to !-pullback on the underlying $\mathcal{O}$-module, the operation of D-module push-forward is less straightforward as it involves taking fiber-wise de Rham cohomology. So, the operation of the IndCoh direct image does something quite non-trivial in this case.

(III) Suppose we have a Lie algebra $\mathfrak{g}$ that acts (by vector fields) on a scheme X. In this case we can create a diagram

$$B(\exp(\mathfrak{g})) \xleftarrow{f_1} B_X(\exp(\mathfrak{g})) \xrightarrow{f_2} X_{\mathrm{dR}},$$

where $B_X(\exp(\mathfrak{g}))$ is an inf-scheme, which is the quotient of X by the action of $\mathfrak{g}$.

Then the composite functor

$$(f_2)_*^{\mathrm{IndCoh}} \circ (f_1)^! : \mathrm{IndCoh}(B(\exp(\mathfrak{g}))) \to \mathrm{IndCoh}(X_{\mathrm{dR}})$$

identifies with the *localization functor*

$$\mathfrak{g}\text{-mod} \to \mathrm{Dmod}(X).$$

---

[4]Properness means the following: to every inf-scheme there corresponds its underlying reduced scheme, and a map between inf-schemes is proper if and only if the map of the underlying reduced schemes is proper in the usual sense.

This third example should be a particularly convincing one: the localization functor, which is usually defined by an explicit formula

$$M \mapsto D_X \underset{U(\mathfrak{g})}{\otimes} M,$$

is given here by the general formalism.

2. How do we do we construct the theory of IndCoh?

Whatever inf-schemes are, for an individual inf-scheme $\mathcal{X}$, the category $\mathrm{IndCoh}(\mathcal{X})$ is bootstrapped from the corresponding categories for schemes by the following procedure:

$$(2.1) \qquad \mathrm{IndCoh}(\mathcal{X}) = \lim_{Z \to \mathcal{X}} \mathrm{IndCoh}(Z).$$

Some explanations are in order.

2.1. What do we mean by limit?

(a) In formula (2.1), the symbol 'lim' appears. This is the limit of categories, but not quite. If we were to literally take the limit in the category of categories, we would obtain utter nonsense. This is a familiar phenomenon: the (literally understood) limit of, say, triangulated categories is not well-behaved. A well-known example of this is that the derived category of sheaves on a space cannot be recovered from the corresponding categories on an open cover. However, this can be remedied if instead of the triangulated categories we consider their higher categorical enhancements, i.e., the corresponding ∞-categories.

So, what we actually mean by 'limit', is the limit taken in the ∞-*category* of ∞-*categories*. That is, in the preceding discussion, all our $\mathrm{IndCoh}(-)$ are actually ∞-categories. In our case, they have a bit more structure: they are k-linear over a fixed ground field k; we call them *DG categories*, and denote the ∞-category of such by DGCat.

Thus, ∞-categories inevitably appear in this book.

(b) The indexing (∞)-category appearing in the expression (2.1) is the (∞)-category opposite to that of schemes Z equipped with a map $Z \to \mathcal{X}$ to our inf-scheme $\mathcal{X}$. The transition functors are given by

$$(Z' \xrightarrow{f} Z) \in \mathrm{Sch}_{/\mathcal{X}} \rightsquigarrow \mathrm{IndCoh}(Z) \xrightarrow{f^!} \mathrm{IndCoh}(Z').$$

So, in order for the expression in (2.1) to make sense we need to make the assignment

$$(2.2) \qquad Z \rightsquigarrow \mathrm{IndCoh}(Z), \quad (Z' \xrightarrow{f} Z) \rightsquigarrow (\mathrm{IndCoh}(Z) \xrightarrow{f^!} \mathrm{IndCoh}(Z'))$$

into a *functor of ∞-categories*

$$(2.3) \qquad \mathrm{IndCoh}^!_{\mathrm{Sch}} : (\mathrm{Sch})^{\mathrm{op}} \to \mathrm{DGCat}.$$

To that end, before we proceed any further, we need to explain what the DG category $\mathrm{IndCoh}(Z)$ is for a scheme Z.

For a scheme Z, the category $\mathrm{IndCoh}(Z)$ will be almost the same as $\mathrm{QCoh}(Z)$. The former is obtained from the latter by a *renormalization* procedure, whose nature we shall now explain.

2.2. Why renormalize? Keeping in mind the examples of $\mathrm{Rep}(G)$ and $\mathfrak{g}$-mod, it is natural to expect that the assignment (2.2) (for schemes, and then also for inf-schemes) should have the following properties:

(i) For every scheme Z, the DG category $\mathrm{IndCoh}(Z)$ should contain infinite direct sums;

(ii) For a map $Z' \xrightarrow{f} Z$, the functor $\mathrm{IndCoh}(Z) \xrightarrow{f^!} \mathrm{IndCoh}(Z')$ should preserve infinite direct sums.

This means that the functor (2.3) takes values in the subcategory of DGCat, where we allow as objects only DG categories satisfying (i)[5] and as 1-morphisms only functors that satisfy (ii)[6].

Let us first try to make this work with the usual QCoh. We refer the reader to Volume I, Chapter 3, where the DG category $\mathrm{QCoh}(\mathcal{X})$ is introduced for an arbitrary *prestack*, and in particular a scheme. However, for a scheme Z, whatever the DG category $\mathrm{QCoh}(Z)$ is, its homotopy category (which is a triangulated category) is the usual (unbounded) derived category of quasi-coherent sheaves on Z.

Suppose we have a map of schemes $Z' \xrightarrow{f} Z$. The construction of the !-pullback functor

$$f^! : \mathrm{QCoh}(Z) \to \mathrm{QCoh}(Z')$$

is quite complicated, except when f is proper. In the latter case, $f^!$, which from now on we will denote by $f^{!,\mathrm{QCoh}}$, is defined to be the *right adjoint* of

$$f_* : \mathrm{QCoh}(Z') \to \mathrm{QCoh}(Z).$$

The only problem is that the above functor $f^{!,\mathrm{QCoh}}$ *does not* preserve infinite direct sums. The simplest example of a morphism for which this happens is

$$f : \mathrm{Spec}(k) \to \mathrm{Spec}(k[t]/t^2)$$

(or the embedding of a *singular* point into any scheme).

The reason for the failure to preserve infinite direct sums is this: the left adjoint of $f^{!,\mathrm{QCoh}}$, i.e., f_*, *does not* preserve compactness. Indeed, f_* does not necessarily send *perfect complexes* on Z' to perfect complexes on Z, unless f is of finite Tor-dimension[7].

So, our attempt with QCoh fails (ii) above.

2.3. Ind-coherent sheaves on a scheme. The nature of the renormalization procedure that produces $\mathrm{IndCoh}(Z)$ out of $\mathrm{QCoh}(Z)$ is to force (ii) from Sect. 2.2 'by hand'.

As we just saw, the problem with $f^{!,\mathrm{QCoh}}$ was that its left adjoint f_* did not send the corresponding subcategories of perfect complexes to one another. However, f_* sends the subcategory

$$\mathrm{Coh}(Z') \subset \mathrm{QCoh}(Z')$$

---

[5]Such DG categories are called *cocomplete*.

[6]Such functors are called *continuous*.

[7]We remark that a similar phenomenon, where instead of the category $\mathrm{QCoh}(\mathrm{Spec}(k[t]/t^2)) = k[t]/t^2$-mod we have the category of representations of a finite group, leads to the notion of Tate cohomology: the trivial representation on $\mathbb{Z}$ is *not* a compact object in the category of representations.

to
$$\mathrm{Coh}(Z) \subset \mathrm{QCoh}(Z),$$
where $\mathrm{Coh}(-)$ denotes the subcategory of bounded complexes, whose cohomology sheaves are *coherent* (as opposed to quasi-coherent).

The category $\mathrm{IndCoh}(Z)$ is defined as the *ind-completion* of $\mathrm{Coh}(Z)$ (see Volume I, Chapter 1, Sect. 7.2 for what this means). The functor f_* gives rise to a functor $\mathrm{Coh}(Z') \to \mathrm{Coh}(Z)$, and ind-extending we obtain a functor
$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Z') \to \mathrm{IndCoh}(Z).$$

Its right adjoint, denoted $f^! : \mathrm{IndCoh}(Z) \to \mathrm{IndCoh}(Z')$ satisfies (ii) from Sect. 2.2.

Are we done? Far from it. First, we need to define the functor
$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Z') \to \mathrm{IndCoh}(Z)$$
for a morphism f that is not necessarily proper. This will not be difficult, and will be done by appealing to t-structures, see Sect. 2.4 below.

What is much more serious is to define $f^!$ for any f. More than that, we need $f^!$ not just for an individual f, but we need the data of (2.2) to be a functor of ∞-categories as in (2.3). Roughly a third of the work in this book goes into the construction of the functor (2.3); we will comment on the nature of this work in Sect. 2.5 and then in Sect. 3 below.

2.4. In what sense is IndCoh **a 'renormalization' of** QCoh**?** The tautological embedding $\mathrm{Coh}(Z) \hookrightarrow \mathrm{QCoh}(Z)$ induces, by ind-extension, a functor
$$\Psi_Z : \mathrm{IndCoh}(Z) \to \mathrm{QCoh}(Z).$$

The usual t-structure on the DG category $\mathrm{Coh}(Z)$ induces one on $\mathrm{IndCoh}(Z)$. The key feature of the functor Ψ_Z is that it is *t-exact*. Moreover, for every fixed n, the resulting functor
$$\mathrm{IndCoh}(Z)^{\geq -n} \to \mathrm{QCoh}(Z)^{\geq -n}$$
is an *equivalence*[8]. The reason for this is that any coherent complex can be approximated by a perfect one up to something in $\mathrm{Coh}(Z)^{< -n}$ for any given n.

In other words, the difference between $\mathrm{IndCoh}(Z)$ and $\mathrm{QCoh}(Z)$ occurs 'somewhere at $-\infty$'. So, this difference can only become tangible in the finer questions of homological algebra (such as convergence of spectral sequences).

However, we do need to address such questions adequately if we want to have a functioning theory, and for the kind of applications we have in mind (see Sect. 1.5 above) this necessitates working with IndCoh rather than QCoh.

As an illustration of how the theory of IndCoh takes something very familiar and unravels it to something non-trivial, consider the IndCoh direct image functor.

In the case of schemes, for a morphism $f : Z' \to Z$, the functor
$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Z') \to \mathrm{IndCoh}(Z)$$
does 'little new' as compared to the usual
$$f_* : \mathrm{QCoh}(Z') \to \mathrm{QCoh}(Z).$$

---

[8]But the functor Ψ_Z is an equivalence on all of $\mathrm{IndCoh}(Z)$ if *and only if* Z is smooth.

Namely, f_*^{IndCoh} is the unique functor that preserves infinite direct sums and makes the diagram

$$
\begin{array}{ccc}
\mathrm{IndCoh}(Z')^{\geq -n} & \xrightarrow[\sim]{\Psi_{Z'}} & \mathrm{QCoh}(Z')^{\geq -n} \\
{\scriptstyle f_*^{\mathrm{IndCoh}}}\Big\downarrow & & \Big\downarrow{\scriptstyle f_*} \\
\mathrm{IndCoh}(Z)^{\geq -n} & \xrightarrow[\sim]{\Psi_{Z}} & \mathrm{QCoh}(Z)^{\geq -n}
\end{array}
$$

commute for every n.

However, as was already mentioned, once we extend the formalism of IndCoh direct image to inf-schemes, we will in particular obtain the de Rham direct image functor. So, it is in the world of inf-schemes that IndCoh shows its full strength.

2.5. Construction of the !-pullback functor. As has been mentioned already, a major component of work in this book is the construction of the functor

$$
\mathrm{IndCoh}^!_{\mathrm{Sch}} : (\mathrm{Sch})^{\mathrm{op}} \to \mathrm{DGCat}
$$

of (2.3).

We already know what $\mathrm{IndCoh}(Z)$ is for an individual scheme. We now need to extend it to morphisms.

For a morphism $f : Z' \to Z$, we can factor it as

$$
(2.4) \qquad\qquad\qquad\qquad Z' \xrightarrow{f_1} \overline{Z}' \xrightarrow{f_2} Z,
$$

where f_1 is an open embedding and f_2 is proper. We then define

$$
f^! : \mathrm{IndCoh}(Z) \to \mathrm{IndCoh}(Z')
$$

to be

$$
f_1^! \circ f_2^!,
$$

where

(i) $f_2^!$ is the *right adjoint* of $(f_2)_*^{\mathrm{IndCoh}}$;

(ii) $f_1^!$ is the *left adjoint* of $(f_1)_*^{\mathrm{IndCoh}}$.

Of course, in order to have $f^!$ as a well-defined functor, we need to show that its definition is independent of the factorization of f as in (2.4). Then we will have to show that the definition is compatible with compositions of morphisms. But this is only the tip of the iceberg.

Since we want to have a functor between ∞-categories, we need to supply the assignment

$$
f \rightsquigarrow f^!
$$

with a *homotopy-coherent* system of compatibilities for n-fold compositions of morphisms, a task which appears infeasible to do 'by hand'.

What we do instead is we prove an existence and uniqueness theorem... not for (2.3), but rather for a more ambitious piece of structure. We refer the reader to Volume I, Chapter 5, Proposition 2.1.4 for the precise formulation. Here we will only say that, in addition to (2.3), this structure contains the data of a functor

$$
(2.5) \qquad\qquad\qquad\qquad \mathrm{IndCoh} : \mathrm{Sch} \to \mathrm{DGCat},
$$

$$Z \rightsquigarrow \mathrm{IndCoh}(Z), \quad (Z' \xrightarrow{f} Z) \rightsquigarrow (\mathrm{IndCoh}(Z') \xrightarrow{f_*^{\mathrm{IndCoh}}} \mathrm{IndCoh}(Z)),$$

as well as *compatibility* between (2.3) and (2.5).

The latter means that whenever we have a Cartesian square

(2.6)
$$\begin{array}{ccc} Z_1' & \xrightarrow{g'} & Z' \\ f_1 \downarrow & & \downarrow f \\ Z_1 & \xrightarrow{g} & Z \end{array}$$

there is a canonical isomorphism of functors, called base change:

(2.7) $$(f_1)_*^{\mathrm{IndCoh}} \circ (g')^! \simeq g^! \circ f_*^{\mathrm{IndCoh}}.$$

2.6. Enter DAG. The appearance of the Cartesian square (2.6) heralds another piece of 'bad news'. Namely, Z_1' must be the fiber product

$$Z_1 \underset{Z}{\times} Z'.$$

But what category should we take this fiber product in? If we look at the example

$$\begin{array}{ccc} \mathrm{pt} \underset{\mathbb{A}^1}{\times} \mathrm{pt} & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathbb{A}^1, \end{array}$$

(here $\mathrm{pt} = \mathrm{Spec}(k)$, $\mathbb{A}^1 = \mathrm{Spec}(k[t])$), we will see that the fiber product $\mathrm{pt} \underset{\mathbb{A}^1}{\times} \mathrm{pt}$ *cannot* be taken to be the point-scheme, i.e., it cannot be the fiber product in the category of usual (=classical) schemes. Rather, we need to take

$$\mathrm{pt} \underset{\mathbb{A}^1}{\times} \mathrm{pt} = \mathrm{Spec}(k \underset{k[t]}{\otimes} k),$$

where the tensor product is understood in the *derived* sense, i.e.,

$$k \underset{k[t]}{\otimes} k = k[\epsilon], \quad \deg(\epsilon) = -1.$$

This is to say that in building the theory of IndCoh, we cannot stay with classical schemes, but rather need to enlarge our world to that of *derived algebraic geometry*.

So, unless the reader has already guessed this, in all the previous discussion, the word 'scheme' had to be understood as 'derived scheme'[9] (although in the main body of the book we say just 'scheme', because everything is derived).

However, this is not really 'bad news'. Since we are already forced to work with ∞-categories, passing from classical algebraic geometry to DAG does not add a new level of complexity. But it does add *a lot* of new techniques, for example in anything that has to do with deformation theory (see Chapter 1).

Moreover, many objects that appear in geometric representation theory naturally belong to DAG (e.g., Springer fibers, moduli of local systems on a curve, moduli of vector bundles on a surface). That is, these objects are *not* classical, i.e.,

---

[9]Technically, for whatever has to do with IndCoh, we need to add the adjective 'laft'='locally almost of finite type', see Volume I, Chapter 2, Sect. 3.5 for what this means.

we cannot ignore their derived structure if we want to study their scheme-theoretic (as opposed to topological) properties. So, we would have wanted to do DAG in any case.

Here are two particular examples:

(I) Consider the category of D-modules (resp., perverse) sheaves on the double quotient

$$I\backslash G((t))/I,$$

where G is a connected reductive group, $G((t))$ is the corresponding loop group (considered as an ind-scheme) and $I \subset G((t))$ is the Iwahori subgroup. Then Bezrukavnikov's theory (see [**Bez**]) identifies this category with the category of ad-equivariant ind-coherent (resp., coherent) sheaves on the *Steinberg scheme* (for the Langlands dual group). But what do we mean by the Steinberg scheme? By definition, this is the fiber product

$$(2.8) \qquad\qquad\qquad \widetilde{\mathcal{N}} \underset{\mathfrak{g}}{\times} \widetilde{\mathcal{N}},$$

where $\widetilde{\mathcal{N}}$ is the Springer resolution of the nilpotent cone. However, in order for this equivalence to hold, the fiber product in (2.8) needs be understood in the *derived* sense.

(II) Let X be a smooth and complete curve. Let $\mathrm{Pic}(X)$ be the *Picard stack* of X, i.e., the stack parameterizing line bundles on X. Let $\mathrm{LocSys}(X)$ be the stack parameterizing 1-dimensional local systems on X. The Fourier-Mukaï-Laumon transform defines an equivalence

$$\mathrm{Dmod}(\mathrm{Pic}(X)) \simeq \mathrm{QCoh}(\mathrm{LocSys}(X)).$$

However, in order for this equivalence to hold, we need to understand $\mathrm{LocSys}(X)$ as a *derived stack*.

2.7. Back to inf-schemes. The above was a somewhat lengthy detour into the constructions of the theory of IndCoh on schemes. Now, if $\mathcal{X}$ is an inf-scheme, the category $\mathrm{IndCoh}(\mathcal{X})$ is defined by the formula (2.1).

Thus, informally, an object $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ is a family of assignments

$$(Z \xrightarrow{x} \mathcal{X}) \rightsquigarrow \mathcal{F}_{Z,x} \in \mathrm{IndCoh}(Z)$$

(here Z is a scheme) plus

$$(Z' \xrightarrow{f} Z) \in \mathrm{Sch}_{/\mathcal{X}} \rightsquigarrow f^!(\mathcal{F}_{Z,x}) \simeq \mathcal{F}_{Z',x'},$$

along with a homotopy-coherent compatibility data for compositions of morphisms.

For a map $g : \mathcal{X}' \to \mathcal{X}$, the functor

$$g^! : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X}')$$

is essentially built into the construction. Recall, however, that our goal is to also have the functor

$$g_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}') \to \mathrm{IndCoh}(\mathcal{X}).$$

The construction of the latter requires some work (which occupies most of Volume II, Chapter 3). What we show is that there exists a unique system of such functors such that for every commutative (but not necessarily Cartesian) diagram

$$
\begin{array}{ccc}
Z' & \xrightarrow{\ i'\ } & \mathcal{X}' \\
f\downarrow & & \downarrow g \\
Z & \xrightarrow{\ i\ } & \mathcal{X}
\end{array}
$$

with Z, Z' being schemes and the morphisms i, i' proper, we have an isomorphism

$$
g_*^{\mathrm{IndCoh}} \circ (i')_*^{\mathrm{IndCoh}} \simeq i_*^{\mathrm{IndCoh}} \circ f^{\mathrm{IndCoh}},
$$

where i_*^{IndCoh} (resp., $(i')_*^{\mathrm{IndCoh}}$) is the left adjoint of $i^!$ (resp., $(i')^!$).

Amazingly, this procedure contains the de Rham push-forward functor as a particular case.

3. What is actually done in this book?

This book consists of two volumes. The first Volume consists of three Parts and an Appendix and the second Volume consists of two Parts. Each Part consists of several Chapters. The Chapters are designed so that they can be read independently from one another (in a sense, each Chapter is structured as a separate paper with its own introduction that explains what this particular chapter does).

Below we will describe the contents of the different Parts and Chapters from several different perspectives: (a) goals and role in the overall project; (b) practical implications; (c) nature of work; (d) logical dependence.

3.1. The contents of the different parts.

Volume I, Part I is called 'preliminaries', and it is really preliminaries.

Volume I, Part II builds the theory of IndCoh on schemes.

Volume I, Part III develops the formalism of *categories of correspondences*; it is used as a 'black box' in the key constructions in Volume I, Part II and Volume II, Part I: this is our tool of bootstrapping the theory of IndCoh out of a much smaller amount of data.

Volume I, Appendix provides a sketch of the theory of $(\infty, 2)$-categories, which, in turn, is crucially used in Volume I, Part III.
Volume II, Part I defines the notion of *inf-scheme* and extends the formalism of IndCoh from schemes to inf-schemes, and in that it achieves one of the two main goals of this book.

Volume II, Part II consists of applications of the theory of IndCoh: we consider formal moduli problems, Lie theory and infinitesimal differential geometry; i.e., exactly the things one needs for geometric representation theory. Making these constructions available is the second of our main goals.

3.2. Which chapters should a practically minded reader be interested in? Not all the Chapters in this book make an enticing read; some are downright technical and tedious. Here is, however, a description of the 'cool' things that some of the Chapters do:

None of the material in Volume I, Part I alters the pre-existing state of knowledge.

Volume I, Chapters 4 and 5 should not be a difficult read. They construct the theory of IndCoh on schemes (the hard technical work is delegated to Volume I, Chapter 7). The reader cannot avoid reading these chapters if he/she is interested in the applications of IndCoh: one has to have an idea of what IndCoh is in order to use it.

Volume I, Chapters 6 is routine. The only really useful thing from it is the functor

$$\Upsilon_Z : \mathrm{QCoh}(Z) \to \mathrm{IndCoh}(Z),$$

given by tensoring an object of $\mathrm{QCoh}(Z)$ with the dualizing complex $\omega_Z \in \mathrm{IndCoh}(Z)$. Extract this piece of information from Sects. 3.2-3.3 and move on.

Volume I, Chapter 7 introduces the formalism of correspondences. The idea of the category of correspondences is definitely something worth knowing. We recommend the reader to read Sect. 1 in its entirety, then understand the universal property stated in Sect. 3, and finally get an idea about the two extension theorems, proved in Sects. 4 and 5, respectively. These extension theorems are the mechanism by means of which we construct IndCoh as a functor out of the category of correspondences in Volume I, Chapter 5.

Volume I, Chapter 8 proves a rather technical extension theorem, stated in Sect. 1; we do not believe that the reader will gain much by studying its proof. This theorem is key to the extension of IndCoh from schemes to inf-schemes in Volume II, Chapter 3.

Volume I, Chapter 9 is routine, except for one observation, contained in Sects. 2.2-2.3: the natural involution on the category of correspondences encodes *duality*. In fact, this is how we construct Serre duality on $\mathrm{IndCoh}(Z)$ and Verdier duality on $\mathrm{Dmod}(Z)$ where Z is a scheme (or inf-scheme), see Volume I, Chapter 5, Sect. 4.2, Chapter 3, Sect. 6.2, and Chapter 4, Sect. 2.2, respectively.

Volume I, Chapter 10 introduces the notion of $(\infty, 2)$-category and some basic constructions in the theory of $(\infty, 2)$-categories. This Chapter is not very technical (mainly because it omits most proofs) and might be of independent interest.

Volume I, Chapter 11 does a few more technical things in the theory of $(\infty, 2)$-categories. It introduces the $(\infty, 2)$-*category of* $(\infty, 2)$-*categories*, denoted $2\text{-}\mathrm{Cat}$. We then discuss the straightening/unstraightening procedure in the $(\infty, 2)$-categorical context and the $(\infty, 2)$-categorical Yoneda lemma. The statements of the results from this Chapter may be of independent interest.

Volume I, Chapter 12 discusses the notion of adjunction in the context of $(\infty, 2)$-categories. The main theorem in this Chapter explicitly constructs the *universal adjointable functor* (and its variants), and we do believe that this is of interest beyond the particular goals of this book.

Volume II, Chapter 1 is background on deformation theory. The reason it is included in the book is that the notion of inf-scheme is based on deformation theory.

However, the reader may find the material in Sects. 1-7 of this Chapter useful without any connection to the contents of the rest of the book.

Volume II, Chapter 2 introduces inf-schemes. It is quite technical. So, the practically minded reader should just understand the definition (Sect. 3.1) and move on.

Volume II, Chapter 3 bootstraps the theory of IndCoh from schemes to inf-schemes. It is not too technical, and should be read (for the same reason as Volume I, Chapters 4 and 5). The hard technical work is delegated to Volume I, Chapter 8.

Volume II, Chapter 4 explains how the theory of crystals/D-modules follows from the theory of IndCoh on inf-schemes. Nothing in this Chapter is very exciting, but it should not be a difficult read either.

Volume II, Chapter 5 is about formal moduli problems. It proves a pretty strong result, namely, the equivalence of categories between formal groupoids acting on a given prestack $\mathcal{X}$ (assumed to admit deformation theory) and formal moduli problems *under* $\mathcal{X}$.

Volume II, Chapter 6 is a digression on the general notion of Lie algebra and Koszul duality in a symmetric monoidal DG category. It gives a nice interpretation of the universal enveloping algebra of a Lie algebra of $\mathfrak{g}$ as the homological Chevalley complex of the Lie algebra obtained by *looping* $\mathfrak{g}$. The reader may find this Chapter useful and independently interesting.

Volume II, Chapter 7 develops Lie theory in the context of inf-schemes. Namely, it establishes an equivalence of categories between group inf-schemes (over a given base $\mathcal{X}$) and Lie algebras in $\mathrm{IndCoh}(\mathcal{X})$. One can regard this result as one of the main applications of the theory developed hereto.

Volume II, Chapters 8 and 9 use the theory developed in the preceding Chapters for 'differential calculus' in the context of DAG. We discuss Lie algebroids and their universal envelopes, the procedure of deformation to the normal cone, etc. For example, the notion of n-th infinitesimal neighborhood developed in Volume II, Chapter 9 gives rise to the Hodge filtration.

3.3. The nature of the technical work. The substance of mathematical thought in this book can be roughly split into three modes of cerebral activity: (a) making constructions; (b) overcoming difficulties of homotopy-theoretic nature; (c) dealing with issues of *convergence*.

Mode (a) is hard to categorize or describe in general terms. This is what one calls 'the fun part'.

Mode (b) is something much better defined: there are certain constructions that are obvious or easy for ordinary categories (e.g., define categories or functors by an explicit procedure), but require some ingenuity in the setting of higher categories. For many readers that would be the least fun part: after all it is clear that the thing should work, the only question is how to make it work without spending another 100 pages.

Mode (c) can be characterized as follows. In low-tech terms it consists of showing that certain spectral sequences converge. In a language better adapted for our needs, it consists of proving that in some given situation we can swap a limit

and a colimit (the very idea of IndCoh was born from this mode of thinking). One can say that mode (c) is a sort of analysis within algebra. Some people find it fun.

Here is where the different Chapters stand from the point of view of the above classification:

Volume I, Chapter 1 is (b) and a little of (c).

Volume I, Chapter 2 is (a) and a little of (c).

Volume I, Chapter 3 is (c).

Volume I, Chapter 4 is (a) and (c).

Volume I, Chapter 5 is (a).

Volume I, Chapter 6 is (b).

Volume I, Chapters 7-9 are (b).

Volume I, Chapters 10-12 are (b).

Volume II, Chapter 1 is (a) and a little of (c).

Volume II, Chapter 2 is (a) and a little of (c).

Volume II, Chapter 3 is (a).

Volume II, Chapter 4 is (a).

Volume II, Chapter 5 is (a).

Volume II, Chapter 6 is (c) and a little of (b).

Volume II, Chapter 7 is (c) and a little of (a).

Volume II, Chapters 8 and 9 are (a).

3.4. Logical dependence of chapters. This book is structured so that Volume I prepares the ground and Volume II reaps the fruit. However, below is a scheme of the logical dependence of chapters, where we allow a 5% *skip margin* (by which we mean that the reader skips certain things[10] and comes back to them when needed).

3.4.1.   Volume I, Chapter 1 reviews ∞-categories and higher algebra. Read it only if you have no prior knowledge of these subjects. In the latter case, here is what you will need in order to understand the constructions in the main body of the book:

Read Sects. 1-2 to get an idea of how to operate with ∞-categories (this is a basis for everything else in the book).

Read Sects. 5-7 for a summary of *stable* ∞-categories: this is what our QCoh($-$) and IndCoh($-$) are; forget on the first pass about the additional structure of k-linear DG category (the latter is discussed in Sect. 10).

Read Sects. 3-4 for a summary of monoidal structures and duality in the context of higher category theory. You will need it for this discussion of Serre duality and for Volume I, Chapter 6.

Sects. 8-9 are about algebra in (symmetric) monoidal stable ∞-categories. You will need it for Volume II, Part II of the book.

---

[10]These are things that can be taken on faith without compromising the overall understanding of the material.

Volume I, Chapter 2 introduces DAG proper. If you have not seen any of it before, read Sect. 1 for the (shockingly general, yet useful) notion of prestack. Every category of geometric objects we will encounter in this book (e.g., (derived) schemes, Artin stacks, inf-schemes, etc.) will be a full subcategory of the ∞-category of prestacks. Proceed to Sect. 3.1 for the definition of derived schemes. Skip all the rest.

Volume I, Chapter 3 introduces QCoh on prestacks. Even though the main focus of this book is the theory of ind-coherent sheaves, the latter theory takes a significant input and interacts with that of quasi-coherent sheaves. If you have not seen this before, read Sect. 1 and then Sects. 3.1-3.2.

3.4.2. In Volume I, Chapter 4 we develop the elementary aspects of the theory of IndCoh on schemes: we define the DG category $\mathrm{IndCoh}(Z)$ for an individual scheme Z, construct the IndCoh direct image functor, and also the !-pullback functor for proper morphisms. This Chapter uses the material from Volume I, Part I mentioned above. You will need the material from this chapter in order to proceed with the reading of the book.

Volume I, Chapter 5 builds on Volume 1, Chapter 4, and accomplishes (modulo the material delegated to Volume I, Chapter 7) one of the main goals of this book. We construct IndCoh as a *functor out of the category of correspondences.* In particular, we construct the functor (2.3). The material from this Chapter is also needed for the rest of the book.

In Volume I, Chapter 6 we study the interaction between IndCoh and QCoh. For an individual scheme Z we have an action of $\mathrm{QCoh}(Z)$ (viewed as a monoidal category) on $\mathrm{IndCoh}(Z)$. We study how this action interacts with the formalism of correspondences from Volume I, Chapter 5, and in particular with the operation of !-pullback. The material in this Chapter uses the formalism of monoidal categories and modules over them from Volume I, Chapter 1, as well as the material from Volume I, Chapter 5. Skipping Volume I, Chapter 6 will not impede your understanding of the rest of the book, so it might be a good idea to do so on the first pass.

3.4.3. Volume I, Part II develops the theory of categories of correspondences. It plays a service role for Volume I, Chapter 6 and Volume II, Chapter 3, and relies on the theory of $(\infty, 2)$-categories, developed in Volume I, Appendix.

3.4.4. Volume I, Appendix develops the theory of $(\infty, 2)$-categories. It plays a service role for Volume I, Part III.

Volume I, Chapters 11 and 12 rely on Volume I, Chapter 10, but can be read independently of one another.

3.4.5. Volume II, Chapter 1 introduces deformation theory. It is needed for the definition of inf-schemes and, therefore, for proofs of any results about inf-schemes (that is, for Volume II, Chapter 2). We will also need it for the discussion of formal moduli problems in Volume II, Chapter 5. The prerequisites for Volume II, Chapter

1 are Volume I, Chapters 2 and 3, so it is (almost)[11] independent of the material from Volume I, Part II.

In Volume II, Chapter 2 we introduce inf-schemes and some related notions (ind-schemes, ind-inf-schemes). The material here relies in that of Volume II, Chapter 1, and will be needed in Volume II, Chapter 3.

In Volume II, Chapter 3 we construct the theory of IndCoh on inf-schemes. The material here relies on that from Volume I, Chapter 5 and Volume II, Chapter 2 (and also a tedious general result about correspondences from Volume I, Chapter 8). Thus, Volume II, Chapter 3 achieves one of our goals, the later being making the theory of IndCoh on inf-schemes available. The material from Volume II, Chapter 3 will (of course) be used when will apply the theory of IndCoh, in Volume II, Chapter 4 and 7–9.

In Volume II, Chapter 4 we apply the material from Volume II, Chapter 3 in order to develop a proper framework for crystals (=D-modules), together with the forgetful/induction functors that related D-modules to $\mathcal{O}$-modules. The material from this Chapter will not be used later, except for the extremely useful notion of the de Rham prestack construction $\mathcal{X} \rightsquigarrow \mathcal{X}_{\mathrm{dR}}$.

3.4.6. In Volume II, Chapter 5 we prove a key result that says that in the category of prestacks that admit deformation theory, the operation of taking the quotient with respect to a formal groupoid is well-defined. The material here relies on that from Volume II, Chapter 1 (at some point we appeal to a proposition from Volume II, Chapter 3, but that can be avoided). So, the main result from Volume II, Chapter 5 is independent of the discussion of IndCoh.

Volume II, Chapter 6 is about Lie algebras (or more general operad algebras) in symmetric monoidal DG categories. It only relies on the material from Volume I, Chapter 1, and is independent of the preceding Chapters of the book (no DAG, no IndCoh). The material from this Chapter will be used for the subsequent Chapters in Volume II, Part II.

3.4.7. *A shortcut.* As has been mentioned earlier, Volume II, Chapters 7–9 are devoted to applications of IndCoh to 'differential calculus'. This 'differential calculus' occurs on prestacks that admit deformation theory.

If one really wants to use arbitrary such prestacks, one needs the entire machinery of IndCoh provided by Volume II, Chapter 3. However, if one is content with working with inf-schemes (which would suffice for the majority of applications), much less machinery would suffice:

The cofinality result from Chapter 3, Sect. 4.3 implies that we can bypass the entire discussion of correspondences, and only use the material from Volume I, Chapter 4, i.e., IndCoh on schemes and !-pullbacks for proper (in fact, finite) morphisms.

3.4.8. Volume II, Chapters 7-9 form a logical succession. As input from the preceding chapters they use Volume II, Chapter 3 (resp., Volume I, Chapter 5 (see Sect. 3.4.7 above), Volume II, Chapter 1 and Volume II, Chapters 5–6.

---

[11]Whenever we want to talk about *tangent* (as opposed to *cotangent*) spaces, we have to use IndCoh rather than QCoh, and these parts in Volume II, Chapter 1 use the material from Volume I, Chapter 5.

Acknowledgements

Introduction

1. What is done in Volume II?

In this volume, we will apply the theory of IndCoh developed in Volume I to do geometry (more precisely, to do what we understand by 'doing geometry').

Namely, we will introduce the class of geometric objects of interest, called *inf-schemes* (or relative versions thereof), and study the corresponding categories of ind-coherent sheaves. These are exactly the categories that one encounters in representation-theoretic situations.

We will perform various operations with inf-schemes (such as taking the quotient with respect to a groupoid), and we will study what such operations do to the corresponding categories of ind-coherent sheaves.

1.1. As was already mentioned, our basic class of geometric objects is that of inf-schemes, denoted $\mathrm{infSch}_{\mathrm{laft}}$. Just as any other class of geometric objects in this book, $\mathrm{infSch}_{\mathrm{laft}}$ is a full subcategory in the ambient category of prestacks. I.e., an inf-scheme is a contravariant functor on $\mathrm{Sch}^{\mathrm{aff}}$ that satisfies some *conditions* (rather than having some additional structure).

As is often the case in algebraic geometry, along with a particular class of objects, there is the corresponding relative notion. I.e., we also introduce what it means for a map of prestacks to be *inf-schematic*.

What are inf-schemes? The definition is surprisingly simple. These are prestacks (technically, locally almost of finite type) whose underlying reduced prestack is a (reduced) scheme[1], that *admit deformation theory*. The latter is a condition that guarantees reasonable infinitesimal behavior; we refer the reader to Chapter 1 of this volume for the precise definition of what it means to admit deformation theory.

Thus, one can informally say that the class of inf-schemes contains all prestacks that are schemes 'up to something infinitesimal, but controllable'. For example, all (derived) schemes, the de Rham prestacks of schemes and formal schemes are all examples of inf-schemes.

1.2. As was explained in Volume I, Chapter 5, once we have the theory of IndCoh on schemes (almost of finite type), functorial with respect to the operation of !-pullback, we can extend it to all prestacks (locally almost of finite type). In particular, we obtain the theory of IndCoh on inf-schemes, functorial with respect to !-pullbacks.

---

[1]I.e., when we evaluate our prestack on reduced affine schemes, the result is representable by a (reduced) scheme.

We proceed to define the functor of IndCoh-direct image for maps between inf-schemes that satisfies base change with respect to !-pullback. Furthermore, this leads to the operation of IndCoh-direct image for *inf-schematic* maps between prestacks (locally almost of finite type).

The IndCoh categories and the functors of !-pullback and IndCoh-direct image describe many of representation-theoretic categories and functors between them that arise in practice.

1.3. As an illustration of the utility of inf-schemes, we proceed to develop *Lie theory* in this context.

Let $\mathcal{X}$ be a base prestack (locally almost of finite type). On the one hand, we consider the category of *formal* group-objects over $\mathcal{Y}$. I.e., these are group-objects in the category of prestacks $\mathcal{Y}$ equipped with a map $f : \mathcal{Y} \to \mathcal{X}$, such that f is inf-schematic and induces an isomorphism at the reduced level. Denote this category by $\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$.

On the other hand, we consider the category $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$ of Lie algebra objects in the (symmetric monoidal) category $\mathrm{IndCoh}(\mathcal{X})$. We establish a canonical equivalence

(1.1) $$\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \simeq \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})).$$

I.e., this is an equivalence between formal groups and Lie algebras in full generality. We can view the equivalence (1.1) as a justification for the notion of inf-scheme (or rather, inf-schematic map): we need those in order to define the category $\mathrm{FormMod}_{/\mathcal{X}}$.

We show that given a group-object $\mathcal{G} \in \mathrm{FormMod}_{/\mathcal{X}}$, one can form its *classifying space*,

$$B_{\mathcal{X}}(\mathcal{G}) \in \mathrm{FormMod}_{/\mathcal{X}},$$

which is equipped with a section $s : \mathcal{X} \to B_{\mathcal{X}}(\mathcal{G})$ (i.e., it is pointed), so that $\mathcal{G}$ is recovered as the loop-object of $B_{\mathcal{X}}(\mathcal{G})$ in the category $\mathrm{FormMod}_{/\mathcal{X}}$. The above functor

$$B_{\mathcal{X}} : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$$

is equivalence.

We show that there is a canonical equivalence

(1.2) $$\mathrm{IndCoh}(B_{\mathcal{X}}(\mathcal{G})) \simeq \mathfrak{g}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})),$$

where the latter is the category of $\mathfrak{g}$-modules in the symmetric monoidal category $\mathrm{IndCoh}(\mathcal{X})$, where $\mathfrak{g}$ is the object of $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$ corresponding to $\mathcal{G}$ via (1.1).

With respect to this equivalence, the functors $s^!$ and s_*^{IndCoh} correspond to the forgetful and "free module" functors for the category of $\mathfrak{g}$-modules. Moreover, for the natural map $p : B_{\mathcal{X}}(\mathcal{G}) \to \mathcal{X}$, the functors $p^!$ and p_*^{IndCoh} correspond to the trivial $\mathfrak{g}$-module and Lie algebra homology functors, respectively. In short, Lie theory works at the level of representations as it should.

1.4. Let $\mathcal{X}$ be a prestack (locally almost of finite type that admits deformation theory). One of the basic objects involved in doing 'differential geometry' on $\mathcal{X}$ is that of Lie algebroid. Here comes an unpleasant surprise, though:

We have not been able to define the notion of Lie algebroid purely *algebraically*. Namely, the classical definition of Lie algebroid involves some binary operations that satisfy some relations, and we were not able to make sense of those in our context of derived algebraic geometry.

Instead, we *define* the notion of Lie algebroid via geometry: we set the category LieAlgbroid$(\mathcal{X})$ to be, by definition, that of formal groupoids over $\mathcal{X}$. This definition is justified by the equivalence (1.1), which says that Lie algebras are the same as formal groups.

We show that Lie algebroids defined in this way indeed behave in the way we expect Lie algebroids to behave. For example, we have a pair of adjoint functors

$$\text{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \rightleftarrows \text{LieAlgbroid}(\mathcal{X}),$$

(here $T(\mathcal{X}) \in \text{IndCoh}(\mathcal{X})$ is the tangent complex[2] of $\mathcal{X}$), where the right adjoint forgets the algebroid structure, and the left adjoint is the functor of *free Lie algebroid*.

We also show that the category LieAlgbroid$(\mathcal{X})$ is equivalent to the category FormMod$_{\mathcal{X}/}$ of formal moduli problems *under* $\mathcal{X}$, i.e., to the category of prestacks $\mathcal{Y}$ (locally almost of finite type that admit deformation theory) equipped with a map $f : \mathcal{X} \to \mathcal{Y}$ such that f is inf-schematic and induces an isomorphism at the reduced level.

For example, we show that under the equivalence

$$\text{(1.3)} \qquad\qquad \text{LieAlgbroid}(\mathcal{X}) \simeq \text{FormMod}_{\mathcal{X}/},$$

the functor of free Lie algebroid corresponds to the functor of square-zero extension.

Generalizing (1.2), we show that if $\mathfrak{L} \in \text{LieAlgbroid}(\mathcal{X})$ corresponds to $\mathcal{Y} \in \text{FormMod}_{\mathcal{X}/}$, we have a canonical equivalence

$$\text{(1.4)} \qquad\qquad \text{IndCoh}(\mathcal{Y}) \simeq \mathfrak{L}\text{-mod}(\text{IndCoh}(\mathcal{X})),$$

where the left-hand side is the appropriately defined category of objects of $\text{IndCoh}(\mathcal{X})$, equipped with an action of $\mathfrak{L}$.

A basic example of a Lie algebroid on $\mathcal{X}$ is the tangent algebroid $\mathcal{T}(\mathcal{X})$, whose underlying ind-coherent sheaf is the tangent complex. The corresponding formal moduli problem under $\mathcal{X}$ is $\mathcal{X}_{\text{dR}}$. By definition, $\text{IndCoh}(\mathcal{X}_{\text{dR}})$ is the category of D-modules on $\mathcal{X}$. As a consequence of the equivalence (1.4), we obtain that the category of D-modules on $\mathcal{X}$ is given by modules in $\text{IndCoh}(\mathcal{X})$ over the derived version of the ring of differential operators (built as the enveloping algebra of the algebroid $\mathcal{T}(\mathcal{X})$).

Here is a typical construction from representation theory that uses the theory of Lie algebroids. Let $\mathcal{X}$ be as above, and let $\mathfrak{g}$ be a Lie algebra (i.e., a Lie algebra

---

[2]Another advantage of the theory of ind-coherent sheaves is that a prestack $\mathcal{X}$ (locally almost of finite type that admits deformation theory) admit a *tangent complex*, which is an object of $\text{IndCoh}(\mathcal{X})$, while the more traditional *cotangent complex* is an object of the pro-category, and thus is more difficult to work with.

object in the category Vect of chain complexes of vector spaces over our ground field) that *acts* on $\mathcal{X}$. In this case, we can form a Lie algebroid $\mathfrak{g}_{\mathcal{X}}$ on $\mathcal{X}$, and a functor

$$\mathfrak{g}\text{-mod}(\text{Vect}) \to \mathfrak{g}_{\mathcal{X}}\text{-mod}(\text{IndCoh}(\mathcal{X})).$$

Composing with the induction functor for the map $\mathfrak{g}_{\mathcal{X}} \to \mathcal{T}(\mathcal{X})$ and using the equivalence (1.4), we construct the *localization functor*

$$\mathfrak{g}\text{-mod}(\text{Vect}) \to \text{IndCoh}(\mathcal{X}_{\text{dR}}) \simeq \text{Dmod}(\mathcal{X}).$$

1.5. At the end of this volume, we develop some elements of 'infinitesimal differential geometry'. Namely, we address the following question:

Many objects of differential-geometric nature come equipped with natural filtrations. For example, we have the filtration on the ring of differential operators (according to the order), or the filtration on a formal completion of a scheme along a subscheme (the n-th infinitesimal neighborhoods). We wish to have similar structures in the general context of prestacks (locally almost of finite type that admit deformation theory). However, as is always the case in higher category theory and derived algebraic geometry, we cannot define these filtrations 'by hand'.

Instead, we use the following idea: a filtered object (of linear nature) is the same as a family of such objects over $\mathbb{A}^1$ that is equivariant with respect to the action of $\mathbb{G}_m$ by dilations.

We implement this idea in geometry. Namely, we show that given a $\mathcal{Y} \in$ FormMod$_{\mathcal{X}/}$, we can canonically construct its *deformation to the normal cone*, which is a family

(1.5) $$\mathcal{Y}_t \in \text{FormMod}_{\mathcal{X}/}, \quad t \in \mathbb{A}^1,$$

that deforms the original $\mathcal{Y}$ to a vector bundle situation.

We show (which by itself might not be so well-known even in the context of usual algebraic geometry) that this deformation gives rise to all the various filtrations that we are interested in.

In its turn, the deformation (1.5) also needs to be constructed by a functorial procedure (rather than an explicit formula). We construct it using a certain geometric device, explained in Chapter 9, Sect. 2.

2. What do we use from Volume I?

2.1. One thing is unavoidable: we use the language of higher category theory. So, the reader in encouraged to become familiar with the contents of Chapter 1, Sects. 1 and 2 of Volume I.

Here are some of the most essential pieces of notation. For the remainder of this section, all references are to Volume I.

We denote by 1-Cat the $(\infty, 1)$-category of $(\infty, 1)$-categories, and by Spc its full subcategory that consists of spaces (a.k.a. ∞-groupoids).

For a pair of $(\infty, 1)$-categories $\mathbf{C}$ and $\mathbf{D}$, we let Funct$(\mathbf{C}, \mathbf{D})$ denote the $(\infty, 1)$-category of functors between them.

For an $(\infty, 1)$-category $\mathbf{C}$, and a pair objects $\mathbf{c}_0, \mathbf{c}_1 \in \mathbf{C}$, we denote by Maps$_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1)$ the space of maps from $\mathbf{c}_0$ to $\mathbf{c}_1$.

2.2. Much of the time, we will work in a linear context, by which we mean the world of DG categories. Note that we only consider 'large,' i.e. cocomplete (and in fact, presentable) DG categories. We refer the reader to Chapter 1, Sect. 10 for the definition.

We let Vect denote the DG category of chain complexes of vector spaces (over a fixed ground field k, assumed to be of characteristic zero).

Given a pair of DG categories $\mathbf{C}$ and $\mathbf{D}$, we will write $\mathrm{Funct}(\mathbf{C}, \mathbf{D})$ for the DG category of k-linear functors between them; this clashes with the notation introduced earlier that denoted all functors, but we believe that this is unlikely to lead to confusion.

We write $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{C}, \mathbf{D})$ for the full subcategory of $\mathrm{Funct}(\mathbf{C}, \mathbf{D})$ that consists of *continuous* (i.e., colimit preserving) functors. We denote by $\mathrm{DGCat}_{\mathrm{cont}}$ the $(\infty, 1)$-category formed by DG categories and continuous linear functors between them.

For a given DG category $\mathbf{C}$ and a pair of objects $\mathbf{c}_0, \mathbf{c}_1 \in \mathbf{C}$, we will write $\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1) \in \mathrm{Vect}$ for the chain complex of maps between them. The objects $\mathcal{M}aps_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1) \in \mathrm{Vect}$ and $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}_1) \in \mathrm{Spc}$ are related by the Dold-Kan functor, see *loc. cit.*

For a pair of DG categories $\mathbf{C}$ and $\mathbf{D}$, we denote by $\mathbf{C} \otimes \mathbf{D}$ their tensor product.

2.3. The basic notions pertaining to derived algebraic geometry are set up in Chapter 2.

We denote by PreStk the $(\infty, 1)$-category of all prestacks, i.e., accessible functors

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}.$$

For this volume, it is of crucial importance to know the definition of the full subcategory

$$\mathrm{PreStk}_{\mathrm{laft}} \subset \mathrm{PreStk}$$

that consists of prestacks that are locally almost of finite type, see Chapter 2, Sect. 1.7 for the definition.

Derived schemes are introduced in Chapter 2, Sect. 3; the corresponding category is denoted by Sch. Henceforth, we will omit the word 'derived' and refer to derived schemes as schemes.

For a prestack $\mathcal{X}$, we denote by $\mathrm{QCoh}(\mathcal{X})$ the DG category of quasi-coherent sheaves on it; see Chapter 3, Sect. 1 for the definition.

2.4. One cannot read this volume without knowing what ind-coherent sheaves are. For an individual scheme X (almost of finite type), the category $\mathrm{IndCoh}(X)$ is defined in Chapter 4, Sect. 1. The basic functoriality is given by the functor of IndCoh-direct image

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y),$$

for a map $f : X \to Y$, see Chapter 4, Sect. 2.

The machinery of the IndCoh functor is fully developed in Chapter 5, where we construct the !-pullback functor

$$f^! : \text{IndCoh}(Y) \to \text{IndCoh}(X),$$

for f as above.

We denote by $\omega_X \in \text{IndCoh}(X)$ the dualizing object, i.e., the pullback of the generator

$$k \in \text{Vect} \simeq \text{IndCoh}(\text{pt})$$

under the tautological projection $X \to \text{pt}$.

The assignment $X \rightsquigarrow \text{IndCoh}(X)$ (with !-pullbacks) is extended from schemes (almost of finite type) to all of $\text{PreStk}_{\text{laft}}$ by the procedure of left Kan extension. Moreover, we make IndCoh a functor out of the *2-category of correspondences*,

$$\text{Corr}(\text{PreStk}_{\text{laft}})^{\text{sch \& proper}}_{\text{sch-qc;all}} \to \text{DGCat}^{\text{2-Cat}}_{\text{cont}}.$$

In the above formula, $\text{DGCat}^{\text{2-Cat}}_{\text{cont}}$ is the 2-categorical enhancement of $\text{DGCat}_{\text{cont}}$, see Chapter 1, Sect. 10.3.9.

One piece of notation from Chapter 6 that the reader might need is the functor

$$\Upsilon_{\mathcal{X}} : \text{QCoh}(\mathcal{X}) \to \text{IndCoh}(\mathcal{X}),$$

defined for any $\mathcal{X} \in \text{PreStk}_{\text{laft}}$, and given by tensoring a given object of $\text{QCoh}(\mathcal{X})$ by the dualizing object $\omega_{\mathcal{X}} \in \text{IndCoh}(\mathcal{X})$.

2.5. Chapter 7 introduces the category of correspondences. The idea is that given an $(\infty, 1)$-category $\mathbf{C}$ and three classes of morphisms $vert, horiz$ and adm, we introduce an $(\infty, 2)$-category $\text{Corr}(\mathbf{C})^{adm}_{vert,horiz}$, whose objects are the same as those of $\mathbf{C}$, but where 1-morphisms from $\mathbf{c}_0$ to $\mathbf{c}_1$ are *correspondences*

$$
\begin{array}{ccc}
\mathbf{c}_{0,1} & \xrightarrow{\alpha} & \mathbf{c}_0 \\
\downarrow{\scriptstyle \beta} & & \\
\mathbf{c}_1 & &
\end{array}
$$

where $\alpha \in horiz$ and $\beta \in vert$. For a pair of correspondences $(\mathbf{c}_{0,1}, \alpha, \beta)$ and $(\mathbf{c}'_{0,1}, \alpha', \beta')$, the space of 2-morphisms between them is that of commutative diagrams

For a pair of correspondences $(\mathbf{c}_{0,1}, \alpha, \beta)$ and $(\mathbf{c}'_{0,1}, \alpha', \beta')$, we want the space of maps between them to be that of commutative diagrams

(2.1)

with $\gamma \in adm$.

2.6. The use of other notions and notations from Volume I is sporadic and the reader can look it up, using the index of notation of Volume I when needed.

Part I

Inf-schemes

Introduction

1. Why inf-schemes?

1.1. The primary new geometric object considered in this book is the notion of *inf-scheme*. Let us start with the definition: an inf-scheme is a prestack $\mathcal{X}$ such that:

(a) $\mathcal{X}$ is laft (locally almost of finite type, see Volume I, Chapter 2, Sect. 1.7 for what this means);

(b) $\mathcal{X}$ admits deformation theory (i.e., has reasonable infinitesimal properties, see Chapter 1 of this volume or Sect. 2 of this Introduction);

(c) The underlying reduced prestack $^{\text{red}}\mathcal{X}$ is a (reduced) scheme.

Let us explain what are the favorable properties enjoyed by inf-schemes and how one is led to this definition.

1.2. Our initial goal was to have a geometric framework in which we could talk simultaneously about $\text{QCoh}(X)$ and $\text{Dmod}(X)$ (where $X \in \text{Sch}_{\text{aft}}$) equipped with the pair of adjoint functors of forgetting the D-module structure to that of an $\mathcal{O}$-module, and inducing an $\mathcal{O}$-module to a D-module.

However, as was explained in [**GaRo2**], if we replace $\text{QCoh}(X)$ by $\text{IndCoh}(X)$, the resulting adjoint pair has much better properties. So what we really want to consider is the functors

$$(1.1) \qquad \mathbf{ind}_X : \text{IndCoh}(X) \leftrightarrows \text{Dmod}(X) : \mathbf{oblv}_X,$$

and their compatibility with the direct and inverse functors on $\text{IndCoh}(-)$ and $\text{Dmod}(-)$ for maps between schemes.

According to [**GaRo2**], the category $\text{Dmod}(X)$ is *defined* as $\text{IndCoh}(X_{\text{dR}})$, where X_{dR} is the de Rham prestack of X (i.e., $\text{Maps}(S, X_{\text{dR}}) = \text{Maps}(^{\text{red}}S, X)$).

So it is natural to set up the sought-for theory as IndCoh of a certain class of prestacks that contains schemes and de Rham prestacks of schemes. We would like the adjoint pair (1.1) to be given by the push-forward/pullback adjunct

$$(p_{\text{dR},X})_*^{\text{IndCoh}} : \text{IndCoh}(X) \leftrightarrows \text{IndCoh}(X_{\text{dR}}) : (p_{\text{dR},X})^!,$$

where $p_{\text{dR},X}$ denotes the tautological map $X \to X_{\text{dR}}$.

1.3. Let us try to be minimalistic and consider only prestacks $\mathcal{X}$ such that $^{\text{red}}\mathcal{X}$ is a (reduced) scheme. Let us denote the sought-for class of prestacks by $\mathcal{C}$, and let us list some constructions that we would like to be possible within $\mathcal{C}$.

(i) For a map $f : \mathcal{X}_1 \to \mathcal{X}_2$ between objects of $\mathcal{C}$ we should have a well-defined push-forward functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}_1) \to \mathrm{IndCoh}(\mathcal{X}_2),$$

that satisfies base change with respect to the !-pullback. I.e., IndCoh restricted to $\mathcal{C}$ should extend to a functor out of the category of correspondences on $\mathcal{C}$.

(ii) Since we want to talk about base change, $\mathcal{C}$ should contain fiber products. Now, for a scheme X, the fiber product $X \underset{X_{\mathrm{dR}}}{\times} X$ is the formal completion $X^\wedge$ of X in $X \times X$. Hence, it is natural to ask that $\mathcal{C}$ contain *formal schemes*, i.e., *ind-schemes*, whose underlying reduced prestacks are (reduced) schemes.

(iii) Having included in $\mathcal{C}$ all formal schemes, one's appetite grows a little more. Let $\mathcal{G}$ be a formal groupoid over a scheme X that belongs to $\mathcal{C}$. One would like to be able to form the quotient of X by $\mathcal{G}$, which is still a prestack in our class $\mathcal{C}$. For example, the quotient of X of $X^\wedge$ should give us back X_{dR}.

(iv) Finally, we would like to have a description of *formal group-objects* in $\mathcal{C}$ over a scheme X in terms of their Lie algebras. As will be explained in Chapter 1, the latter, being tangent spaces at the unit section, are objects of $\mathrm{IndCoh}(X)$. So by a 'Lie algebra' we should understand a Lie algebra in the symmetric monoidal category $\mathrm{IndCoh}(X)$ with respect to the $\overset{!}{\otimes}$-tensor product.

1.4.   As we will eventually see in Part I of this volume, properties (iii) and (iv) will force us to include into our class $\mathcal{C}$ all inf-schemes $\mathcal{X}$, defined as above.

However, one can consider it a strike of luck that as general a definition as one given in Sect. 1.1 above produces a workable notion, i.e., IndCoh on inf-schemes has the properties mentioned above.

2. Deformation theory

The definition of inf-schemes involves the notion of *admitting deformation theory*. In Chapter 1 of this part of the book we make a review of deformation theory.

2.1.   A prestack $\mathcal{X}$ is said to admit deformation theory if:

(i) $\mathcal{X}$ is convergent, i.e., for $S \in \mathrm{Sch}$, the map

$$\mathrm{Maps}(S, \mathcal{X}) \to \lim_n \mathrm{Maps}(^{\leq n}S, \mathcal{X})$$

is an isomorphism. (In other words, the values of $\mathcal{X}$ on all affine schemes are completely determined by its values on eventually coconnective affine schemes.)

(ii) For a push-out diagram

$$
\begin{array}{ccc}
S_1 & \longrightarrow & S_2 \\
\downarrow & & \downarrow \\
S_1' & \longrightarrow & S_2',
\end{array}
$$

of affine schemes, where the map $S_1 \to S_1'$ (and, hence, also $S_2 \to S_2'$) is a *nilpotent embedding*, the corresponding diagram

$$
\begin{array}{ccc}
\mathrm{Maps}(S_1, \mathcal{X}) & \longleftarrow & \mathrm{Maps}(S_2, \mathcal{X}) \\
\uparrow & & \uparrow \\
\mathrm{Maps}(S_1', \mathcal{X}) & \longleftarrow & \mathrm{Maps}(S_2', \mathcal{X}),
\end{array}
$$

is a pull-back diagram.

In condition (ii), we remind that a map of affine schemes $S \to S'$ is said to be nilpotent embedding if the corresponding map of classical schemes ${}^{\mathrm{cl}}S \to {}^{\mathrm{cl}}S'$ is a closed embedding with a nilpotent ideal of definition.

We also remind that if

$$
S_1 = \mathrm{Spec}(A_1), \ \ S_2 = \mathrm{Spec}(A_2), \ \ S_1' = \mathrm{Spec}(A_1'), \ \ S_2' = \mathrm{Spec}(A_2'),
$$

then to be a push-out diagram means that the map

$$
A_2' \to A_1' \underset{A_1}{\times} A_2
$$

should be an isomorphism in $\mathrm{Vect}^{\leq 0}$.

2.2. The above way of formulating what it means to admit deformation theory may at first appear mysterious (why these push-outs, and who has ever seen push-outs in algebraic geometry anyway?). And indeed, the more common definition, and the one we give in Chapter 1, Sect. 7 is different (but, of course, equivalent). The advantage of the definition given above is that it is concise.

It turns out that the infinitesimal behavior of a prestack that admits deformation theory is goverened by its *pro-cotangent complex*, where the latter is a functorial assignment for any

$$
S \xrightarrow{x} \mathcal{X} \quad \in \quad \mathrm{Sch}_{/\mathcal{X}}
$$

of an object $T_x^*(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S^-))$. The precise meaning of the words 'governed by' is explained in the Introduction to Chapter 1.

Here we just say informally that, say when $\mathcal{X}$ is laft, the knowledge of the values of $\mathcal{X}$ on *reduced affine schemes* and some *linear* data (expressible in terms of the pro-cotangent complex of $\mathcal{X}$) allows to recover the values of $\mathcal{X}$ on all schemes.

2.3. Going back to inf-schemes, requiring the condition that they admit deformation theory makes them reasonable objects: by condition (c) in Sect. 1.1, the values of an inf-scheme $\mathcal{X}$ on a reduced scheme are given by a scheme $X = {}^{\mathrm{red}}\mathcal{X}$, and when we want evaluate $\mathcal{X}$ on an arbitrary scheme S, the fibers of the map

$$
\mathrm{Maps}(S, \mathcal{X}) \to \mathrm{Maps}({}^{\mathrm{red}}S, \mathcal{X}) \simeq \mathrm{Maps}({}^{\mathrm{red}}S, X)
$$

are controlled by linear data.

3. Inf-schemes

In Chapter 2 we introduce inf-schemes and study their basic properties. The main results of this chapter are Theorems 4.1.3 and 4.2.5. Here we will informally explain what these theorems say.

3.1. Let $\mathcal{X}$ be an inf-scheme such that $^{\mathrm{red}}\mathcal{X} = X \in {}^{\mathrm{red}}\mathrm{Sch}$. Consider the full subcategory

$$(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nil\text{-}isom\,to}\,\mathcal{X}} \subset (\mathrm{Sch})_{/\mathcal{X}}$$

consisting of those maps $Z \to \mathcal{X}$ that are *nil-isomorphisms*, i.e., induce an isomorphism

$$^{\mathrm{red}}Z \to {}^{\mathrm{red}}\mathcal{X} = X.$$

The assertion of Chapter 2, Theorem 4.1.3 (in the guise of Chapter 2, Corollary 4.3.3) is that the resulting map

$$\underset{Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nil\text{-}isom\,to}\,\mathcal{X}}}{\mathrm{colim}} Z \to \mathcal{X}$$

is an isomorphism of prestacks.

The latter means, by definition, that for an *affine* scheme S, the map

$$(3.1) \qquad \underset{Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nil\text{-}isom\,to}\,\mathcal{X}}}{\mathrm{colim}} \mathrm{Maps}(S, Z) \to \mathrm{Maps}(S, \mathcal{X})$$

is an isomorphism.

Equivalently, for an *affine* scheme S and a map $S \to \mathcal{X}$, the category $\mathrm{Factor}(x, \mathrm{aft}, \mathrm{nil\text{-}isom})$ of its factorizations as

$$S \to Z \to \mathcal{X}$$

with $Z \in \mathrm{Sch}_{\mathrm{aft}}$ and the map $Z \to \mathcal{X}$ being a nil-isomorphism, is contractible.

3.2. Let us emphasize, however, that it is *not* true that the map (3.1) is an isomorphism if S is non-affine. Equivalently, it is *not* true that the category $\mathrm{Factor}(x, \mathrm{aft}, \mathrm{nil\text{-}isom})$ is contractible if S is non-affine.

We remark, however, that in Sects. 1 and 2 we study a more restricted class of objects, commonly called *formal schemes* (but we choose to call *nil-schematic ind-schemes*), for which the map (3.1) is an isomorphism (and the category $\mathrm{Factor}(x, \mathrm{aft}, \mathrm{nil\text{-}isom})$ is contractible).

In fact, we consider the full subcategory

$$(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nilp\text{-}embed\,into}\,\mathcal{X}} \subset (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nil\text{-}isom\,to}\,\mathcal{X}},$$

consisting of those $Z \to \mathcal{X}$ that are *nilpotent embeddings.*

We show that if $\mathcal{X}$ is a nil-schematic ind-scheme, then the category $(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nil\text{-}embed\,into}\,\mathcal{X}}$ is *filtered* and the map

$$\underset{Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nilp\text{-}embed\,into}\,\mathcal{X}}}{\mathrm{colim}} \mathrm{Maps}(S, Z) \to \mathrm{Maps}(S, \mathcal{X})$$

is an isomorphism for *any* (i.e., not necessarily affine) $S \in \mathrm{Sch}$.

Equivalently, for a given $S \overset{x}{\to} \mathcal{X}$, the corresponding category $\mathrm{Factor}(x, \mathrm{aft}, \mathrm{nilp\text{-}embed})$ is contractible.

3.3. We will now explain the content of the second main result of this Chapter, namely, Chapter 2, Theorem 4.2.5 (in its guise as Chapter 2, Corollary 4.4.6).

Let $\mathcal{X}$ be an inf-scheme, such that $X := {}^{\mathrm{red}}\mathcal{X}$ is affine. It follows from Theorem 4.1.3 that $\mathcal{X}$, when viewed as a functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc},$$

is completely determined[3] by its restriction to the category

$$\mathrm{Sch}^{\mathrm{aff}} \underset{{}_{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}}{\times} \{X\}.$$

In words, the above category is that of affine schemes, whose reduced subscheme is of finite type and is equipped with a map to X.

Now, Chapter 2, Theorem 4.2.5 is a converse to the above assertion. Namely, it says that any functor

$$\left(\mathrm{Sch}^{\mathrm{aff}} \underset{{}_{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}}{\times} \{X\}\right)^{\mathrm{op}} \to \mathrm{Spc},$$

that satisfies deformation theory-like conditions, gives rise to an inf-scheme $\mathcal{X}$ with ${}^{\mathrm{red}}\mathcal{X} \simeq X$.

4. Ind-coherent sheaves on inf-schemes

Chapter 3 of this part is a central one for this book. In it we study the category IndCoh on inf-schemes.

4.1. What makes this theory manageable is Chapter 2, Theorem 4.1.3 mentioned above. Namely, when we write

$$\mathcal{X} \simeq \underset{\alpha \in A}{\mathrm{colim}}\, Z_\alpha,$$

where $Z_\alpha \in \mathrm{Sch}_{\mathrm{aft}}$ and the transition maps $f_{\alpha,\beta} : Z_\alpha \to Z_\beta$ are nil-isomorphisms, we have:

$$\mathrm{IndCoh}(\mathcal{X}) \simeq \underset{\alpha \in A^{\mathrm{op}}}{\lim}\, Z_\alpha,$$

where the limit is formed with respect to the functors of !-pullback

$$f_{\alpha,\beta} \rightsquigarrow f_{\alpha,\beta}^! : \mathrm{IndCoh}(Z_\beta) \to \mathrm{IndCoh}(Z_\alpha).$$

The above presentation of $\mathrm{IndCoh}(\mathcal{X})$ as a limit tells what the objects and morphisms are in this category. However, since the functors $f_{\alpha,\beta}^!$ admit left adjoints, by Volume I, Chapter 1, Proposition 2.5.7 we also have:

$$(4.1) \qquad \mathrm{IndCoh}(\mathcal{X}) \simeq \underset{\alpha \in A}{\mathrm{colim}}\, \mathrm{IndCoh}(Z_\alpha),$$

where the colimit is formed with respect to the push-forward functors

$$f_{\alpha,\beta} \rightsquigarrow (f_{\alpha,\beta})^{\mathrm{IndCoh}_*} : \mathrm{IndCoh}(Z_\alpha) \to \mathrm{IndCoh}(Z_\beta).$$

The latter presentation tells us what it takes to construct a functor *out of* $\mathrm{IndCoh}(\mathcal{X})$. Namely, such a functor amounts to a compatible family of functors out of $\mathrm{IndCoh}(X_\alpha)$.

---

[3] Technically, by 'completely determined' we mean 'is the left Kan extension from'.

4.2.   The main construction in Chapter 3 is that of the direct image functor. Namely, let $f : \mathcal{X}^1 \to \mathcal{X}^2$ be a map between inf-schemes. We want to construct the functor

$$(4.2) \qquad\qquad f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}^1) \to \mathrm{IndCoh}(\mathcal{X}^2).$$

For example, when $\mathcal{X}^i = X_{\mathrm{dR}}^i$ where $X^i \in \mathrm{Sch}_{\mathrm{ft}}$, the resulting functor will be the de Rham (D-module) direct image.

In Theorem 4.3.2 we show that there exists a functor (4.2) that is *uniquely* characterized by the requirement that whenever

$$
\begin{array}{ccc}
Z^1 & \xrightarrow{\;g_1\;} & \mathcal{X}^1 \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Z^2 & \xrightarrow{\;g_2\;} & \mathcal{X}^2
\end{array}
$$

is a commutative diagram with $Z^i \in \mathrm{Sch}_{\mathrm{aft}}$ and the maps g_i nil-isomorphisms, then the diagram of functors

$$
\begin{array}{ccc}
\mathrm{IndCoh}(Z^1) & \xrightarrow{\;(g_1)_*^{\mathrm{IndCoh}}\;} & \mathrm{IndCoh}(\mathcal{X}^1) \\
{\scriptstyle (f')_*^{\mathrm{IndCoh}}}\downarrow & & \downarrow{\scriptstyle f_*^{\mathrm{IndCoh}}} \\
\mathrm{IndCoh}(Z^2) & \xrightarrow{\;(g_2)_*^{\mathrm{IndCoh}}\;} & \mathrm{IndCoh}(\mathcal{X}^2)
\end{array}
$$

commutes, where the functors $(g_i)_*^{\mathrm{IndCoh}}$ are the ones from the presentation (4.1).

4.3.   Having constructed direct images, we show that they satisfy the proper base change property. Then, by applying the general machinery from Volume I, Chapter 8, Sect. 1, we show that IndCoh, viewed as a functor out of the category of correspondences

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}^{\mathrm{proper}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}^{\mathrm{proper}} \to \mathrm{DGCat}_{\mathrm{cont}}^{2\text{-}\mathrm{Cat}}$$

uniquely extends to a functor

$$(4.3)\quad \mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{corr:all;all}}^{\mathrm{ind\text{-}proper}}} : \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{corr:all;all}}^{\mathrm{ind\text{-}proper}} \to \mathrm{DGCat}_{\mathrm{cont}}^{2\text{-}\mathrm{Cat}} .$$

5. Crystals and D-modules

In Chapter 4 we apply the theory of IndCoh on inf-schemes to construct the theory of D-modules, viewed as a functor

$$(5.1) \qquad \mathrm{Dmod}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}^{\mathrm{proper}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}^{\mathrm{proper}} \to \mathrm{DGCat}_{\mathrm{cont}}^{2\text{-}\mathrm{Cat}} .$$

5.1.   Namely, we stipulate that for $X \in \mathrm{Sch}_{\mathrm{aft}}$

$$(5.2) \qquad\qquad \mathrm{Dmod}(X) := \mathrm{IndCoh}(X_{\mathrm{dR}}).$$

Now, the operation $X \mapsto X_{\mathrm{dR}}$ defines a functor

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{corr:all;all}}^{\mathrm{proper}} \to \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{corr:all;all}}^{\mathrm{ind\text{-}proper}} .$$

Thus, composing this functor with (4.3), we obtain the desired functor (5.1).

5.2. The definition of D-modules as in (5.2) gives also a natural framework for the induction functor

$$(5.3) \qquad \qquad \text{IndCoh}(X) \to \text{Dmod}(X),$$

left adjoint to the tautological forgetful functor.

Namely, the functor (5.3) is the functor of direct image with respect to the tautological morphism

$$X \to X_{\text{dR}}.$$

5.3. In Sect. 4 of this Chapter, we explain why the definition of D-modules (5.2) is the right thing to do.

Namely, we show that when X is a smooth affine scheme, $\text{IndCoh}(X_{\text{dR}})$ does indeed recover the category of modules over the ring Diff_X of differential operators on X.

Moreover, we show that for a map $f : X \to Y$ between smooth schemes, the functors

$$(f_{\text{dR}})^! : \text{IndCoh}(Y_{\text{dR}}) \to \text{IndCoh}(X_{\text{dR}}) \text{ and } (f_{\text{dR}})_*^{\text{IndCoh}} : \text{IndCoh}(X_{\text{dR}}) \to \text{IndCoh}(Y_{\text{dR}})$$

correspond to the usual functors of pullback and push-forward on the corresponding categories of D-modules.

Deformation theory

Introduction

0.1. What does deformation theory do?

0.1.1. *Deformation theory via pullbacks.* 'Admitting deformation theory' refers to a certain property of a prestack. Namely, although the initial definition will be different, according to Proposition 7.2.2, a prestack $\mathcal{X}$ admits deformation theory if:

(a) It is convergent, i.e., for $S \in \mathrm{Sch}^{\mathrm{aff}}$ the map $\mathrm{Maps}(S, \mathcal{X}) \to \varprojlim_n \mathrm{Maps}(^{\leq n}S, \mathcal{X})$ is an isomorphism;

(b) For a push-out diagram of objects of $\mathrm{Sch}^{\mathrm{aff}}$

$$
\begin{array}{ccc}
S_1 & \longrightarrow & S_2 \\
\downarrow & & \downarrow \\
S_1' & \longrightarrow & S_2',
\end{array}
$$

where the map $S_1 \to S_1'$ is a *nilpotent embedding* (i.e., the map of classical affine schemes $^{\mathrm{cl}}S_1 \to {}^{\mathrm{cl}}S_2$ is a closed embedding, whose ideal of definition vanishes to some power), the resulting diagram in Spc

$$
\begin{array}{ccc}
\mathrm{Maps}(S_1, \mathcal{X}) & \longleftarrow & \mathrm{Maps}(S_2, \mathcal{X}) \\
\uparrow & & \uparrow \\
\mathrm{Maps}(S_1', \mathcal{X}) & \longleftarrow & \mathrm{Maps}(S_2', \mathcal{X})
\end{array}
$$

is a pullback diagram.

The reason that this notion is useful is that it allows to study the infinitesimal behavior of $\mathcal{X}$ (i.e., properties of the map $\mathrm{Maps}(S', \mathcal{X}) \to \mathrm{Maps}(S, \mathcal{X})$ whenever $S \to S'$ is a nilpotent embedding) by using *linear* objects. Let us explain this in more detail.

0.1.2. *Pro-cotangent fibers and cotangent complex.* For $S = \mathrm{Spec}(A)$ and an S-point $x : S \to \mathcal{X}$, by considering nilpotent embeddings of the form $S \to S'$ for

$$
S' = S_{\mathcal{I}} := \mathrm{Spec}(A \oplus M), \quad M = \Gamma(S, \mathcal{I}), \quad \mathcal{I} \in \mathrm{QCoh}(S)^{\leq 0},
$$

one shows that the functor

(0.1) $$\mathcal{I} \mapsto \mathrm{Maps}_{S/}(S_{\mathcal{I}}, \mathcal{X}), \quad \mathrm{QCoh}(S)^{\leq 0} \to \mathrm{Spc}$$

is given by a well-defined object

$$
T_x(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-),
$$

called the *pro-cotangent space* of $\mathcal{X}$ at x. We emphasize that the fact that the functor (0.1) comes from such an object is already a non-trivial condition and amounts to this functor commuting with certain pullbacks that are among the pullbacks in Sect. 0.1.1.

Next, one shows that the assignments

$$(0.2) \qquad (S,x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}} \quad \rightsquigarrow \quad T_x^*(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-),$$

are compatible in the sense that for

$$f : S_1 \to S_2, \quad x_2 \in \mathrm{Maps}(S_2, \mathcal{X}), \quad x_1 = x_2 \circ f,$$

the natural map in $\mathrm{Pro}(\mathrm{QCoh}(S_1)^-)$

$$T_{x_1}^*(\mathcal{X}) \to \mathrm{Pro}(f^*)(T_{x_2}^*(\mathcal{X}))$$

is an isomorphism. This follows from the condition in Sect. 0.1.1 applied to the push-out diagram

$$\begin{array}{ccc} S_1 & \xrightarrow{\quad f \quad} & S_2 \\ \downarrow & & \downarrow \\ (S_1)_{\mathcal{I}} & \longrightarrow & (S_2)_{f_*(\mathcal{I})}. \end{array}$$

Furthermore, one shows that for $\mathcal{X} = X \in \mathrm{Sch}$, we have $T_x^*(X) \in \mathrm{QCoh}(S)^{\leq 0}$, so that the assignment (0.2) comes from a well-defined object $T^*(X) \in \mathrm{QCoh}(X)^{\leq 0}$, called the *cotangent complex* of X.

By functoriality, for $(S,x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$, we have a canonically defined map in $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$

$$T_x^*(\mathcal{X}) \to T^*(S),$$

called the *co-differential* of x.

0.1.3. *Square-zero extensions.* Among nilpotent embeddings $S \to S'$ one singles out a particular class, called *square-zero extensions*. Namely, one shows that for every object $\mathcal{I} \in \mathrm{QCoh}(S)^{\leq 0}$ equipped with a map

$$T^*(S) \xrightarrow{\gamma} \mathcal{I}[1]$$

one can canonically attach a nilpotent embedding

$$S \xrightarrow{i} S_{\mathcal{I},\gamma},$$

such that

$$\mathrm{Fib}(\mathcal{O}_{S_{\mathcal{I},\gamma}} \to i_*(\mathcal{O}_S)) \simeq i_*(\mathcal{I}).$$

In fact, the pair $(i, S_{\mathcal{I},\gamma})$ can be uniquely characterized by the property that for a map $f : S \to U$ in $\mathrm{Sch}^{\mathrm{aff}}$, the data of extension of f to a map $S_{\mathcal{I},\gamma} \to U$ is equivalent to that of a *null-homotopy* of the composed map

$$f^*(T^*(U)) \to T^*(S) \xrightarrow{\gamma} \mathcal{I}[1],$$

where the map $f^*(T^*(U)) \to T^*(S)$ is the co-differential of f.

In particular, for $\gamma = 0$ we have tautologically $S_{\mathcal{I},0} = S_{\mathcal{I}}$, where $S_{\mathcal{I}}$ is as in Sect. 0.1.2.

When S is classical and $\mathcal{I} \in \mathrm{QCoh}(S)^{\heartsuit}$, one shows that the above assignment

$$(\mathcal{I}, \gamma) \rightsquigarrow S_{\mathcal{I},\gamma}$$

is an equivalence between $\mathrm{QCoh}^{\heartsuit}_{T^*(S)[-1]/}$ and the category of closed embeddings $S \to S'$ with S' being a classical scheme such that the ideal of definition of S in S' squares to 0.

0.1.4. *From square-zero extensions to all nilpotent extensions.* A fact of crucial importance is that *any* nilpotent embedding $S \to S'$ can be obtained as a composition of square-zero extensions, *up to any given truncation.* More precisely, there exists a sequence of affine schemes

$$S = S_0 \to S_1 \to S_2 \to ... \to S_n \to ... \to S'$$

such that each $S_i \to S_{i+1}$ is a square-zero extension, and for every n there exists m such that the maps

$$^{\leq n}S_m \to {}^{\leq n}S_{m+1} \to ... \to {}^{\leq n}S'$$

are all isomorphisms.

Thus, if $\mathcal{X}$ is *convergent*, then if we can control the map

$$\mathrm{Maps}(S', \mathcal{X}) \to \mathrm{Maps}(S, \mathcal{X})$$

when $S \to S'$ is a square-zero extension, we can control it for any nilpotent embedding.

0.1.5. *Back to deformation theory.* One shows that if $\mathcal{X}$ admits deformation theory, then, given a point $S \overset{x}{\to} \mathcal{X}$, the datum of its extension to a point $S_{\mathcal{I},\gamma} \to \mathcal{X}$ is equivalent to the datum of a null-homotopy for the map

$$(0.3) \qquad\qquad T^*_x(\mathcal{X}) \to T^*(S) \overset{\gamma}{\to} \mathcal{I}[1].$$

The latter fact, combined with Sect. 0.1.4, is the precise expression of the above-mentioned principle that extensions of a given map $S \to \mathcal{X}$ to a map $S' \to \mathcal{X}$ (for a nilpotent embedding $S \to S'$) are controlled by linear objects, the latter being the objects in $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$ appearing in (0.3).

0.2. What is done in this chapter? This chapter splits naturally into two halves: the build-up to the formulation of what it means to admit deformation theory (Sects. 1-6) and consequences of the property of admitting deformation theory (Sects. 7-10).

0.2.1. *Push-outs.* In Sect. 1, we study the operation of push-out on affine schemes. The reason we need to do this is, as was mentioned above, we formulate the property of a prestack $\mathcal{X}$ to admit deformation theory in terms of push-outs.

Note that the operation of push-out is not so ubiquitous in algebraic geometry– we are much more used to pullbacks. In terms of rings, pullbacks are given by

$$(A_1 \leftarrow A \to A_2) \quad \mapsto \quad A_1 \underset{A}{\otimes} A_2,$$

while push-outs by

$$(A_1 \to A \leftarrow A_2) \quad \mapsto \tau^{\leq 0}(A_1 \underset{A}{\times} A_2).$$

The operation of push-out on affine schemes is not so well-behaved (for example, a push-out of affine schemes may not be a push-out in the category of all schemes).

However, there is one case in which it is well-behaved: namely, when we consider push-out diagrams

$$
\begin{array}{ccc}
S_1 & \longrightarrow & S_2 \\
\downarrow & & \downarrow \\
S_1' & \longrightarrow & S_2',
\end{array}
$$

in which $S_1 \to S_1'$ is a nilpotent embedding.

0.2.2. *(Pro)-cotangent spaces.* In Sect. 2 we define what it means for a prestack $\mathcal{X}$ to admit a (pro)-cotangent space at a given S-point

$$S \xrightarrow{x} \mathcal{X}.$$

By definition, a (pro)-cotangent space, if it exists, is an object

$$T_x^*(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-).$$

The definition is given in terms of the notion of split-square zero extension, $S \to S_{\mathcal{F}}$, see Sect. 0.1.2.

We shall say that $\mathcal{X}$ admits a *cotangent space* at x if $T_x^*(\mathcal{X})$ actually belongs to $\mathrm{QCoh}(S)^-$.

One result in this section that goes beyond definitions is Proposition 2.5.3 that gives the expression of the (pro)-cotangent space of a prestack $\mathcal{X}$ that is itself given as a colimit of prestacks $\mathcal{X}_\alpha$. Namely, if both $\mathcal{X}$ and all $\mathcal{X}_\alpha$ admit (pro)-cotangent spaces, then the (pro)-cotangent space of $\mathcal{X}$ is the limit of the (pro)-cotangent spaces of $\mathcal{X}_\alpha$ (as is natural to expect).

0.2.3. *The tangent space.* In Sect. 3 we discuss various conditions that make sense for objects of $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$ that one can impose on pro-cotangent spaces. We would like to draw the reader's attention to two of these properties: one is *convergence* and the other is *laft*-ness.

An object $\Phi \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)$ is said to be *convergent* it for $\mathcal{F} \in \mathrm{QCoh}(S)^-$, the map

$$\mathrm{Maps}(\Phi, \mathcal{F}) \to \lim_n \mathrm{Maps}(\Phi, \tau^{\geq -n}(\mathcal{F}))$$

is an isomorphism.

It follows almost tautologically that if $\mathcal{X}$ is convergent (in the sense of Sect. 0.1.1), then its pro-cotangent spaces are convergent in the above sense.

An object $\Phi \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)$ is said to be *laft* if it is convergent and for every m_1, m_2, the functor

$$\mathcal{F} \mapsto \mathrm{Maps}(\Phi, \mathcal{F}), \quad \mathcal{F} \in \mathrm{QCoh}(S)^{\geq -m_1, \leq m_2}$$

commutes with filtered colimits.

Again, it follows almost tautologically that if $\mathcal{X}$ is laft, then its pro-cotangent spaces are laft in the above sense.

If S is itself laft, the full subcategory

$$\mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}} \subset \mathrm{Pro}(\mathrm{QCoh}(S)^-)$$

has the following nice interpretation: Serre duality identifies it with the opposite of the category $\mathrm{IndCoh}(S)$.

So, instead of thinking of $T_x^*(\mathcal{X})$ as an object of $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$, we can think of its formal dual, denoted

$$T_x(\mathcal{X}) \in \mathrm{IndCoh}(S),$$

and called the *tangent* space of $\mathcal{X}$ at x.

0.2.4. *The naive tangent space.* We want to emphasize that in our interpretation, the tangent space is *not* the *naive* dual of the (pro)-cotangent space, but rather the *Serre* dual. One can define the naive duality functor

$$(0.4) \qquad\qquad (\mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{QCoh}(S)$$

by sending

$$\Phi \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}} \quad \mapsto \quad \Phi(\mathcal{O}_S),$$

where $\Phi(\mathcal{O}_S)$ is regarded as an A-module if $S = \mathrm{Spec}(A)$.

However, the above functor is the composition of the Serre duality equivalence

$$(\mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{IndCoh}(S),$$

followed by the functor

$$\mathrm{IndCoh}(S) \to \mathrm{QCoh}(S), \quad \mathcal{F} \mapsto \underline{\mathrm{Hom}}(\omega_S, \mathcal{F}),$$

while the latter fails to be conservative (even for S eventually coconnective).

So, in general, the naive duality functor (0.4) loses information, and hence it is not a good idea to think of the tangent space as an object of $\mathrm{QCoh}(S)$ equal to the naive dual of the (pro)-cotangent case.

In the case when S is eventually coconnective, one can explicitly describe a full subcategory inside $\mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}}$ on which the naive duality functor (0.4) is fully faithful:

This is the image of the fully faithful embedding

$$\mathrm{QCoh}(S) \to (\mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}})^{\mathrm{op}},$$

given by

$$\mathcal{F} \in \mathrm{QCoh}(S) \mapsto \Phi_\mathcal{F}, \quad \Phi_\mathcal{F}(\mathcal{F}_1) = \operatorname*{colim}_n \Gamma(S, \mathcal{F} \otimes \tau^{\geq -n}(\mathcal{F}_1)), \quad \mathcal{F}_1 \in \mathrm{QCoh}(S)^-.$$

0.2.5. *The (pro)-cotangent complex.* In Sect. 4 we impose a condition on a prestack $\mathcal{X}$ that its (pro)-cotangent spaces are compatible under pullbacks, see Sect. 0.1.2. If this condition is satisfied, we say that $\mathcal{X}$ *admits a (pro)-cotangent complex.*

If $\mathcal{X}$ is laft, we show (using Sect. 0.2.3 above) that if it admits a pro-cotangent complex, then it admits a *tangent complex*, which is an object of $\mathrm{IndCoh}(\mathcal{X})$.

0.2.6. *Square-zero extensions.* In Sect. 5 we introduce the category of square-zero extensions of a scheme, already mentioned in Sect. 0.1.3 above.

By definition, the category of square-zero extensions of X, denoted $\mathrm{SqZ}(X)$, is

$$((\mathrm{QCoh}(X)^{\leq 0})_{T(X)[-1]/})^{\mathrm{op}}.$$

As was explained in Sect. 0.1.3 we have a functor

$$\mathrm{RealSqZ} : \mathrm{SqZ}(X) \to (\mathrm{Sch})_{X/}.$$

We note, however, that unless X is classical and we restrict ourselves to the part of $\mathrm{SqZ}(X)$ that corresponds to $((\mathrm{QCoh}(X)^{\heartsuit})_{T(X)[-1]/})^{\mathrm{op}}$, the functor $\mathrm{RealSqZ}$ is *not* fully faithful. I.e., being a square-zero extension is not a condition but additional structure.

We proceed to study several crucial pieces of structure pertaining to square-zero extensions:

(i) A canonical structure of square-zero extension on an $(n+1)$-coconnective scheme of square-zero extension of its n-coconnective truncation;

(ii) The approximation of any nilpotent embedding by a series of square-zero extensions, already mentioned in Sect. 0.1.4;

(iii) Functoriality of square-zero extension under push-forwards: given a square-zero extension $X_1 \to X_1'$ by means of $\mathcal{I} \in \mathrm{QCoh}(X_1)^{\leq 0}$ and an affine morphism $f : X_1 \to X_2$, we obtain a canonically defined structure of square-zero extension by means of $f_*(\mathcal{I})$ on $X_2 \to X_2' := X_1' \underset{X_1}{\sqcup} X_2$.

(iv) Functoriality of square-zero extension under pullbacks: given a square-zero extension $X_2 \to X_2'$ by means of $\mathcal{I} \in \mathrm{QCoh}(X_2)^{\leq 0}$ and a map $f' : X_1' \to X_2'$, we obtain a canonically defined structure of square-zero extension on $X_2 \underset{X_2'}{\times} X_1' =: X_1 \to X_1'$ by

means of $f^*(\mathcal{I})$, where f is the resulting map $X_1 \to X_2$.

0.2.7. *Infinitesimal cohesiveness.* In Sect. 6 we define what it means for a prestack $\mathcal{X}$ to be *infinitesimally cohesive*. Namely, we say that $\mathcal{X}$ is infinitesimally cohesive if whenever $S \to S'$ is a square-zero extension of affine schemes given by

$$T^*(S) \overset{\gamma}{\to} \mathcal{I}[1], \quad \mathcal{I} \in \mathrm{QCoh}(S)^{\leq 0},$$

and $x : S \to \mathcal{X}$ is a map, the (naturally defined) map from the space of extensions of x to a map $x' : S' \to \mathcal{X}$ to the space of *null-homotopies* of the composed map

$$T_x^*(\mathcal{X}) \to T^*(S) \overset{\gamma}{\to} \mathcal{I}[1]$$

is an isomorphism.

We explain that this property can also be interpreted as the fact that $\mathcal{X}$ takes certain push-outs in $\mathrm{Sch}^{\mathrm{aff}}$ to pullbacks in Spc.

0.2.8. *Finally: deformation theory!* In Sect. 7 we finally introduce what it means for a prestack $\mathcal{X}$ to admit deformation theory. We define it as follows: we say that $\mathcal{X}$ admits deformation theory if:

(a) It is convergent;

(b') It admits a pro-cotangent complex;

(b") It is infinitesimally cohesive.

However, as was mentioned in Sect. 0.1.1, in Proposition 7.2.2, we show that one can replace conditions (b') and (b") by just one condition (b) from Sect. 0.1.1, namely that $\mathcal{X}$ takes certain push-out to pullbacks.

We proceed to study some properties of prestacks associated with the notion of admitting deformation theory:

(i) We introduce and study the notion of *formal smoothness* of a prestack;

(ii) We show that for any integer k, prestacks that are k-Artin stacks admit deformation theory.

0.2.9. *Consequences of admitting deformation theory.* In Sect. 8 we derive some consequences of the fact that a given prestack admits deformation theory:

(i) If $\mathcal{X}_0$ is a classical prestack and $i : \mathcal{X}_0 \to {}^{\mathrm{cl}}\mathcal{X}$ is a nilpotent embedding, then if $\mathcal{X}_0$ satisfies étale descent, then so does $\mathcal{X}$;

(ii) In the above situation, if $f : \mathcal{X} \to \mathcal{X}'$ is a map where $\mathcal{X}'$ admits deformation theory such that for any classical affine scheme S and a map $x_0 : S \to \mathcal{X}_0$, the map

$$T^*_{f \circ i \circ x_0}(\mathcal{X}') \to T^*_{i \circ x_0}(\mathcal{X})$$

is an isomorphism, then f itself is an isomorphism.

0.2.10. *Deformation theory and laft-ness.* In Sect. 9 we prove two assertions related to the interaction of deformation theory with the property of a prestack to be laft (locally almost of finite type).

The first assertion, Theorem 9.1.2, gives the following infinitesimal criterion to determine whether $\mathcal{X}$ is laft. Namely, it says that a prestack $\mathcal{X}$ admitting deformation theory is laft if and only if:

(i) ${}^{\mathrm{cl}}\mathcal{X}$ is locally of finite type;
(ii) For any classical scheme of finite type S and a point $x : S \to {}^{\mathrm{cl}}\mathcal{X}$, we have $T^*_x(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}}$,

The second assertion, Theorem 9.1.4, says that if $\mathcal{X}$ admits deformation theory, its laft-ness property implies something stronger than for arbitrary laft prestacks. Namely, it says that $\mathcal{X}$, when viewed as a functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}$$

is the left Kan extension from

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \subset (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}.$$

I.e., for any $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$, the space of factoring x as

$$S \to U \to \mathcal{X}, \quad U \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$$

is contractible.

Note that for arbitrary prestacks such a property holds not on all $S \in \mathrm{Sch}^{\mathrm{aff}}$ but only on truncated ones.

0.2.11. *Square-zero extensions of prestacks.* In Sect. 10 we define the notion of square-zero extension of a given prestack $\mathcal{X}$ by means of $\mathcal{I} \in \mathrm{QCoh}(\mathcal{X})^{\leq 0}$. This is defined via the functoriality of square-zero extensions under pullbacks, see Sect. 0.2.6(iv).

Assuming that $\mathcal{X}$ admits deformation theory, we prove that under certain circumstances, the map from $\mathrm{SqZ}(\mathcal{X}, \mathcal{I})$ to $\mathrm{Maps}(T^*(\mathcal{X}), \mathcal{I}[1])$ is an isomorphism, and that any prestack obtained as a square-zero extension of $\mathcal{X}$ itself admits deformation theory.

1. Push-outs of schemes

In this subsection we study the operation of push-out on schemes. This operation is not so ubiqutous in algebraic geometry. However, it is crucial for deformation theory. In fact, deformation theory is defined in terms of compatibility with certain push-outs.

1.1. Push-outs in the category of affine schemes.
In this subsection we describe what push-outs look like in the category of affine schemes.

1.1.1.   Let
$$i \mapsto X_i, \quad i \in \mathbf{I}$$
be an $\mathbf{I}$-diagram in $\mathrm{Sch}^{\mathrm{aff}}$ for some $\mathbf{I} \in 1\text{-Cat}$.

Let Y denote its colimit in the category $\mathrm{Sch}^{\mathrm{aff}}$. I.e., if $X_i = \mathrm{Spec}(A_i)$, then $Y = \mathrm{Spec}(B)$, where
$$B = \lim_i A_i,$$
where the limit is taken in the category of *connective* commutative k-algebras.

REMARK 1.1.2. Note that in the above formula $B = \tau^{\leq 0}(B')$, where
$$B' = \lim_i A_i,$$
the limit is taken in the category $\mathrm{ComAlg}(\mathrm{Vect})$ of all commutative k-algebras. Note also that the forgetful functor
$$\mathbf{oblv}_{\mathrm{Com}} : \mathrm{ComAlg}(\mathrm{Vect}) \to \mathrm{Vect}$$
commutes with limits, so that it is easy to understand what B' looks like.

The functor $\tau^{\leq 0}$ of connective truncation used above is also very explicit. Namely, by definition, the category of connective commutative k-algebras is $\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0})$, and the functor $\tau^{\leq 0}$ is the right adjoint to the embedding
$$\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}) \to \mathrm{ComAlg}(\mathrm{Vect}).$$
This functor makes the diagram

$$
\begin{array}{ccc}
\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}) & \xleftarrow{\ \tau^{\leq 0}\ } & \mathrm{ComAlg}(\mathrm{Vect}) \\
{\scriptstyle \mathbf{oblv}_{\mathrm{Com}}} \downarrow & & \downarrow {\scriptstyle \mathbf{oblv}_{\mathrm{Com}}} \\
\mathrm{Vect}^{\leq 0} & \xleftarrow{\ \tau^{\leq 0}\ } & \mathrm{Vect}
\end{array}
$$

commute.

1.1.3. In particular, consider a diagram $X_1 \leftarrow X \rightarrow X_2$ in $\mathrm{Sch}^{\mathrm{aff}}$ and set

$$Y := X_1 \underset{X}{\sqcup} X_2,$$

where the push-out is taken in $\mathrm{Sch}^{\mathrm{aff}}$. I.e., if $X_i = \mathrm{Spec}(A_i)$ and $X = \mathrm{Spec}(A)$, then $Y = \mathrm{Spec}(B)$, where

$$B := A_1 \underset{A}{\times} A_2.$$

Note that if $X \rightarrow X_1$ is a closed embedding, then so is the map $X_2 \rightarrow Y$.

1.2. The case of closed embeddings. In this subsection we show that if we take push-outs with respect to maps that are closed embeddings, then this operation is well-behaved.

1.2.1. Suppose we are in the context of Sect. 1.1.3. We observe the following:

LEMMA 1.2.2. *Suppose that both maps $X \rightarrow X_i$ are closed embeddings. Then:*

(a) *The Zariski topology on Y is induced by that on $X_1 \sqcup X_2$.*

(b) *For open affine subschemes $\overset{\circ}{X}_i \subset X_i$ such that $\overset{\circ}{X}_1 \cap X = \overset{\circ}{X}_2 \cap X =: \overset{\circ}{X}$, and the corresponding (by point (a)) open subscheme $\overset{\circ}{Y} \subset Y$, the map*

$$\overset{\circ}{X}_1 \underset{\overset{\circ}{X}}{\sqcup} \overset{\circ}{X}_2 \rightarrow \overset{\circ}{Y}$$

is an isomorphism, where the push-out is taken in $\mathrm{Sch}^{\mathrm{aff}}$.

(c) *The diagram*

$$\begin{array}{ccc} X & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & Y \end{array}$$

is also a push-out diagram also in Sch.

1.2.3. From here we obtain:

COROLLARY 1.2.4. *Let $X_1 \leftarrow X \rightarrow X_2$ be a diagram in Sch, where both maps $X_i \rightarrow X$ are closed embeddings. Then:*

(a) *The push-out $Y := X_1 \underset{X}{\sqcup} X_2$ in Sch exists.*

(b) *The Zariski topology on Y is induced by that on $X_1 \sqcup X_2$.*

(c) *For open subschemes $\overset{\circ}{X}_i \subset X_i$ such that $\overset{\circ}{X}_1 \cap X = \overset{\circ}{X}_2 \cap X =: \overset{\circ}{X}$, and the corresponding (by point (b)) open subscheme $\overset{\circ}{Y} \subset Y$, the map*

$$\overset{\circ}{X}_1 \underset{\overset{\circ}{X}}{\sqcup} \overset{\circ}{X}_2 \rightarrow \overset{\circ}{Y}$$

is an isomorphism.

1.3. The push-out of a closed nil-isomorphism. The situation studied in this subsection is of crucial importance for deformation theory.

1.3.1. We recall (see Volume I, Chapter 4, Sect. 6.1.4) that a map $X \to Y$ is Sch is called a nil-isomorphism if it induces an isomorphism

$$^{\mathrm{red}}X \to {}^{\mathrm{red}}Y,$$

where the notation $^{\mathrm{red}}X$ means the reduced classical scheme underlying cl$X$.

We emphasize that 'nil-isomorphism' *does not* imply 'closed embedding'.

1.3.2. Let

$$X_1 \to X_1'$$

be a *closed* nil-isomorphism of affine schemes, and let $f : X_1 \to X_2$ be a map, where $X_2 \in \mathrm{Sch}^{\mathrm{aff}}$.

Let $X_2' = X_1' \underset{X_1}{\sqcup} X_2$, where the colimit is taken in $\mathrm{Sch}^{\mathrm{aff}}$. Note that the map

$$X_2 \to X_2'$$

is also a closed nil-isomorphism.

We observe:

LEMMA 1.3.3. *In the above situation we have:*

(a) *For an open affine subscheme $\mathring{X}_2 \subset X_2$, $f^{-1}(\mathring{X}_2) =: \mathring{X}_1 \subset X_1$, and the corresponding open affine subschemes $\mathring{X}_i' \subset X_i'$ for $i = 1, 2$, the map*

$$\mathring{X}_1' \underset{\mathring{X}_1}{\sqcup} \mathring{X}_2 \to \mathring{X}_2'$$

is an isomorphism, where the push-out is taken in $\mathrm{Sch}^{\mathrm{aff}}$.

(b) *The diagram*

$$
\begin{array}{ccc}
X_1' & \longrightarrow & X_2' \\
\uparrow & & \uparrow \\
X_1 & \longrightarrow & X_2
\end{array}
$$

is also a push-out diagram in Sch.

1.3.4. As a corollary we obtain:

COROLLARY 1.3.5. *Let $X_1 \to X_1'$ be a closed nil-isomorphism of (not necessarily affine) schemes, and let $f : X_1 \to X_2$ be an affine map between schemes. Then:*

(a) *The push-out $X_2' := X_1' \underset{X_1}{\sqcup} X_2$ in* Sch *exists, the map $X_2 \to X_2'$ is a closed nil-isomorphism (and in particular affine).*

(b) *For an open subscheme $\mathring{X}_2 \subset X_2$, $f^{-1}(\mathring{X}_2) =: \mathring{X}_1 \subset X_1$, and the corresponding open affine subscheme $\mathring{X}_i' \subset X_i'$, the map*

$$\mathring{X}_1' \underset{\mathring{X}_1}{\sqcup} \mathring{X}_2 \to \mathring{X}_2'$$

is an isomorphism, where the push-out is taken in Sch.

(c) *If f is an open embedding, then so is the map $X_1' \to X_2'$.*

PROOF. By Lemma 1.3.3, it suffices to prove the corollary when X_2 is affine, in which case it also follows from Lemma 1.3.3.

$\square$

REMARK 1.3.6. Note that in the situation of Corollary 1.3.5(a), if the map $X_1 \to X_2$ is not affine, the map $X_2 \to X_2'$ is not necessarily a closed embedding. In fact, it can look like a "pinching":

Let $X_2 := \mathbb{A}^2$, and let X_1 be its blow-up at the origin. Let $X_1 \hookrightarrow X_1'$ be the square-zero extension supported on exceptional divisor with ideal $\mathcal{O}(-2)^{\oplus 2}$, given by the canonical element in $\mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-2)^{\oplus 2})$. Then X_2' is the spectrum of the algebra consisting of polynomials with a vanishing derivative at the origin.

1.4. Behavior of quasi-coherent sheaves. In this subsection we will describe how the categories QCoh and IndCoh behave with respect to the operation of push-out.

1.4.1. Let $f : X_1 \to X_2$ and $g_1 : X_1 \to X_1'$ be maps of affine schemes with g_1 being a closed embedding. Let

$$X_2' := X_1' \underset{X_1}{\sqcup} X_2$$

be the push-out in the category $\mathrm{Sch}^{\mathrm{aff}}$.

We claim:

PROPOSITION 1.4.2. *In the diagram*

(1.1)
$$
\begin{array}{ccc}
\mathrm{QCoh}(X_1) & \xleftarrow{\;g_1^*\;} & \mathrm{QCoh}(X_1') \\
{\scriptstyle f^*}\big\uparrow & & \big\uparrow{\scriptstyle f'^*} \\
\mathrm{QCoh}(X_2) & \xleftarrow{\;g_2^*\;} & \mathrm{QCoh}(X_2'),
\end{array}
$$

the map

$$\mathrm{QCoh}(X_2') \to \mathrm{QCoh}(X_2) \underset{\mathrm{QCoh}(X_1)}{\times} \mathrm{QCoh}(X_1')$$

is fully faithful.

REMARK 1.4.3. We note that even when the maps f and g_1 are closed embeddings, the diagram (1.1) is generally *not*[1] a pullback square in $\mathrm{DGCat}_{\mathrm{cont}}$. Indeed, this fails already for $X_1 = \mathrm{pt}$, $X_2 \simeq X_1' \simeq \mathbb{A}^1$.

That said, one can show that the diagram consisting of perfect complexes

$$
\begin{array}{ccc}
\mathrm{QCoh}(X_1)^{\mathrm{perf}} & \xleftarrow{\;g_1^*\;} & \mathrm{QCoh}(X_1')^{\mathrm{perf}} \\
{\scriptstyle f^*}\big\uparrow & & \big\uparrow{\scriptstyle f'^*} \\
\mathrm{QCoh}(X_2)^{\mathrm{perf}} & \xleftarrow{\;g_2^*\;} & \mathrm{QCoh}(X_2')^{\mathrm{perf}},
\end{array}
$$

is a pullback square, see Chapter 8, Sect. A.2.

We note that there is no contradiction between the above two facts: if

$$\mathbf{C} \to \mathbf{D} \leftarrow \mathbf{E}$$

---

[1] We are grateful to D. Nadler who pointed this out to us, thereby correcting a mistake in the previous version.

is a diagram of compactly generated categories with functors preserving compact objects, the inclusion

$$(\mathbf{C}^c \underset{\mathbf{D}^c}{\times} \mathbf{E}^c) \hookrightarrow (\mathbf{C} \underset{\mathbf{D}}{\times} \mathbf{E})^c$$

is in general *not* an equality (although the corresponding fact is true for *filtered* limits).

PROOF OF PROPOSITION 1.4.2. One readily reduces the assertion to the case when X_2 (and hence also X_1 and X_1') are affine. In the latter case, we will prove the proposition just assuming that the map g_1 is a closed embedding.

We construct a functor

$$\mathrm{QCoh}(X_1') \underset{\mathrm{QCoh}(X_1)}{\times} \mathrm{QCoh}(X_2) \to \mathrm{QCoh}(X_2'),$$

right adjoint to the tautological functor

$$(f'^* \times g_2^*) : \mathrm{QCoh}(X_2') \to \mathrm{QCoh}(X_1') \underset{\mathrm{QCoh}(X_1)}{\times} \mathrm{QCoh}(X_2),$$

by sending a datum

$$(\mathcal{F}_1' \in \mathrm{QCoh}(X_1'), \ \mathcal{F}_2 \in \mathrm{QCoh}(X_2), \ \mathcal{F}_1 \in \mathrm{QCoh}(X_1), \ g_1^*(\mathcal{F}_1') \simeq \mathcal{F}_1 \simeq f^*(\mathcal{F}_2))$$

to

$$\mathrm{Fib}\left(f_*'(\mathcal{F}_1') \oplus g_{2*}(\mathcal{F}_2) \to h_*(\mathcal{F}_1)\right),$$

where h denotes the map $X_1 \to X_2'$.

We claim that the unit of the adjunction is an isomorphism. Indeed, we have to check that for $\mathcal{F} \in \mathrm{QCoh}(X_2')$, the map

$$\mathcal{F} \to \mathrm{Fib}\left(f_*' \circ f'^*(\mathcal{F}) \oplus g_{2*} \circ g_2^*(\mathcal{F}) \to h_* \circ h^*(\mathcal{F})\right)$$

is an isomorphism. However, the above map is obtained by tensoring $\mathcal{F}$ with the corresponding map for $\mathcal{F} = \mathcal{O}_X$. Hence, since X is affine, it suffices to check that the map

$$\Gamma(X_2', \mathcal{O}_{X_2'}) \to \mathrm{Fib}\left(\Gamma(X_1', \mathcal{O}_{X_1'}) \oplus \Gamma(X_2, \mathcal{O}_{X_2}) \to \Gamma(X_1, \mathcal{O}_{X_1})\right)$$

is an isomorphism. However, the latter follows from the construction of the push-out

$$\square$$

1.4.4.   Assume now that in the above situation, X_1, X_2, X_1' belong to $\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}}$ and the map $f : X_1 \to X_2$ is finite so that X_2' also belongs to $\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}}$.

We claim:

PROPOSITION 1.4.5. *In the diagram*

(1.2)
$$\begin{array}{ccc} \mathrm{IndCoh}(X_1) & \xleftarrow{\;g_1^!\;} & \mathrm{IndCoh}(X_1') \\ {\scriptstyle f^!}\big\uparrow & & \big\uparrow{\scriptstyle f'^!} \\ \mathrm{IndCoh}(X_2) & \xleftarrow{\;g_2^!\;} & \mathrm{IndCoh}(X_2'), \end{array}$$

the map

$$\mathrm{IndCoh}(X_2') \to \mathrm{IndCoh}(X_2) \underset{\mathrm{IndCoh}(X_1)}{\times} \mathrm{IndCoh}(X_1')$$

is fully faithful.

REMARK 1.4.6. Unlike the situation with QCoh, in Chapter 8, Sect. A.1, we will see that (1.2) *is* a pullback square[2].

PROOF OF PROPOSITION 1.4.5. We construct the *left* adjoint to the

$$(f'^! \times g_2^!) : \mathrm{IndCoh}(X_2') \to \mathrm{IndCoh}(X_1') \underset{\mathrm{IndCoh}(X_1)}{\times} \mathrm{IndCoh}(X_2),$$

by sending a datum

$$(\mathcal{F}_1' \in \mathrm{IndCoh}(X_1'), \ \mathcal{F}_2 \in \mathrm{IndCoh}(X_2), \ \mathcal{F}_1 \in \mathrm{IndCoh}(X_1), \ g_1^!(\mathcal{F}_1') \simeq \mathcal{F}_1 \simeq f^!(\mathcal{F}_2))$$

to

$$\mathrm{coFib}\left(h_*^{\mathrm{IndCoh}}(\mathcal{F}_1) \to (f')_*^{\mathrm{IndCoh}}(\mathcal{F}_1') \oplus (g_2)_*^{\mathrm{IndCoh}}(\mathcal{F}_2)\right).$$

We claim that the co-unit of the adjunction is an isomorphism. I.e., we claim that for $\mathcal{F} \in \mathrm{IndCoh}(X_2')$, the map

$$(1.3) \quad \mathrm{coFib}\left(h_*^{\mathrm{IndCoh}} \circ h^!(\mathcal{F}) \to (f')_*^{\mathrm{IndCoh}} \circ (f')^!(\mathcal{F}) \oplus (g_2)_*^{\mathrm{IndCoh}} \circ g_2^!(\mathcal{F})\right) \to \mathcal{F}$$

is an isomorphism.

Note that in order to check this, it is enough to take $\mathcal{F} \in \mathrm{Coh}(X_2')$, in which case, both sides in (1.3) belong to $\mathrm{IndCoh}(X_2')^+$. Hence, it is enough to show that the map (1.3) becomes an isomorphism after applying the functor

$$\Gamma^{\mathrm{IndCoh}}(X_2', -) : \mathrm{IndCoh}(X_2') \to \mathrm{Vect}.$$

Denote

$$A_i = \Gamma(X_i, \mathcal{O}_{X_i}), \ A_i' = \Gamma(X_i', \mathcal{O}_{X_i'}), \quad i = 1, 2.$$

Denote $\mathcal{M} \coloneqq \Gamma^{\mathrm{IndCoh}}(X_2', \mathcal{F})$. After applying $\Gamma^{\mathrm{IndCoh}}(X_2', -)$ to the left-hand side in (1.3) we obtain

$$\mathrm{coFib}\left(\mathcal{M}aps_{A_2'\text{-mod}}(A_1, M) \to \mathcal{M}aps_{A_2'\text{-mod}}(A_1', M) \oplus \mathcal{M}aps_{A_2'\text{-mod}}(A_2, M)\right),$$

and that maps isomorphically to M, since

$$A_2' \to \mathrm{Fib}(A_1' \oplus A_2 \to A_1)$$

is an isomorphism.

$$\square$$

2. (pro)-cotangent and tangent spaces

In this section we define what it means for a prestack to admit a *(pro)-cotangent space* at a given S-point, where $S \in \mathrm{Sch}^{\mathrm{aff}}$. The definition is given in terms of the construction known as the *split zero extension*.

2.1. Split square-zero extensions. In this subsection we review the construction of split zero extensions; see [**Lu2**, Sect. 7.3.4].

---

[2]We are again grateful to D. Nadler, who pointed out a gap in the proof of this assertion in an earlier version.

2.1.1.   Let S be an object of $\mathrm{Sch}^{\mathrm{aff}}$. There is a natural functor

$$\mathrm{RealSplitSqZ} : (\mathrm{QCoh}(S)^{\leq 0})^{\mathrm{op}} \to (\mathrm{Sch}^{\mathrm{aff}})_{S/}$$

that assigns to $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0}$ the corresponding *split square-zero extension* $\mathrm{RealSplitSqZ}(\mathcal{F})$, also denoted by $S_{\mathcal{F}}$.

Namely, if $S = \mathrm{Spec}(A)$ and $\Gamma(S, \mathcal{F}) = M \in A\text{-mod}$,

$$S_{\mathcal{F}} = \mathrm{Spec}(A \oplus M).$$

2.1.2.   One can show (see [**Lu2**, Theorem 7.3.4.13] or Chapter 6, Proposition 1.8.3) that the functor $\mathrm{RealSplitSqZ}$ defines an equivalence

$$\mathrm{QCoh}(S)^{\leq 0} \simeq \mathrm{ComMonoid}(((\mathrm{Sch}^{\mathrm{aff}})_{S//S})^{\mathrm{op}}),$$

where $\mathrm{ComMonoid}(-)$ denotes the category of commutative monoids in a given $(\infty, 1)$-category, see Volume I, Chapter 1, Sect. 3.3.3.

2.1.3.   The following is nearly tautological:

LEMMA 2.1.4.   *The functor*

$$\mathrm{RealSplitSqZ} : (\mathrm{QCoh}(S)^{\leq 0})^{\mathrm{op}} \to (\mathrm{Sch}^{\mathrm{aff}})_{S/}$$

commutes with colimits.

In addition, as in Lemma 1.3.3 one shows:

LEMMA 2.1.5.   *The composite functor*

$$\mathrm{RealSplitSqZ} : (\mathrm{QCoh}(S)^{\leq 0})^{\mathrm{op}} \to (\mathrm{Sch}^{\mathrm{aff}})_{S/} \to \mathrm{Sch}_{S/}$$

also commutes with colimits.

2.1.6. *Terminology.* In what follows, for an affine scheme S, we will also use the notation

$$\mathrm{SplitSqZ}(S) := (\mathrm{QCoh}(S)^{\leq 0})^{\mathrm{op}},$$

so that $\mathrm{RealSplitSqZ}$ is a functor

$$\mathrm{SplitSqZ}(S) \to (\mathrm{Sch}^{\mathrm{aff}})_{S/}.$$

2.2. The condition of admitting a (pro)-cotangent space at a point. The condition that a given prestack admit a (pro)-cotangent space at a point means that it is *infinitesimally linearizable*, i.e., defines an exact (=excisive) functor on split square-zero extensions.

2.2.1.   Let $\mathcal{X}$ be an arbitrary object of PreStk, and let (S, x) be an object of $(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$.

We consider the functor $\mathrm{QCoh}(S)^{\leq 0} \to \mathrm{Spc}$, given by

(2.1) $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0} \mapsto \mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}) \in \mathrm{Spc}.$

2.2.2. Let $\mathcal{F}_1 \to \mathcal{F}_2$ be a map in $\mathrm{QCoh}(S)^{\leq 0}$, such that $H^0(\mathcal{F}_1) \to H^0(\mathcal{F}_2)$ is a surjection. Set

$$\mathcal{F} := 0 \underset{\mathcal{F}_2}{\times} \mathcal{F}_1.$$

By assumption, $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0}$.

Note that by Lemma 2.1.4,

$$(2.2) \qquad\qquad S \underset{S_{\mathcal{F}_2}}{\sqcup} S_{\mathcal{F}_1} \to S_{\mathcal{F}}$$

is a push-out diagram in $\mathrm{Sch}^{\mathrm{aff}}$ (and, by Lemma 2.1.5 or Lemma 1.3.3(b), also in Sch).

Consider the corresponding map

$$(2.3) \qquad \mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}) \to * \underset{\mathrm{Maps}_{S/}(S_{\mathcal{F}_2}, \mathcal{X})}{\times} \mathrm{Maps}_{S/}(S_{\mathcal{F}_1}, \mathcal{X}).$$

DEFINITION 2.2.3. *Let $\mathcal{X}$ be an object of* PreStk. *We shall say that $\mathcal{X}$ admits a* pro-cotangent *space at the point x, if the map* (2.3) *is an isomorphism for all $\mathcal{F}_1 \to \mathcal{F}_2$ as above.*

2.2.4. For example, from Lemma 1.3.3(b) we obtain:

COROLLARY 2.2.5. *If $\mathcal{X} = X \in$ Sch, then X admits a* pro-cotangent *space at any $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/X}$.*

2.2.6. Suppose that $\mathcal{X}$ admits a pro-cotangent space at x. Note that the functor (2.1) can be extended to a functor

$$(2.4) \qquad\qquad \mathrm{QCoh}(S)^- \to \mathrm{Spc},$$

by sending $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq k}$ to

$$\Omega^i(\mathrm{Maps}_{S/}(S_{\mathcal{F}[i]}, \mathcal{X}))$$

for $i \geq k$. The fact that this is well-defined is guaranteed by the isomorphism (2.3).

In addition, the isomorphism (2.3) implies that the functor (2.4) is exact. Hence, it is pro-corepresentable by an object of $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$. In what follows we shall denote this object by

$$T_x^*(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)$$

and refer to it as the *pro-cotangent space to $\mathcal{X}$ at x*.

2.2.7. Let us recall (see [**Lu1**, Corollary 5.3.5.4]) that for any (accessible) $(\infty, 1)$-category $\mathbf{C}$ with finite limits, the category $(\mathrm{Pro}(\mathbf{C}))^{\mathrm{op}}$ is the full subcategory of $\mathrm{Funct}(\mathbf{C}, \mathrm{Spc})$ that consists of (accessible) functors that preserve finite limits.

Recall also (see Volume I, Chapter 1, Sects. 7.2.1) that if $\mathbf{C}$ is stable, we can identify this category with that of exact functors

$$\mathbf{C} \to \mathrm{Sptr},$$

by composing with the forgetful functor $\Omega^\infty : \mathrm{Sptr} \to \mathrm{Spc}$.

Finally, if $\mathbf{C}$ is a k-linear DG category (such in our case of interest $\mathrm{QCoh}(S)^-$)), we can identify it also with that of k-linear exact functors

$$\mathbf{C} \to \mathrm{Vect},$$

by composing with the Dold-Kan functor

$$\text{Dold-Kan}^{\text{Sptr}} : \text{Vect} \to \text{Sptr},$$

see Volume I, Chapter 1, Sect. 10.2.)

Given an object $\Phi \in \text{Pro}(\mathbf{C})$, the corresponding functor $\mathbf{C} \to \text{Vect}$ is explicitly given by

$$\mathcal{F} \mapsto \mathcal{M}aps_{\text{Pro}(\mathbf{C})}(\Phi, \mathcal{F}),$$

where we regard $\text{Pro}(\mathbf{C})$ also as a k-linear DG category, and $\mathbf{C}$ as its full subcategory.

2.2.8. Suppose that

$$(2.5)$$

$$\begin{array}{ccc} \mathcal{F}_1' & \longrightarrow & \mathcal{F}_1 \\ \downarrow & & \downarrow \\ \mathcal{F}_2' & \longrightarrow & \mathcal{F}_2 \end{array}$$

is a pullback diagram *in the category* $\text{QCoh}(S)$, where all objects belong to $\text{QCoh}(S)^{\leq 0}$. I.e., we have $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_2' \in \text{QCoh}(S)^{\leq 0}$, and we require that

$$\mathcal{F}_1' := \mathcal{F}_1 \underset{\mathcal{F}_2}{\times} \mathcal{F}_2'$$

also belongs to $\text{QCoh}(S)^{\leq 0}$.

Note that by Lemma 2.1.4,

$$\begin{array}{ccc} S_{\mathcal{F}_1'} & \longleftarrow & S_{\mathcal{F}_1} \\ \uparrow & & \uparrow \\ S_{\mathcal{F}_2'} & \longleftarrow & S_{\mathcal{F}_2} \end{array}$$

is a push-out diagram in $(\text{Sch}^{\text{aff}})_{S/}$.

For a given map $x : S \to \mathcal{X}$, consider the corresponding map

$$(2.6) \qquad \text{Maps}_{S/}(S_{\mathcal{F}_1'}, \mathcal{X}) \to \text{Maps}_{S/}(S_{\mathcal{F}_1}, \mathcal{X}) \underset{\text{Maps}_{S/}(S_{\mathcal{F}_2}, \mathcal{X})}{\times} \text{Maps}_{S/}(S_{\mathcal{F}_2'}, \mathcal{X}).$$

We have:

LEMMA 2.2.9. *Suppose $\mathcal{X}$ admits a pro-cotangent space at x. Then* (2.6) *is an isomorphism.*

PROOF. Follows from the commutation of $\mathcal{M}aps(T_x^*(\mathcal{X}), -)$ with finite limits. $\square$

2.2.10. We end this subsection with the following definition:

DEFINITION 2.2.11. *Let $\mathcal{X}$ be an object of* PreStk. *We shall say that $\mathcal{X}$ admits a* cotangent *space at* $(S, x) \in (\text{Sch}^{\text{aff}})_{/\mathcal{X}}$ *if it admits a pro-cotangent space, and* $T_x^*(\mathcal{X})$ *belongs to*

$$\text{QCoh}(S)^- \subset \text{Pro}(\text{QCoh}(S)^-).$$

2.3. The condition of admitting (pro)-cotangent space*s.*

2.3.1. We give the following definition:

DEFINITION 2.3.2. *Let $\mathcal{X}$ be an object of* PreStk.

(a) *We shall say that $\mathcal{X}$ admits* pro-cotangent spaces, *if admits a pro-cotangent space for every* $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$.

(b) *We shall say that $\mathcal{X}$ admits* cotangent spaces, *if admits a cotangent space for every* $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$.

2.3.3. For example, Corollary 2.2.5 can be reformulated as saying that any scheme admits pro-cotangent spaces.

REMARK 2.3.4. We shall soon see that every $X \in \mathrm{Sch}$ actually admits *cotangent* spaces, see Proposition 3.2.6.

2.3.5. Zariski gluing allows us to extend the construction of split square-zero extensions to schemes that are not necessarily affine. Thus, for $Z \in \mathrm{Sch}$, we obtain a well-defined functor:

$$\mathrm{RealSplitSqZ} : (\mathrm{QCoh}(Z)^{\leq 0})^{\mathrm{op}} \to \mathrm{Sch}_{Z/}, \quad \mathcal{F} \mapsto Z_{\mathcal{F}}.$$

Let $\mathcal{X}$ be an object of PreStk that admits pro-cotangent spaces. Assume also that $\mathcal{X}$ is a sheaf in the Zariski topology.

Fix a map $x : Z \to \mathcal{X}$. It follows formally that the functor

$$\mathrm{QCoh}(Z)^{\leq 0} \to \mathrm{Spc}, \quad \mathcal{F} \mapsto \mathrm{Maps}_{Z/}(Z_{\mathcal{F}}, \mathcal{X})$$

is pro-corepresentable by an object

$$T_x^*(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(Z)^-).$$

2.4. The relative situation. The same definitions apply when we work over a fixed prestack $\mathcal{X}_0$.

2.4.1. For $\mathcal{X} \in \mathrm{PreStk}_{/\mathcal{X}_0}$ and $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$, we shall say that $\mathcal{X}$ admits a pro-cotangent space at x relative to $\mathcal{X}_0$ if in the situation of Sect. 2.2.2, the diagram

$$
\begin{array}{ccc}
\mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}) & \longrightarrow & * \underset{\mathrm{Maps}_{S/}(S_{\mathcal{F}_2}, \mathcal{X})}{\times} \mathrm{Maps}_{S/}(S_{\mathcal{F}_1}, \mathcal{X}) \\
\downarrow & & \downarrow \\
\mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}_0) & \longrightarrow & * \underset{\mathrm{Maps}_{S/}(S_{\mathcal{F}_2}, \mathcal{X}_0)}{\times} \mathrm{Maps}_{S/}(S_{\mathcal{F}_1}, \mathcal{X}_0)
\end{array}
$$

is a pullback square, i.e., if the fibers of the map

$$\mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}) \to * \underset{\mathrm{Maps}_{S/}(S_{\mathcal{F}_2}, \mathcal{X})}{\times} \mathrm{Maps}_{S/}(S_{\mathcal{F}_1}, \mathcal{X})$$

map isomorphically to the fibers of the map

$$\mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}_0) \to * \underset{\mathrm{Maps}_{S/}(S_{\mathcal{F}_2}, \mathcal{X}_0)}{\times} \mathrm{Maps}_{S/}(S_{\mathcal{F}_1}, \mathcal{X}_0).$$

2.4.2.   If this condition holds, we will denote by

$$T_x^*(\mathcal{X}/\mathcal{X}_0) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)$$

the object that pro-corepresents the functor

$$\mathcal{F} \mapsto \mathrm{Maps}_{S/}(S_\mathcal{F}, \mathcal{X}) \underset{\mathrm{Maps}_{S/}(S_\mathcal{F}, \mathcal{X}_0)}{\times} *.$$

2.4.3.   Note that if in the above situation $\mathcal{X}_0$ admits a pro-cotangent space at $x_0 : S \to \mathcal{X} \to \mathcal{X}_0$, then $\mathcal{X}$ admits a pro-cotangent space at x if and only if $\mathcal{X}$ admits a pro-cotangent space at x relative to $\mathcal{X}_0$ and

$$T_x^*(\mathcal{X}/\mathcal{X}_0) \simeq \mathrm{coFib}(T_{x_0}^*(\mathcal{X}_0) \to T_x^*(\mathcal{X})).$$

2.4.4.   The next assertion easily results from the definitions:

LEMMA 2.4.5.   *A prestack $\mathcal{X}$ admits pro-cotangent spaces relative to $\mathcal{X}_0$ if and only if for every $S_0 \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}_0}$, the prestack $S_0 \underset{\mathcal{X}_0}{\times} \mathcal{X}$ admits pro-cotangent spaces.*

2.5. Describing the pro-cotangent space as a limit. In this subsection we will study (pro)-cotangent spaces of prestacks that are presented as colimits.

2.5.1.   Let $\mathcal{X}$ be an object of PreStk, written as

$$(2.7) \qquad\qquad\qquad\qquad \mathcal{X} = \operatorname*{colim}_{a \in A} \mathcal{X}_a,$$

where the colimit is taken in PreStk.

Assume that each $\mathcal{X}_a$ admits pro-cotangent spaces. We wish to express the pro-cotangent spaces of $\mathcal{X}$ (if they exist) in terms of those of $\mathcal{X}_a$.

2.5.2.   For $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$, let $A_{x/}$ denote the category co-fibered over A, whose fiber over a given $a \in A$ is the space of factorizations of x as

$$S \xrightarrow{x_a} \mathcal{X}_a \to \mathcal{X}.$$

We claim:

PROPOSITION 2.5.3.   *Suppose that $T_x^*(\mathcal{X})$ exists and that the category $A_{x/}$ is sifted[3]. Then the natural map*

$$(2.8) \qquad\qquad\qquad T_x^*(\mathcal{X}) \to \lim_{(a,x_a) \in (A_{x/})^{\mathrm{op}}} T_{x_a}^*(\mathcal{X}_a),$$

where the limit is taken in $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$, is an isomorphism.

PROOF.   We need to show that for $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0}$, the map

$$(2.9) \qquad\qquad \operatorname*{colim}_{(a,x_a) \in A_{x/}} \mathcal{M}aps(T_{x_a}^*(\mathcal{X}_a), \mathcal{F}) \to \mathcal{M}aps(T_x^*(\mathcal{X}), \mathcal{F}),$$

where the colimit is taken in Vect, is an isomorphism.

Denote $V_a := \mathcal{M}aps(T_{x_a}^*(\mathcal{X}_a), \mathcal{F})$, $V := \mathcal{M}aps(T_x^*(\mathcal{X}), \mathcal{F})$. We claim that it is enough to show that for every $n \in \mathbb{N}$, the resulting composite map

$$(2.10) \qquad\qquad \operatorname*{colim}_{(a,x_a) \in A_{x/}} \tau^{\leq n}(V_a) \to \tau^{\leq n}(\operatorname*{colim}_{(a,x_a) \in A_{x/}} V_a) \to \tau^{\leq n}(V)$$

is an isomorphism.

_____

[3]See [**Lu1**, Sect. 5.5.8] for what this means.

Indeed, if (2.10) is an isomorphism, then the map

$$\operatorname*{colim}_{(a,x_a)\in A_{x/}} V_a \simeq \operatorname*{colim}_{(a,x_a)\in A_{x/}} \operatorname*{colim}_n \tau^{\leq n}(V_a) \simeq \operatorname*{colim}_n \operatorname*{colim}_{(a,x_a)\in A_{x/}} \tau^{\leq n}(V_a) \to \operatorname*{colim}_n \tau^{\leq n}(V) \simeq V$$

is an isomorphism as well.

We note that the composite map in (2.10) can be interpreted as the shift by $[-n]$ of the map

$$(2.11) \qquad \operatorname*{colim}_{(a,x_a)\in A_{x/}} \tau^{\leq 0}\left(\mathcal{M}aps(T^*_{x_a}(\mathcal{X}_a), \mathcal{F}[n])\right) \to \tau^{\leq 0}\left(\mathcal{M}aps(T^*_x(\mathcal{X}), \mathcal{F}[n])\right)$$

in $\mathrm{Vect}^{\leq 0}$. So, it suffices to show that the map (2.11) is an isomorphism.

Now, using the assumption that $A_{x/}$ is sifted and the fact that the functor

$$\mathrm{Dold\text{-}Kan} : \mathrm{Vect}^{\leq 0} \to \mathrm{Spc}$$

commutes with sifted colimits (see Volume I, Chapter 1, Sect. 10.2.3), when we apply it to (2.11), we obtain the map

$$(2.12) \qquad \operatorname*{colim}_{(a,x_a)\in A_{x/}} \mathrm{Maps}_{S/}(S_{\mathcal{F}[n]}, \mathcal{X}_a) \to \mathrm{Maps}_{S/}(S_{\mathcal{F}[n]}, \mathcal{X}).$$

Thus, since Dold-Kan is conservative, we obtain that it suffices to show that (2.12) is an isomorphism.

Hence, it remains to show that for a $S' \in \mathrm{Sch}^{\mathrm{aff}}_{S/}$, the map

$$\operatorname*{colim}_{(a,x_a)\in A_{x/}} \mathrm{Maps}_{S/}(S', \mathcal{X}_a) \to \mathrm{Maps}_{S/}(S', \mathcal{X})$$

is an isomorphism. However, this follows from the isomorphism (2.7).

$\square$

2.5.4.   We now claim:

LEMMA 2.5.5. *Suppose that in the situation of (2.7), the category A is filtered. Then $\mathcal{X}$ admits pro-cotangent spaces, and there is a canonical isomorphism*

$$T^*_x(\mathcal{X}) \to \operatorname*{lim}_{(a,x_a)\in (A_{x/})^{\mathrm{op}}} T^*_{x_a}(\mathcal{X}_a).$$

PROOF. As in (2.11), we have an identification

$$\mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}) \simeq \operatorname*{colim}_{(a,x_a)\in A_{x/}} \mathrm{Dold\text{-}Kan}\left(\tau^{\leq 0}\left(\mathcal{M}aps(T^*_{x_a}(\mathcal{X}_a), \mathcal{F})\right)\right),$$

functorial in $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0}$.

First, we note that the filteredness assumption on A implies that all the categories $A_{x/}$ are filtered and in particular sifted. Hence,

$$\operatorname*{colim}_{(a,x_a)\in A_{x/}} \mathrm{Dold\text{-}Kan}\left(\tau^{\leq 0}\left(\mathcal{M}aps(T^*_{x_a}(\mathcal{X}_a), \mathcal{F})\right)\right) \simeq$$

$$\simeq \mathrm{Dold\text{-}Kan}\left(\operatorname*{colim}_{(a,x_a)\in A_{x/}} \tau^{\leq 0}\left(\mathcal{M}aps(T^*_{x_a}(\mathcal{X}_a), \mathcal{F})\right)\right).$$

Since $A_{x/}$ is filtered, the functor $\tau^{\leq 0} : \mathrm{Vect} \to \mathrm{Vect}^{\leq 0}$ commutes with colimits along $A_{x/}$, and we obtain:

$$\operatorname*{colim}_{(a,x_a)\in A_{x/}} \tau^{\leq 0}\left(\mathcal{M}aps(T^*_{x_a}(\mathcal{X}_a), \mathcal{F})\right) \simeq \tau^{\leq 0}\left(\operatorname*{colim}_{(a,x_a)\in A_{x/}} \mathcal{M}aps(T^*_{x_a}(\mathcal{X}_a), \mathcal{F})\right).$$

This implies the assertion of the lemma.

$\square$

3. Properties of (pro)-cotangent spaces

By definition, the (pro)-cotangent space of a prestack at a given S-point is an object of $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$. One can impose the condition that the (pro)-cotangent space belong to a given subcategory of $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$, and obtain more restricted infinitesimal behavior. In this section, we will study various such conditions.

3.1. Connectivity conditions. The first type of condition is obtained by requiring that the (pro)-cotangent space be bounded above.

3.1.1. We start with the following observation: let $\mathbf{C}$ be a stable $(\infty, 1)$-category, in which case the category $\mathrm{Pro}(\mathbf{C})$ is also stable [4].

Assume now that $\mathbf{C}$ is endowed with a t-structure. In this case $\mathrm{Pro}(\mathbf{C})$ also inherits a t-structure, so that its connective subcategory $\mathrm{Pro}(\mathbf{C})^{\leq 0}$ consists of those left-exact[5] functors

$$\mathbf{C} \to \mathrm{Spc}$$

that map $\mathbf{C}^{>0}$ to $* \in \mathrm{Spc}$.

Equivalently, if we interpret objects of $\mathrm{Pro}(\mathbf{C})$ as exact functors

$$\mathbf{C} \to \mathrm{Sptr},$$

the subcategory $\mathrm{Pro}(\mathbf{C})^{\leq 0}$ consists of those functors that send $\mathbf{C}^{\geq 0}$ to the subcategory $\mathrm{Sptr}^{\geq 0} \subset \mathrm{Sptr}$.

Clearly,

$$\mathbf{C} \cap \mathrm{Pro}(\mathbf{C})^{\leq n} = \mathbf{C}^{\leq n},$$

as subcategories of $\mathbf{C}$.

3.1.2. Restriction along $\mathbf{C}^{\leq n} \hookrightarrow \mathbf{C}$ defines a functor

$$(3.1) \qquad\qquad \mathrm{Pro}(\mathbf{C})^{\leq n} \to \mathrm{Pro}(\mathbf{C}^{\leq n}),$$

LEMMA 3.1.3. *The functor* (3.1) *is an equivalence.*

Similarly, for any $m \leq n$, the natural functor

$$\mathrm{Pro}(\mathbf{C})^{\geq m, \leq n} \to \mathrm{Pro}(\mathbf{C}^{\geq m, \leq n})$$

is an equivalence.

In what follows we shall denote by $\mathrm{Pro}(\mathbf{C})_{\text{event-conn}}$ the full subcategory of $\mathrm{Pro}(\mathbf{C})$ equal to $\underset{n}{\cup} \mathrm{Pro}(\mathbf{C})^{\leq n}$. I.e., $\mathrm{Pro}(\mathbf{C})_{\text{event-conn}}$ is the same thing as $\mathrm{Pro}(\mathbf{C})^-$.

---

[4]Note, however, that even if $\mathbf{C}$ is presentable, the category $\mathrm{Pro}(\mathbf{C})$ is not, so caution is required when applying such results as the adjoint functor theorem.

[5]We recall that a functor is said to be left-exact if it commutes with finite limits. This notion has nothing to do with t-structures.

3.1.4.   We give the following definitions:

DEFINITION 3.1.5. *Let $\mathcal{X}$ be an object of* PreStk.

(a) *We shall say that $\mathcal{X}$ admits a $(-n)$-connective pro-cotangent (resp., cotangent) space at x if it admits a pro-cotangent (resp., cotangent) space at x and $T_x^*(\mathcal{X}) \in$ $\mathrm{Pro}(\mathrm{QCoh}(S)^{\leq n})$.*

(a') *We shall say that $\mathcal{X}$ admits an eventually connective pro-cotangent space at x if it admits a $(-n)$-connective pro-cotangent space at x for some n.*

(b) *We shall that $\mathcal{X}$ admits $(-n)$-connective pro-cotangent (resp., cotangent) spaces, if it admits a $(-n)$-connective pro-cotangent (resp., cotangent) space for every $(S,x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$.*

(b') *We shall that $\mathcal{X}$ admits locally eventually connective pro-cotangent spaces, if it admits an eventually connective pro-cotangent space for every $(S,x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$.*

(c) *We shall that $\mathcal{X}$ admits uniformly eventually connective pro-cotangent (resp., cotangent) spaces, if there exists an integer $n \in \mathbb{Z}$ such that $\mathcal{X}$ admits a $(-n)$-connective pro-cotangent (resp., cotangent) space for every $(S,x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$.*

3.1.6.   Tautologically, if $\mathcal{X}$ admits a pro-cotangent space at x, then this pro-cotangent space is $(-n)$-connective if and only if for some/any $i \geq 0$ and $\mathcal{F} \in$ $\mathrm{QCoh}(S)^{\geq -i, \leq 0}$, the space

$$\mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X})$$

is $(n+i)$-truncated.

3.1.7.   Let us consider separately the case when $n = 0$ (in this case, we shall say 'connective' instead of '0-connective'). Almost tautologically, we have:

LEMMA 3.1.8. *A prestack $\mathcal{X}$ admits a connective pro-cotangent space at $x : S \to \mathcal{X}$ if and only if the functor (2.1) commutes with finite limits (equivalently, takes pullbacks to pullbacks).*

REMARK 3.1.9. The point of Lemma 3.1.8 is that the condition of admitting a connective pro-cotangent space is stronger than that of just admitting a pro-cotangent space: the former requires that functor (2.1) take any pullback square in $\mathrm{QCoh}(S)^{\leq 0}$ to a pullback square, while the latter does so only for those pullback squares in $\mathrm{QCoh}(S)^{\leq 0}$ that stay pullback squares in all of $\mathrm{QCoh}(S)$.

3.1.10.   From Lemma 1.3.3(b) we obtain:

COROLLARY 3.1.11. *Every $\mathcal{X} = X \in \mathrm{Sch}$ admits connective pro-cotangent spaces.*

3.2. Pro-cotangent vs cotangent. Assume that $\mathcal{X}$ admits a $(-n)$-connective pro-cotangent space at x. We wish to give a criterion for when $\mathcal{X}$ admits a cotangent space at x. In this case, $T_x^*(\mathcal{X})$ would be an object of $\mathrm{QCoh}(S)^{\leq n}$.

3.2.1.   We have:

LEMMA 3.2.2. *Let $\mathbf{C}$ be as in Sect. 3.1.1, and let $\mathbf{c}$ be an object of $\mathrm{Pro}(\mathbf{C}^{\leq n})$. Assume that $\mathbf{C}$ contains filtered limits and retracts. Then $\mathbf{c}$ belongs to $\mathbf{C}^{\leq n}$ if and only if the corresponding functor $\mathbf{C}^{\leq n} \to \mathrm{Spc}$ commutes with filtered limits.*

3.2.3. From the lemma we obtain:

COROLLARY 3.2.4. *If $\mathcal{X}$ admits a $(-n)$-connective pro-cotangent space at x, then it admits a cotangent space at x if and only if the functor (2.1) commutes with filtered limits.*

3.2.5. We now claim:

PROPOSITION 3.2.6. *Any $\mathcal{X} = X \in \mathrm{Sch}$ admits connective cotangent spaces.*

PROOF. According to Corollaries 3.1.11, we only need to show that the composite functor

$$\mathrm{RealSplitSqZ} : (\mathrm{QCoh}(S)^{\leq 0})^{\mathrm{op}} \to (\mathrm{Sch}^{\mathrm{aff}})_{S/} \to \mathrm{Sch}_{S/} \to \mathrm{Sch}$$

commutes with filtered colimits. However, taking into account Lemma 2.1.5, this follows from the fact that the forgetful functor $\mathrm{Sch}_{S/} \to \mathrm{Sch}$ commutes with colimits *indexed by any contractible category.*

$\square$

3.3. The convergence condition. The convergence condition says that the value of the (pro)-cotangent space on a given $\mathcal{F} \in \mathrm{QCoh}(S)^-$ is determined by the cohomological truncations $\tau^{\geq -n}(\mathcal{F})$. It is the infinitesimal version of the condition of convergence on a prestack itself.

3.3.1. For $S \in \mathrm{Sch}^{\mathrm{aff}}$, we let

$$^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^-) \subset \mathrm{Pro}(\mathrm{QCoh}(S)^-)$$

denote the full subcategory spanned by objects Φ that satisfy the following convergence condition:

We require that when $\Phi \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)$ is viewed as a functor $\mathrm{QCoh}(S)^{\leq 0} \to \mathrm{Spc}$, then for any $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0}$, the map

$$\Phi(\mathcal{F}) \to \lim_n \Phi(\tau^{\geq -n}(\mathcal{F}))$$

be an isomorphism.

We note the analogy between this definition and the notion of convergence for objects of PreStk, see Volume I, Chapter 2, Sect. 1.4.

3.3.2. The following is nearly tautological:

LEMMA 3.3.3. *Suppose that $\mathcal{X} \in \mathrm{PreStk}$ is convergent, and suppose that it admits a pro-cotangent space at $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$. Then $T_x^*(\mathcal{X})$ belongs to $^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^-)$.*

In addition, we have:

LEMMA 3.3.4. *Suppose that $\mathcal{X} \in \mathrm{PreStk}$ is convergent. Then in order to test whether $\mathcal{X}$ admits pro-cotangent spaces (resp., $(-n)$-connective pro-cotangent spaces), it is enough to do so for (S, x) with S eventually coconnective and check that (2.3) is an isomorphism for $\mathcal{F}_i \in \mathrm{QCoh}(S)^{>-\infty, \leq 0}$, $i = 1, 2$.*

Similarly, we have the following extension of Lemma 2.4.5:

LEMMA 3.3.5. *Let $\pi : \mathcal{X} \to \mathcal{X}_0$ be a morphism in $^{\mathrm{conv}}\mathrm{PreStk}$. Then in order to check that $\mathcal{X}$ admits pro-cotangent spaces relative to $\mathcal{X}_0$, it sufficient to check that for every $S_0 \in (^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}_0}$, the fiber product $S_0 \underset{\mathcal{X}_0}{\times} \mathcal{X}$ admits pro-cotangent spaces.*

3.4. The almost finite type condition. In this subsection we introduce another condition on an object of $\mathrm{Pro}(\mathrm{QCoh}(X)^-)$, namely, that it be 'almost of finite type'.

3.4.1. For a scheme X, let

$$\mathrm{Pro}(\mathrm{QCoh}(X)^-)_{\mathrm{laft}} \subset \mathrm{Pro}(\mathrm{QCoh}(X)^-)$$

denote the full subcategory consisting of objects Φ satisfying the following two conditions:

(1) $\Phi \in {}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(X)^-)$;
(2) For every $m \geq 0$, the resulting functor $\Phi : \mathrm{QCoh}(X)^{\geq-m,\leq 0} \to \mathrm{Spc}$ commutes with filtered colimits.

We note the analogy between the above definition and the corresponding definition for prestacks, see Volume I, Chapter 2, Sect. 1.7.

3.4.2. From now until the end of this subsection we will assume that $X \in \mathrm{Sch}_{\mathrm{aft}}$. In particular, we have a well-defined (non-cocomplete) DG subcategory

$$\mathrm{Coh}(X) \subset \mathrm{QCoh}(X).$$

3.4.3. Here is a more explicit interpretation of Condition (2) in Sect. 3.4.1 in the eventually connective case.

Let X be an object of $\mathrm{Sch}_{\mathrm{aft}}$, and let Φ be an object of $\mathrm{Pro}(\mathrm{QCoh}(X)^{\leq n})$ for some n. We have:

LEMMA 3.4.4. *The following conditions are equivalent:*

(a) *For every $m \geq 0$, the functor $\Phi : \mathrm{QCoh}(X)^{\geq-m,\leq 0} \to \mathrm{Spc}$ commutes with filtered colimits.*

(b) *For every $m \geq 0$, the truncation $\tau^{\geq-m}(\Phi)$ belongs to the full subcategory*

$$\mathrm{Pro}(\mathrm{Coh}(X)^{\geq-m,\leq n}) \subset \mathrm{Pro}(\mathrm{QCoh}(X)^{\geq-m,\leq n}).$$

(c) *The cohomologies of Φ belong to $\mathrm{Pro}(\mathrm{Coh}(X)^\heartsuit) \subset \mathrm{Pro}(\mathrm{QCoh}(X)^\heartsuit)$.*

3.4.5. Note that restriction along $\mathrm{Coh}(X) \hookrightarrow \mathrm{QCoh}(X)$ defines a functor

(3.2) $$\mathrm{Pro}(\mathrm{QCoh}(X)^-) \to \mathrm{Pro}(\mathrm{Coh}(X)).$$

We claim:

PROPOSITION 3.4.6. *The functor* (3.2) *defines an equivaence*

$$\mathrm{Pro}(\mathrm{QCoh}(X)^-)_{\mathrm{laft}} \to \mathrm{Pro}(\mathrm{Coh}(X)).$$

REMARK 3.4.7. Note the analogy between this proposition and the corresponding assertion in Volume I, Chapter 2, Proposition 1.7.6.

PROOF OF PROPOSITION 3.4.6. We construct the inverse functor as follows.

Given $\widetilde{\Phi} \in \mathrm{Pro}(\mathrm{Coh}(X))$, viewed as a functor

$$\mathrm{Coh}(X)^{\leq 0} \to \mathrm{Spc},$$

we construct a functor

$$\Phi^b : \mathrm{QCoh}(X)^{>-\infty,\leq 0} \to \mathrm{Spc},$$

as the *left Kan extension* of $\widetilde{\Phi}$ under

$$\mathrm{Coh}(X)^{\leq 0} \hookrightarrow \mathrm{QCoh}(X)^{>-\infty,\leq 0}.$$

We define the sought-for functor $\Phi : \mathrm{QCoh}(X)^{\leq 0} \to \mathrm{Spc}$ as the *right Kan extension* of Φ^b under

$$\mathrm{QCoh}(X)^{>-\infty,\leq 0} \hookrightarrow \mathrm{QCoh}(X)^{\leq 0}.$$

Explicitly,

$$\Phi(\mathcal{F}) = \lim_{m} \Phi^b(\tau^{\geq -m}(\mathcal{F})).$$

It is easy to check that the construction $\widetilde{\Phi} \mapsto \Phi$ is the inverse to (3.2).

$\square$

COROLLARY 3.4.8. *For* $X \in \mathrm{Sch}_{\mathrm{aft}}$ *there exists a canonical equivalence*

$$(\mathrm{Pro}(\mathrm{QCoh}(X)^-)_{\mathrm{laft}})^{\mathrm{op}} \simeq \mathrm{IndCoh}(X).$$

PROOF. Follows from the canonical equivalence between $(\mathrm{Pro}(\mathrm{Coh}(X)))^{\mathrm{op}}$ and $\mathrm{IndCoh}(X)$ given by Serre duality

$$\mathbb{D}_S^{\mathrm{Serre}} : (\mathrm{Coh}(X))^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Coh}(X),$$

see Volume I, Chapter 5, Sect. 4.2.10.

$\square$

The following results from the construction:

LEMMA 3.4.9.

(a) *Under the equivalence of Corollary 3.4.8, the full subcategory of* $(\mathrm{Pro}(\mathrm{QCoh}(X)^-)_{\mathrm{laft}})^{\mathrm{op}}$ *corresponding to*

$$\mathrm{Pro}(\mathrm{QCoh}(X)^-)_{\mathrm{laft}} \cap \mathrm{Pro}(\mathrm{QCoh}(X)^-)_{\mathrm{event\text{-}conn}} \subset \mathrm{Pro}(\mathrm{QCoh}(X)^-)$$

maps onto $\mathrm{IndCoh}(X)^+ \subset \mathrm{IndCoh}(X)$.

(b) *Under the equivalence of Corollary 3.4.8, the full subcategory of* $(\mathrm{Pro}(\mathrm{QCoh}(X)^-)_{\mathrm{laft}})^{\mathrm{op}}$ *corresponding to*

$$\mathrm{Pro}(\mathrm{QCoh}(X)^-)_{\mathrm{laft}} \cap \mathrm{QCoh}(X)^- \subset \mathrm{Pro}(\mathrm{QCoh}(X)^-)$$

maps onto the full subcategory of $\mathrm{IndCoh}(X)^+$, *consisting of objects with* coherent *cohomologies.*

3.5. Prestacks locally almost of finite type. In this subsection, we will study what the 'locally almost of finite type' condition on a prestack implies about its (pro)-cotangent spaces.

3.5.1.   The definition of the subcategory

$$\mathrm{PreStk}_{\mathrm{laft}} \subset \mathrm{PreStk}$$

implies:

LEMMA 3.5.2. *Suppose that* $\mathcal{X} \in \mathrm{PreStk}$ *belongs to* $\mathrm{PreStk}_{\mathrm{laft}}$, *and suppose that it admits a pro-cotangent space at* $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$. *Then* $T_x^*(\mathcal{X})$ *belongs to*

$$\mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}} \subset \mathrm{Pro}(\mathrm{QCoh}(S)^-).$$

Moreover, we have:

LEMMA 3.5.3. *Suppose that* $\mathcal{X}$ *belongs to* $\mathrm{PreStk}_{\mathrm{laft}}$. *Then the condition on* $\mathcal{X}$ *to have pro-cotangent spaces is enough to check on* (S, x) *with* $S \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$ *and* $\mathcal{F}_i \in \mathrm{Coh}(S)^{\leq 0}$, $i = 1, 2$.

We also have following extension of Lemma 2.4.5:

LEMMA 3.5.4. *Let $\pi : \mathcal{X} \to \mathcal{X}_0$ be a morphism in $\mathrm{PreStk}_{\mathrm{laft}}$. Then in order to check that $\mathcal{X}$ admits pro-cotangent spaces relative to $\mathcal{X}_0$, it sufficient to check that for every $S_0 \in (^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{X}_0}$, the fiber product $S_0 \underset{\mathcal{X}_0}{\times} \mathcal{X}$ admits pro-cotangent spaces.*

3.5.5. Suppose that $\mathcal{X} = X \in \mathrm{Sch}_{\mathrm{laft}}$, and let $(x : S \to X) \in (\mathrm{Sch}^{\mathrm{aff}})_{/X}$ with $S \in \mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}}$. Consider the object

$$T_x^*(X) \in \mathrm{QCoh}(S)^{\leq 0}.$$

From Lemma 3.4.4 we obtain:

COROLLARY 3.5.6. *The object object $T_x^*(X)$ has coherent cohomologies.*

3.5.7. *The tangent space.* Using Corollary 3.4.8 and Lemma 3.5.2, we obtain that if $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ admits a pro-cotangent space at x for $(S,x) \in (\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}})_{/\mathcal{X}}$, then it admits a well-defined *tangent* space

$$T_x(\mathcal{X}) \in \mathrm{IndCoh}(S).$$

Namely, we let $T_x(\mathcal{X})$ be the object of $\mathrm{IndCoh}(S)$ corresponding to $T_x^*(\mathcal{X})$ via the contravariant equivalence of Corollary 3.4.8.

4. The (pro)-cotangent complex

A prestack admits a *pro-cotangent complex* if it admits pro-cotangent spaces that are compatible under the operation of pullback. We will study this notion in this section.

4.1. Functoriality of (pro)-cotangent spaces. In this section we define what it means for a prestack to admit a (pro)-cotangent complex. We reformulate this definition as compatibility with a certain type of push-outs.

4.1.1. Let $f : S_1 \to S_2$ be a map of affine schemes. Consider the functor

$$f^* : \mathrm{QCoh}(S_2)^- \to \mathrm{QCoh}(S_1)^-,$$

and let $\mathrm{Pro}(f^*)$ denote the resulting functor

$$\mathrm{Pro}(\mathrm{QCoh}(S_2)^-) \to \mathrm{Pro}(\mathrm{QCoh}(S_1)^-).$$

Note that we when regard $\mathrm{Pro}(\mathrm{QCoh}(S_i)^-)$ as a full subcategory of (the opposite of)

$$\mathrm{Funct}(\mathrm{QCoh}(S_i)^{\leq 0}, \mathrm{Spc}),$$

the functor $\mathrm{Pro}(f^*)$ is induced by the functor

$$\mathrm{LKE}_{f^*} : \mathrm{Funct}(\mathrm{QCoh}(S_2)^{\leq 0}, \mathrm{Spc}) \to \mathrm{Funct}(\mathrm{QCoh}(S_1)^{\leq 0}, \mathrm{Spc}).$$

Even more explicitly, for $\mathcal{F}_1 \in \mathrm{QCoh}(S_1)^{\leq 0}$ and $\Phi_2 \in \mathrm{Pro}(\mathrm{QCoh}(S_2)^-)$, we have:

(4.1) $$((\mathrm{Pro}(f^*)(\Phi_2)))(\mathcal{F}_1) = \Phi_2(f_*(\mathcal{F}_1)).$$

4.1.2.   Note also (see [**Lu2**, Theorem 7.3.4.18]) that we have a commutative diagram of functors

(4.2)
$$
\begin{array}{ccc}
\mathrm{QCoh}(S_1)^{\leq 0} & \xrightarrow{\ f_*\ } & \mathrm{QCoh}(S_2)^{\leq 0} \\
\Big\downarrow{\scriptstyle \mathrm{RealSplitSqZ}} & & \Big\downarrow{\scriptstyle \mathrm{RealSplitSqZ}} \\
(\mathrm{Sch}^{\mathrm{aff}})_{S_1/} & \longrightarrow & (\mathrm{Sch}^{\mathrm{aff}})_{S_2/},
\end{array}
$$

where the bottom horizontal arrow is given by push-out.

4.1.3.   Let $\mathcal{X}$ be an object of PreStk that admits pro-cotangent spaces. Let $f : S_1 \to S_2$ be a map of affine schemes. Let $x_2 : S_2 \to \mathcal{X}$ and denote $x_1 := x_2 \circ f$.

From (4.2), for $\mathcal{F}_1 \in \mathrm{QCoh}(S_1)^{\leq 0}$, we obtain a canonically defined map

(4.3) $\mathrm{Maps}_{S_2/}((S_2)_{f_*(\mathcal{F}_1)}, \mathcal{X}) \to \mathrm{Maps}_{S_1/}((S_1)_{\mathcal{F}_1}, \mathcal{X}),$

which depends functorially on $\mathcal{F}_1$.

We can interpret the map (4.3) as a map

(4.4) $T^*_{x_1}(\mathcal{X}) \to \mathrm{Pro}(f^*)(T^*_{x_2}(\mathcal{X}))$

in $\mathrm{Pro}(\mathrm{QCoh}(S_1)^-)$.

DEFINITION 4.1.4.   *We shall say that $\mathcal{X}$ admits a* pro-cotangent complex *if it admits pro-cotangent spaces* and *the map (4.4) is an isomorphism for any $(S_2, x_2 : S_2 \to \mathcal{X})$ and f as above.*

4.1.5.   Equivalently, $\mathcal{X}$ admits a *pro-cotangent complex* if it admits pro-cotangent spaces and the map (4.3) is an isomorphism for any $(S_2, x_2 : S_2 \to \mathcal{X})$, f and $\mathcal{F}_1$ as above.

Still equivalently, from (4.2), we obtain that $\mathcal{X}$ admits a *pro-cotangent complex* it admits pro-cotangent spaces and takes push-outs of the form

$$(S_1)_{\mathcal{F}_1} \underset{S_1}{\sqcup} S_2,$$

where $(S_1)_{\mathcal{F}_1}$ is a split square-zero extension of S_1, to pullbacks in Spc.

REMARK 4.1.6.   Note that both the condition of admitting pro-cotangent spaces and a pro-cotangent complex are expressed as the property of taking certain push-outs in $\mathrm{Sch}^{\mathrm{aff}}$ to pullbacks in Spc.

4.1.7.   *The cotangent complex.*   We give the following definition:

DEFINITION 4.1.8.   *We shall say that $\mathcal{X}$ admits a* cotangent complex *it admits cotangent spaces and a pro-cotangent complex.*

In other words, we require that for every (S, x), the object $T^*_x(\mathcal{X})$ belong to $\mathrm{QCoh}(S)^-$, and that for a map $f : S_1 \to S_2$, the resulting canonical map

$$T^*_{x_1}(\mathcal{X}) \to f^*(T^*_{x_2}(\mathcal{X}))$$

be an isomorphism in $\mathrm{QCoh}(S_1)^-$.

Thus, if $\mathcal{X}$ admits a cotangent complex, the assignment

$$(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}} \rightsquigarrow T^*_x(\mathcal{X}) \in \mathrm{QCoh}(S)$$

defines an object of $\mathrm{QCoh}(\mathcal{X})$, which we shall denote by $T^*(\mathcal{X})$ and refer to as the *cotangent complex* of $\mathcal{X}$.

4.1.9. Let $Z \in \mathrm{Sch}$, regarded as a prestack. We already know that Z admits cotangent spaces. Moreover, from Lemma 1.3.3(b), it follows that the maps (4.3) are isomorphisms. Hence, we obtain that Z admits a cotangent complex.

4.1.10. *The relative situation.* The same definitions apply in the relative situation, when we consider prestacks and affine schemes over a given prestack $\mathcal{X}_0$.

The analog of Lemma 2.4.5 holds when we replace 'cotangent spaces' by 'cotangent complex'.

4.2. Conditions on the (pro)-cotangent complex. In this subsection we introduce various conditions that one can impose on the (pro)-cotangent complex of a prestack.

4.2.1. *Connectivity conditions.*

DEFINITION 4.2.2.

(a) *We shall say that $\mathcal{X}$ admits an $(-n)$-connective pro-cotangent complex (resp., cotangent complex) if it admits $(-n)$-connective pro-cotangent spaces (resp., cotangent spaces) and a pro-cotangent complex.*

(b) *We shall say that $\mathcal{X}$ admits a locally eventually connective pro-cotangent complex if it admits a pro-cotangent complex and its pro-cotangent spaces are eventually connective.*

(c) *We shall say that $\mathcal{X}$ admits a uniformly eventually connective pro-cotangent complex (resp., cotangent complex) if there exists an integer n such that $\mathcal{X}$ admits an $(-n)$-connective pro-cotangent complex (resp., cotangent complex).*

For example, we obtain that any $X \in \mathrm{Sch}$, regarded as an object of PreStk, admits a connective cotangent complex.

4.2.3. *The (pro)-cotangent complex in the convergent/finite type case.* Suppose now that $\mathcal{X}$ is convergent (resp., belongs to $\mathrm{PreStk}_{\mathrm{laft}}$). By Lemma 3.3.4 (resp., Lemma 3.5.3), the condition that $\mathcal{X}$ admit pro-cotangent spaces is sufficient to test on affine schemes that are eventually coconnective (resp., eventually coconnective and of finite type).

Similarly, we have:

LEMMA 4.2.4.

(a) *Assume that $\mathcal{X}$ is convergent. Then $\mathcal{X}$ admits a pro-cotangent complex if and only if it admits pro-cotangent spaces, and the map (4.3) is an isomorphism for $S_1, S_2 \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$ and $\mathcal{F}_1 \in \mathrm{QCoh}(S_1)^{>-\infty,\leq 0}$.*

(b) *Assume that $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$. Then $\mathcal{X}$ admits a pro-cotangent complex if and only if it admits pro-cotangent spaces, and for any map $f : S_1 \to S_2$ in $({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}}$ and $\mathcal{F}_1 \in \mathrm{Coh}(S_1)^{\leq 0}$, the map*

$$\operatorname*{colim}_{\mathcal{F}_2 \in \mathrm{Coh}(S_2)^{\leq 0}, f^*(\mathcal{F}_2) \to \mathcal{F}_1} \mathrm{Maps}_{S_2/}((S_2)_{\mathcal{F}_2}, \mathcal{X}) \to \mathrm{Maps}_{S_1/}((S_1)_{\mathcal{F}_1}, \mathcal{X})$$

is an isomorphism in Spc.

In addition:

LEMMA 4.2.5. *Let $\pi : \mathcal{X} \to \mathcal{X}_0$ be a morphism in $^{\mathrm{conv}}\mathrm{PreStk}$ (resp., $\mathrm{PreStk}_{\mathrm{laft}}$). Then in order to check that $\mathcal{X}$ admits a pro-cotangent complex relative to $\mathcal{X}_0$, it is sufficient to check that for any $S_0 \in {^{<\infty}}(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}_0}$ (resp., $S_0 \in ({^{<\infty}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}_0}$), the fiber product $S_0 \underset{\mathcal{X}_0}{\times} \mathcal{X}$ admits a pro-cotangent complex.*

4.2.6. *Cotangent vs (pro)-cotangent.* We observe the following:

PROPOSITION 4.2.7. *Let $\mathcal{X}$ be convergent (resp., locally almost of finite type) and admit a locally eventually connective pro-cotangent complex. Suppose that $\mathcal{X}$ admits cotangent spaces for all $S \to \mathcal{X}$ with $S \in {^{<\infty}}\mathrm{Sch}^{\mathrm{aff}}$ (resp., $S \in {^{<\infty}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$). Then $\mathcal{X}$ admits a cotangent complex.*

PROOF. First, we note that the assertion in the locally almost of finite type case follows formally from that in the convergent case.

To prove the latter we need to show the following. Let $\mathcal{T}$ be an object of $\mathrm{Pro}(\mathrm{QCoh}(S)^{\leq 0})$, such that for every truncation $i_n : {^{\leq n}}S \to S$, we have

$$(\mathrm{Pro}(i_n^*))(\mathcal{T}) \in \mathrm{QCoh}(^{\leq n}S)^{\leq 0}.$$

Then $\mathcal{T} \in \mathrm{QCoh}(S)^{\leq 0}$.

This follows from the next general observation (which is a particular case of Volume I, Chapter 3, Proposition 3.6.10):

LEMMA 4.2.8. *The functors $\{i_n^*\}$ define an equivalence*

$$\mathrm{QCoh}(S)^{\leq 0} \to \lim_n \mathrm{QCoh}(^{\leq n}S)^{\leq 0}.$$

$\square$

4.3. The pro-cotangent complex as an object of a category. In this subsection we will show that for a prestack $\mathcal{X}$ locally almost of finite type that admits a (pro)-cotangent complex, there exists a *tangent* complex, which is naturally an object of $\mathrm{IndCoh}(\mathcal{X})$.

4.3.1. Let $\mathcal{X}$ be a prestack. We define the category

$$\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}$$

as

$$\lim_{(S,x)\in(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}} \mathrm{Pro}(\mathrm{QCoh}(S)^-).$$

Let us emphasize that $\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}$ is *not* the same as $\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)$ where the latter is the pro-completion of the category

$$\mathrm{QCoh}(\mathcal{X})^- := \underset{n}{\cup} \mathrm{QCoh}(\mathcal{X})^{\leq n}.$$

We have a fully faithful embedding

$$\mathrm{QCoh}(\mathcal{X})^- \to \mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}},$$

given by $\mathrm{QCoh}(S)^- \hookrightarrow \mathrm{Pro}(\mathrm{QCoh}(S)^-)$ for every $(S,x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$.

4.3.2. By definition, if $\mathcal{X}$ admits a pro-cotangent complex, then we have a well-defined object

$$T^*(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}},$$

whose value on every $(S,x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$ is $T_x^*(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)$.

4.3.3.   Let
$$^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}} \subset \mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}$$
be the full subcategory equal to
$$\lim_{(S,x)\in(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}} {}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^-).$$

We note:

LEMMA 4.3.4. *Assume that $\mathcal{X}$ is convergent. Then the restriction functor*
$$^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}} = \lim_{(S,x)\in(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}} {}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^-) \to$$
$$\to \lim_{(S,x)\in(^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}} {}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^-)$$
is an equivalence.

4.3.5.   By Lemma 3.3.3, if $\mathcal{X}$ is convergent and admits a pro-cotangent complex, we have
$$T^*(\mathcal{X}) \in {}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}.$$

4.3.6.   Assume now that $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$. By Lemma 4.3.4, we can rewrite

(4.5) $$^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}} \simeq \lim_{(S,x)\in(^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}}} {}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^-).$$

Let
$$\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}_{\mathrm{laft}} \subset {}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}$$
be the full subcategory equal, in terms of (4.5), to
$$\lim_{(S,x)\in(^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}}} \mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}} \subset \lim_{(S,x)\in(^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}}} {}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^-).$$

By Lemma 3.5.2, obtain that if $\mathcal{X}$ belongs to $\mathrm{PreStk}_{\mathrm{laft}}$ and admits a pro-cotangent complex, we have
$$T^*(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}_{\mathrm{laft}}.$$

4.4. The tangent complex.

4.4.1.   Assume again that $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$. By Corollary 3.4.8 and the convergence property of IndCoh (see Volume I, Chapter 5, Sect. 3.4.1), we obtain:

COROLLARY 4.4.2. *There exists a canonically defined equivalence*
$$\left(\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}_{\mathrm{laft}}\right)^{\mathrm{op}} \simeq \mathrm{IndCoh}(\mathcal{X}).$$

4.4.3.   Assume now that admits a pro-cotangent complex. We obtain that there exists a canonically defined object
$$T(\mathcal{X}) \in \mathrm{IndCoh}(\mathcal{X}),$$
which is obtained from $T^*(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}_{\mathrm{laft}}$ via the equivalence of Corollary 4.4.2 above.

Concretely, $T(\mathcal{X})$ is given by the assignment
$$(S,x) \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{X}} \rightsquigarrow T_x(\mathcal{X})$$
(see Sect. 3.5.7 for the notation $T_x(\mathcal{X})$).

We shall refer to $T(\mathcal{X})$ as the *tangent complex* of $\mathcal{X}$.

4.5. The (co)differential. We will now introduce another basic structure associated with the pro-cotangent complex, namely, the co-differential map.

4.5.1.   Let S be an object of $\mathrm{Sch}^{\mathrm{aff}}$. By the above, we have a canonical object

$$T^*(S) \in \mathrm{QCoh}(S).$$

We claim that there is a canonical map of schemes under S:

(4.6) $\mathfrak{d}: S_{T^*(S)} \to S,$

where $T^*(S)$ is regarded as an object of $\mathrm{QCoh}(S)^{\leq 0}$.

Indeed, the map $\mathfrak{d}$ corresponds to the identity map on the left-hand side in the isomorphism

$$\mathrm{Maps}(T^*(S), T^*(S)) \simeq \mathrm{Maps}_{S/}(S_{T^*(S)}, S),$$

where we take the target prestack $\mathcal{X}$ to be S, and the map $x: S \to \mathcal{X}$ to be the identity map.

4.5.2.   Let $\mathcal{X}$ be an object of PreStk that admits pro-cotangent spaces, and let $x: S \to \mathcal{X}$ be a map. We claim that there is a canonical map in $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$.

$$(dx)^*: T_x^*(\mathcal{X}) \to T^*(S).$$

The map $(dx)^*$ corresponds via the isomorphism

$$\mathrm{Maps}(T_x^*(\mathcal{X}), T^*(S)) \simeq \mathrm{Maps}_{S/}(S_{T^*(S)}, \mathcal{X}),$$

to the map

$$S_{T^*(S)} \xrightarrow{\mathfrak{d}} S \xrightarrow{x} \mathcal{X}.$$

We shall refer to $(dx)^*$ as the *codifferential* of x.

4.5.3.   Assume for a moment that $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ and $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$. In this case $(dx)^*$ corresponds to a canonically defined map in $\mathrm{IndCoh}(Z)$,

$$dx: T(S) \to T_x(\mathcal{X}),$$

which we shall refer to as the *differential* of x.

4.5.4.   Finally, let us note that the construction of the map $\mathfrak{d}$ is local in the Zariski topology. Hence, we obtain that it is well-defined for any $X \in \mathrm{Sch}$, which is not necessarily affine:

$$\mathfrak{d}: X_{T^*(X)} \to X,$$

4.6. The value of the (pro)-cotangent complex on a non-affine scheme.
In this subsection we will study the pullback of the (pro)-cotangent complex of a prestack to a non-affine scheme.

4.6.1.   Let Z be a scheme.

$$(4.7) \qquad\qquad\qquad Z = \operatorname*{colim}_{a \in A} U_a$$

where $U_a \in \mathrm{Sch}^{\mathrm{aff}}$, the maps $U_\alpha \to Z$ are open embeddings, and where the colimit is taken in Sch.

We have a pair of mutually adjoint functors

$$(4.8) \qquad\qquad \lim_{a \in A} \mathrm{Pro}(\mathrm{QCoh}(U_a)^-) \rightleftarrows \mathrm{Pro}(\mathrm{QCoh}(Z)^-),$$

where the functor $\leftarrow$ is given by left Kan extension along each j_a^*, and the functor $\rightarrow$ sends a compatible family

$$\{\Phi_a \in \mathrm{Pro}(\mathrm{QCoh}(U_a)^-)\}$$

to the functor $\Phi : \mathrm{QCoh}(Z)^- \to \mathrm{Vect}$ given by

$$\Phi(\mathcal{F}) := \lim_{a \in A} \Phi_a(j_a^*(\mathcal{F})).$$

We have:

LEMMA 4.6.2. *Let Z be quasi-compact. Then functors in* (4.8) *are mutually inverse equivalences.*

PROOF. Follows easily from the fact that we can replace the limit over the category A by a finite limit.   □

4.6.3.   Let $\mathcal{X}$ be a prestack that admits a pro-cotangent complex. Assume that $\mathcal{X}$ is a sheaf in the Zariski topology. Let Z be a quasi-compact scheme.

Let $x : Z \to \mathcal{X}$ be a map. Recall that according to Sect. 2.3.5, we have a well-defined object

$$T_x^*(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(Z)^-).$$

The fact that the map (4.4) is an isomorphism implies that $T^*(\mathcal{X})$ gives rise to a well-defined object

$$(4.9) \qquad\qquad \{T_{x|_{U_a}}^*(\mathcal{X})\} \in \lim_{a \in A} \mathrm{Pro}(\mathrm{QCoh}(U_a)^-).$$

By definition, we have:

LEMMA 4.6.4. *The object $T_x^*(\mathcal{X})$ is canonically isomorphic to the image of $\{T_{x|_{U_a}}^*(\mathcal{X})\}$ under the functor $\rightarrow$ in* (4.8).

In particular, from Lemma 4.6.2 we obtain:

COROLLARY 4.6.5. *Let $f : Z_1 \to Z_2$ be a map in $(\mathrm{Sch}_{\mathrm{qc}})_{/\mathcal{X}}$. Then the canonical map*

$$T_{x_1}^*(\mathcal{X}) \to \mathrm{Pro}(f^*)(T_{x_2}^*(\mathcal{X}))$$

is an isomorphism in $\mathrm{Pro}(\mathrm{QCoh}(Z_1)^-)$.

4.6.6.   The Zariski-locality of the construction in Sect. 4.5.2 implies that that there exists a canonically defined map

$$(dx)^* : T_x^*(\mathcal{X}) \to T^*(Z).$$

Furthermore, if $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ and $Z \in \mathrm{Sch}_{\mathrm{aft}}$, $(dx)^*$ corresponds to a map in $\mathrm{IndCoh}(Z)$,

$$dx : T(Z) \to T_x(\mathcal{X}).$$

5. Digression: square-zero extensions

The notion of square-zero extension is central to deformation theory. It allows to obtain nilpotent embeddings of a scheme by iterating a certain linear construction.

5.1. The notion of square-zero extension. In this subsection we introduce (following [**Lu2**, Sect. 7.4.1]) the notion of square-zero extension and study its basic properties.

5.1.1.   Let X be an object of Sch. The category of *square-zero extensions* is by definition

$$\left((\mathrm{QCoh}(X)^{\leq -1})_{T^*(X)/}\right)^{\mathrm{op}}.$$

There is a naturally defined functor

(5.1) $\mathrm{RealSqZ} : \left((\mathrm{QCoh}(X)^{\leq -1})_{T^*(X)/}\right)^{\mathrm{op}} \to \mathrm{Sch}_{X/}$

that sends

$$T^*(X) \xrightarrow{\gamma} \mathcal{F} \in (\mathrm{QCoh}(X)^{\leq -1})_{T^*(X)/}$$

to

$$X' := X \underset{X_{\mathcal{F}}}{\sqcup} X,$$

where the two maps $X_{\mathcal{F}} \rightrightarrows \mathcal{F}$ are the tautological projection $X_{\mathcal{F}} \xrightarrow{pr} X$, and the map

$$X_{\mathcal{F}} \xrightarrow{\gamma} X_{T^*(X)} \xrightarrow{\partial} X,$$

respectively. The map $X \to X'$ corresponds to the first factor in $X \underset{X_{\mathcal{F}}}{\sqcup} X$; it is a closed nil-isomorphism.

5.1.2.   Here is a functorial interpretation of the functor (5.1):

Given $X \in \mathrm{Sch}$, let $\mathrm{Sch}_{X/, \mathrm{inf\text{-}closed}}$ be the full subcategory of $\mathrm{Sch}_{X/}$, spanned by those $f : X \to Y$, for which the codifferential

$$(df)^* : T_f^*(Y) = f^*(T^*(Y)) \to T^*(X)$$

induces a surjection on H^0. I.e., $T^*(X/Y) \in \mathrm{QCoh}(X)^{\leq -1}$.

We have a functor

(5.2) $\mathrm{Sch}_{X/, \mathrm{inf\text{-}closed}} \to ((\mathrm{QCoh}(X)^{\leq -1})_{T^*(X)/})^{\mathrm{op}}, \quad (Y, f) \mapsto T^*(X/Y).$

Unwinding the definitions, we see that the functor RealSqZ of (5.1) is the left adjoint of (5.2).

5.1.3.   The following observation may be helpful in parsing the above construction of the functor RealSqZ. Let $\mathcal{F}$ be an object of $\mathrm{QCoh}(X)^{\leq -1}$, and let γ_1, γ_2 be maps $T^*(X) \to \mathcal{F}$. We have:

LEMMA 5.1.4. *There is a canonical isomorphism in* $(\mathrm{Sch})_{X/}$

$$X \underset{0, X_{\mathcal{F}}, \gamma}{\sqcup} X \simeq X \underset{\gamma_1, X_{\mathcal{F}}, \gamma_2}{\sqcup} X,$$

where X maps to both sides via the left copy of X in the push-out, and $\gamma = \gamma_1 - \gamma_2$.

PROOF. By definition, the left-hand and the-right side are the co-equalizers in $\mathrm{Sch}_{X/}$ of the maps

$$X_{\mathcal{F}} \rightrightarrows X,$$

equal to $(pr, \mathfrak{d} \circ \gamma)$ and $(\mathfrak{d} \circ \gamma_1, \mathfrak{d} \circ \gamma_2)$, respectively.

Given a map $x : X \to Y$, where $Y \in \mathrm{Sch}$, in each of the two cases, the datum of a map

$$\mathrm{co\text{-}Eq}(X_{\mathcal{F}} \rightrightarrows X) \to Y$$

in $\mathrm{Sch}_{X/}$ is equivalent to that of a map

$$T_x^*(Y) \to \mathrm{Eq}(T^*(X) \rightrightarrows \mathcal{F}),$$

where $T_x^*(Y) \to T^*(X)$ is $(dx)^*$, and the maps $T^*(X) \rightrightarrows \mathcal{F}$ are

$$(0, \gamma) \text{ and } (\gamma_1, \gamma_2),$$

respectively. This makes the assertion of the lemma manifest.

□

5.1.5.   We shall denote the category

$$\left((\mathrm{QCoh}(X)^{\leq -1})_{T^*(X)/} \right)^{\mathrm{op}}$$

also by $\mathrm{SqZ}(X)$, and refer to its objects as *square-zero extensions of* X. Compare this with the notation $\mathrm{SplitSqZ}(X)$ in Sect. 2.1.6.

Thus, RealSqZ is a functor

$$\mathrm{SqZ}(X) \to \mathrm{Sch}_{X/}.$$

We shall say that $(X \hookrightarrow X') \in \mathrm{Sch}_{X/}$ has a structure of square-zero extension if it given as the image of an object of $\mathrm{SqZ}(X)$ under the functor RealSqZ.

Note, however, that in general, the functor RealSqZ is *not* fully faithful.

5.1.6.   For a fixed $\mathcal{F} \in \mathrm{QCoh}(X)^{\leq -1}$, we shall refer to the category (in fact, space)

$$\mathrm{Maps}(T^*(X), \mathcal{F})$$

as that of *square-zero extensions of X by means of* $\mathcal{I} := \mathcal{F}[-1]$.

The reason for this terminology is the following. Let

$$(X \overset{i}{\hookrightarrow} X') = \mathrm{RealSqZ}(T^*(X) \to \mathcal{F}).$$

Then from the construction of X' as a push-out it follows that we have a fiber sequence in $\mathrm{QCoh}(X')$:

(5.3) $i_*(\mathcal{I}) \to \mathcal{O}_{X'} \to i_*(\mathcal{O}_X),$

where i denotes the closed embedding $X \to X'$. I.e., $\mathcal{I}$ is the 'ideal' of X inside X'.

5.1.7. Finally, let us note that we have the following pullback diagram of categories:

$$(\mathrm{QCoh}(X)^{\leq 0})^{\mathrm{op}} \longrightarrow \left((\mathrm{QCoh}(X)^{\leq -1})_{T^*(X)/}\right)^{\mathrm{op}}$$

$$\scriptstyle{=}\Big\downarrow \qquad\qquad\qquad\qquad \scriptstyle{=}\Big\downarrow$$

$$\mathrm{SplitSqZ}(X) \longrightarrow \mathrm{SqZ}(X)$$

$$\scriptstyle{\mathrm{RealSplitSqZ}}\Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow\scriptstyle{\mathrm{RealSqZ}}$$

$$(\mathrm{Sch})_{X//X} \longrightarrow (\mathrm{Sch})_{X/},$$

where the top horizontal arrow is the functor

$$\mathcal{F} \in \mathrm{QCoh}(X)^{\leq 0} \quad \mapsto \quad (T^*(X) \overset{0}{\to} \mathcal{F}[1]) \in (\mathrm{QCoh}(X)^{\leq -1})_{T^*(X)/}.$$

5.2. Functoriality of square-zero extensions. In this subsection we will study how square-zero extensions behave under push-outs along affine morphisms.

5.2.1. Let $f : X_1 \to X_2$ be an *affine* map in Sch. We claim that there is a canonically defined functor

$$(5.4) \qquad\qquad (\mathrm{QCoh}(X_1)^{\leq -1})_{T^*(X_1)/} \to (\mathrm{QCoh}(X_2)^{\leq -1})_{T^*(X_2)/}$$

that makes the diagram

$$\mathrm{SqZ}(X_1) \longrightarrow \mathrm{SqZ}(X_2)$$

$$\scriptstyle{\mathrm{RealSqZ}}\Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow\scriptstyle{\mathrm{RealSqZ}}$$

$$\mathrm{Sch}_{X_1/} \longrightarrow \mathrm{Sch}_{X_2/}$$

commute, where the functor $\mathrm{Sch}_{X_1/} \to \mathrm{Sch}_{X_2/}$ is given by push-out:

$$(X_1 \hookrightarrow X_1') \mapsto (X_2 \hookrightarrow X_2 \underset{X_1}{\sqcup} X_1').$$

Indeed, the functor (5.4) sends $\gamma_1 : T^*(X_1) \to \mathcal{F}_1$ to

$$\gamma_2 : T^*(X_2) \to f_*(\mathcal{F}_1),$$

where γ_2 is obtained by the (f^*, f_*)-adjunction from the composition

$$f^*(T^*(X_2)) \simeq T_f^*(X_2) \overset{(df)^*}{\longrightarrow} T^*(X_1) \overset{\gamma_1}{\longrightarrow} \mathcal{F}_1.$$

Note that the assumption that f be affine was used to ensure that $f_*(\mathcal{F}_1) \in \mathrm{QCoh}(X_2)^{\leq -1}$.

REMARK 5.2.2. Note that for a map $f : X_1 \to X_2$ as above, the diagram

$$\mathrm{SplitSqZ}(X_1) \longrightarrow \mathrm{SplitSqZ}(X_2)$$

$$\Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow$$

$$\mathrm{SqZ}(X_1) \longrightarrow \mathrm{SqZ}(X_2).$$

commutes, where the top horizontal arrow is

$$f_* : \mathrm{QCoh}(X_1)^{\leq 0} \to \mathrm{QCoh}(X_2)^{\leq 0}.$$

5.2.3. The construction in Sect. 5.2.1 makes the assignment

$$X \rightsquigarrow \mathrm{SqZ}(X)$$

into a functor $(\mathrm{Sch})_{\mathrm{affine}} \to 1\text{-Cat}$, where

$$(\mathrm{Sch})_{\mathrm{affine}} \subset \mathrm{Sch}$$

is the 1-full subcategory, where we restrict 1-morphisms to be affine.

Thus we obtain a co-Cartesian fibration

$$(\mathrm{SqZ}(\mathrm{Sch}))_{\mathrm{affine}} \to (\mathrm{Sch})_{\mathrm{affine}},$$

whose fiber over $X \in \mathrm{Sch}$ is $\mathrm{SqZ}(X)$.

5.2.4. In particular, given an affine map $f : X_1 \to X_2$ and objects

$$(T^*(X_i) \overset{\gamma_i}{\to} \mathcal{F}_i) \in \mathrm{SqZ}(X_i), \quad i = 1, 2$$

we obtain a well-defined notion of *map of square-zero extensions*

$$(T^*(X_1) \overset{\gamma_i}{\to} \mathcal{F}_1) \to (T^*(X_2) \overset{\gamma_i}{\to} \mathcal{F}_2),$$

extending f.

By definition, a datum of such a map amounts to a morphism $\mathcal{F}_2 \to f_*(\mathcal{F}_1)$, equipped with a datum of commutativity of the diagram

$$
\begin{array}{ccc}
f^*(T^*(X_2)) & \xrightarrow{(df)^*} & T^*(X_1) \\
{\scriptstyle \gamma_2}\downarrow & & \downarrow{\scriptstyle \gamma_1} \\
f^*(\mathcal{F}_2) & \longrightarrow & \mathcal{F}_1.
\end{array}
$$

In the above circumstances we shall say that for

$$(X_i \hookrightarrow X_i') = \mathrm{RealSqZ}(T^*(X_i) \overset{\gamma_i}{\to} \mathcal{F}_i),$$

the resulting commutative diagram

$$
\begin{array}{ccc}
X_1 & \longrightarrow & X_1' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
X_2 & \longrightarrow & X_2'
\end{array}
$$

has been given a structure of *map of sqaure-zero* extensions.

5.3. Pull-back of square-zero extensions. In this subsection we will show that, in addition to push-outs of square-zero extensions with respect to the source, one can also form pullbacks with respect to maps of the target.

5.3.1.   Note that the category $\mathrm{SqZ}(\mathrm{Sch})_{\mathrm{affine}}$, introduced above, admits a forgetful functor to the category $\mathrm{Funct}([1],(\mathrm{Sch})_{\mathrm{affine}})$ of pairs of schemes $(X \to X')$ and affine maps between them.

The functor

$$\mathrm{targ} : \mathrm{Funct}([1],(\mathrm{Sch})_{\mathrm{affine}}) \to (\mathrm{Sch})_{\mathrm{affine}}, \quad (X \to X') \to X'$$

is a Cartesian fibration (via the formation of fiber products).

We claim:

PROPOSITION 5.3.2.   *The composite functor*

$$(5.5) \qquad \mathrm{SqZ}(\mathrm{Sch})_{\mathrm{affine}} \to \mathrm{Funct}([1],(\mathrm{Sch})_{\mathrm{affine}}) \overset{\mathrm{targ}}{\longrightarrow} (\mathrm{Sch})_{\mathrm{affine}}$$

is a Cartesian fibration, and the forgetful functor

$$\mathrm{SqZ}(\mathrm{Sch})_{\mathrm{affine}} \to \mathrm{Funct}([1],(\mathrm{Sch})_{\mathrm{affine}})$$

sends Cartesian arrows to Cartesian arrows.

5.3.3.   The concrete meaning of this proposition is that if

$$(X \hookrightarrow X') = \mathrm{RealSqZ}(T^*(X) \overset{\gamma_X}{\longrightarrow} \mathcal{F}_X),$$

then for an affine map $Y' \to X'$, the object

$$(X \underset{X'}{\times} Y' =: Y \hookrightarrow Y') \in \mathrm{Sch}_{Y/}$$

has a canonical structure of square-zero extension; moreover as such it satisfies an appropriate universal property (for mapping into it).

5.3.4.   *Proof of Proposition 5.3.2.* In the notations of Sect. 5.3.3, note that γ_X canonically factors as

$$T^*(X) \to T^*(X/X') \overset{\gamma_X'}{\longrightarrow} \mathcal{F}_X.$$

Set $\mathcal{F}_Y := f^*(\mathcal{F}_X)$. We construct the morphism

$$\gamma_Y : T^*(Y) \to \mathcal{F}_Y$$

as the composite

$$T^*(Y) \to T^*(Y/Y') \simeq f^*(T^*(X/X')) \overset{f^*(\gamma_X')}{\longrightarrow} f^*(\mathcal{F}_X).$$

By Sect. 5.1.2, the square-zero extension of Y corresponding to γ_Y is equipped with a canonical map to Y'. This map is an isomorphism by (5.3).

The fact that this square-zero extension satisfies the required universal property is a straightforward verification.

$\square$

5.3.5.   The construction of pullback in Proposition 5.3.2 is local in the Zariski topology. This allows to extend the Cartesian fibration (5.5) to a Cartesian fibration

$$\mathrm{SqZ}(\mathrm{Sch}) \to \mathrm{Sch},$$

i.e., the formation of structure of square-zero extension on the pullback is applicable to *not necessarily affine* morphisms between schemes.

5.4. Square-zero extensions and truncations. In this subsection we will establish a crucial fact that a scheme can be obtained as a succession of square-zero extensions of its n-coconnective truncations.

5.4.1.   We claim (which is essentially [**Lu2**, Theorem 7.4.1.26]):

PROPOSITION 5.4.2.
(a) *For $X \in {}^{\mathrm{cl}}\mathrm{Sch}$, the category of its square-zero extensions by means of objects of $\mathrm{QCoh}(X)^\heartsuit$ is equivalent to that of closed embeddings of classical schemes $X \hookrightarrow X'$, where the ideal of X in X' is such that its square vanishes.*
(b) *For $X_n \in {}^{\leq n}\mathrm{Sch}$, the category of*

$$(X_{n+1} \in {}^{\leq n+1}\mathrm{Sch}, {}^{\leq n}X_{n+1} \simeq X_n)$$

is canonically equivalent to that of square-zero extensions of X_n by objects of

$$\mathrm{QCoh}(X_n)^\heartsuit[n+1] \subset \mathrm{QCoh}(X_n).$$

PROOF.   We will prove point (b), as the proof of point (a) is similar but simpler. We have the fiber sequence

$$i_*(\mathcal{F}[-1]) \to \mathcal{O}_{X_{n+1}} \to i_*(\mathcal{O}_{X_n}),$$

where $\mathcal{F} \in (\mathrm{QCoh}(X_n)^\heartsuit)[n+2]$.

We claim that X_{n+1} has a structure of square-zero extension of X_n, corresponding to a canonically defined map $\gamma : T^*(X_n) \to \mathcal{F}$.

Indeed, consider the fiber sequence

$$T_i^*(X_{n+1}) \overset{(di)^*}{\to} T^*(X_n) \to T^*(X_n/X_{n+1}),$$

and the existence and canonicity of the required map γ follows from the next observation:

$$\begin{cases} H^k(T^*(X_n/X_{n+1})) = 0 \text{ for } k \geq -n-1 \\ H^{-n-2}(T^*(X_n/X_{n+1})) \simeq \mathcal{F}, \end{cases}$$

which in turns results from the following general assertion (see [**Lu2**, Theorem 7.4.3.1]):

LEMMA 5.4.3. *Let $i : X \to Y$ be a closed embedding of schemes. Consider the corresponding fiber sequence*

$$\mathcal{I} \to \mathcal{O}_Y \to i_*(\mathcal{O}_X).$$

Then:
(a) $H^0(T^*(X/Y)) = 0$ *and*

$$H^{-1}(T^*(X/Y)) = H^0(i^*(\mathcal{I}))$$

as objects of $\mathrm{QCoh}(X)^\heartsuit$.
(b) *For $n \geq 0$ we have:*

$$\tau^{\geq -n}(\mathcal{I}) = 0 \Rightarrow \tau^{\geq -n-1}(T^*(X/Y)) = 0.$$

In the latter case ${}^{\mathrm{cl}}X \simeq {}^{\mathrm{cl}}Y$ and

$$H^{-n-2}(T^*(X/Y)) \simeq H^{-n-1}(\mathcal{I})$$

as objects of $\mathrm{QCoh}(X)^\heartsuit \simeq \mathrm{QCoh}(Y)^\heartsuit$. $\qquad\square$

5.4.4.   The assertion of Proposition 5.4.2(b) in particular constructs a functor

(5.6) $^{\leq n+1}\mathrm{Sch} \to \mathrm{SqZ}(\mathrm{Sch})_{\mathrm{affine}} \underset{\mathrm{Sch}\times\mathrm{Sch}}{\times} (^{\leq n}\mathrm{Sch}\times{}^{\leq n+1}\mathrm{Sch}).$

PROPOSITION 5.4.5. *The functor* (5.6) *is the (fully faithful) right adjoint of the forgetful functor*

$$\mathrm{SqZ}(\mathrm{Sch})_{\mathrm{affine}} \underset{\mathrm{Sch}\times\mathrm{Sch}}{\times} (^{\leq n}\mathrm{Sch}\times{}^{\leq n+1}\mathrm{Sch}) \to {}^{\leq n+1}\mathrm{Sch}.$$

PROOF. We construct the unit of the adjunction as follows. Given a square-zero extension

$$(T^*(X) \overset{\gamma}{\longrightarrow} \mathcal{F}), \quad X \in {}^{\leq n}\mathrm{Sch},\ \mathcal{F} \in \mathrm{QCoh}(X)^{\geq -n-2, \leq -1},$$

denote

$$(X \hookrightarrow X') := \mathrm{RealSqZ}(T^*(X) \overset{\gamma}{\longrightarrow} \mathcal{F}),$$

and note that there exists a canonically defined commutative diagram of scheme[6]

$$
\begin{array}{ccc}
X & \longrightarrow & X' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
^{\leq n}X' & \longrightarrow & X'.
\end{array}
$$

Let $(X \hookrightarrow X')$ be given by a map $\gamma : T^*(X) \to \mathcal{F}$, where $\mathcal{F} \in \mathrm{QCoh}(X)^{\geq -n-2, \leq -1}$. We obtain a commutative diagram in $\mathrm{QCoh}(X)$:

$$
\begin{array}{ccc}
f^*(T^*(^{\leq n}X')) & \longrightarrow & f^*(T^*(^{\leq n}X'/X')) \\
{\scriptstyle (df)^*}\downarrow & & \downarrow \\
T^*(X) & \longrightarrow & \mathcal{F}.
\end{array}
$$

We note that $\mathcal{F}$ lives in the cohomological degrees $\geq -n-2$, while, by Lemma 5.4.3, $T^*(^{\leq n}X'/X')$ lives in the cohomological degrees $\leq -n - 2$ with

$$H^{-n-2}(T^*(^{\leq n}X'/X')) \simeq H^{-n-1}(\mathcal{I}),$$

where $T^*(^{\leq n}X') \to \mathcal{I}[1]$ is the map defining the square-zero extension $^{\leq n}X' \hookrightarrow X'$.

Hence, the map $f^*(T^*(^{\leq n}X'/X')) \to \mathcal{F}$ canonically gives rise to a map $f^*(\mathcal{I}[1]) \to \mathcal{F}$, and we obtain a commutative diagram

$$
\begin{array}{ccc}
f^*(T^*(^{\leq n}X')) & \longrightarrow & f^*(\mathcal{I}[1]) \\
{\scriptstyle (df)^*}\downarrow & & \downarrow \\
T^*(X) & \longrightarrow & \mathcal{F},
\end{array}
$$

which defines the sought-for unit for the adjunction. The fact that it satisfies the adjunction axioms is a straightforward check.

$\square$

5.5. Nilpotent embeddings. In this subsection we will show that a nilpotent embedding of a scheme can be obtained as a (infinite) composition of square-zero extensions.

---

[6]In order to unburden the notation, for the duration of this Chapter, for a scheme Y, we will denote by $^{\leq n}Y$ the object of Sch that should be properly denoted by $^L\tau^{\leq n}(Y)$, see Volume I, Chapter 2, Sect. 2.6.2. I.e., this is the n-coconnective truncation of Y, viewed as an object of Sch, rather than $^{\leq n}\mathrm{Sch}$.

5.5.1. We shall say that a map $X \to Y$ of schemes is a *nilpotent embedding* if $^{\mathrm{cl}}X \to {}^{\mathrm{cl}}Y$ is a closed embedding of classical schemes, such that the ideal of $^{\mathrm{cl}}X$ in $^{\mathrm{cl}}Y$ is nilpotent (i.e., there exists a power n that annihilates every section).

5.5.2. We are going to prove the following useful result:

PROPOSITION 5.5.3. *Let $X \to Y$ be a nilpotent embedding of schemes. There exists a sequence of schemes*

$$X = X_0^0 \hookrightarrow X_0^1 \hookrightarrow ... \hookrightarrow X_0^i \hookrightarrow ... \hookrightarrow X_0^n = X_0 \hookrightarrow X_1 \hookrightarrow ... \hookrightarrow X_j \hookrightarrow ... \hookrightarrow Y,$$

such that:

- *Each of the maps $X_0^i \hookrightarrow X_0^{i+1}$, $X_0' \hookrightarrow X_0$ and $X_j \hookrightarrow X_{j+1}$ has a structure of square-zero extension;*
- *For every j, the map $g_j : X_j \to Y$ induces an isomorphism $^{\leq j}X_j \to {}^{\leq j}Y$.*

The rest of this subsection is devoted to the proof of Proposition 5.5.3.

5.5.4. *Step 1.* Let

$$^{\mathrm{cl}}X = X_{\mathrm{cl},0}^0 \hookrightarrow X_{\mathrm{cl},0}^1 \hookrightarrow ... \hookrightarrow X_{\mathrm{cl},0}^k \hookrightarrow ... X_{\mathrm{cl},0}^n = {}^{\mathrm{cl}}Y$$

be a sequence of square-zero extensions of classical schemes. It exists by the assumption that the ideal of the closed embedding $^{\mathrm{cl}}X \to {}^{\mathrm{cl}}Y$ is nilpotent. Set

$$X_0^i := X_0^0 \underset{^{\mathrm{cl}}X_{\mathrm{cl},0}^0}{\sqcup} X_{\mathrm{cl},0}^k.$$

By construction, $g_0 : X_0 \to Y$ induces an isomorphism $^{\mathrm{cl}}X_0 \to {}^{\mathrm{cl}}Y$.

5.5.5. *Step 2.* Starting from $g_0 : X_0 \to Y$, we shall construct $g_1 : X_1 \to Y$ using the following general procedure. The same procedure constructs $g_{i+1} : X_{i+1} \to Y$ starting from $g_i : X_i \to Y$.

Let $h : Z \to Y$ be a map that induces an isomorphism of the underlying classical schemes, and such that $T^*(Z/Y)$ lives in the cohomological degrees $\leq -(k+1)$ with $k \geq 0$.

We will construct a map $f : Z \hookrightarrow Z'$ with a structure of square-zero extension by an object $\mathcal{J} \in \mathrm{QCoh}(Z)^\heartsuit[k]$, and an extension of the map h to a map $h' : Z' \to Y$ so that h' such that $T^*(Z/Y)$ lives in the cohomological degrees $\leq -(k+2)$. (Hence, by Lemma 5.4.3, the 'ideal' of Z' in Y lives in degrees $\leq -(k+1)$, and in particular, $^{\leq k}Z \to {}^{\leq k}Y$ is an isomorphism.)

Namely, consider the fiber sequence

$$T_h^*(Y) \to T^*(Z) \to T^*(Z/Y),$$

and take

$$\mathcal{J} := H^{-k-1}(T^*(Z/Y))[k] = \tau^{\geq -(k+1)}(T^*(Z/Y))[-1].$$

We let the sought-for square-zero extension $Z \hookrightarrow Z'$ be given by the composite map

$$T^*(Z) \to T^*(Z/Y) \to \mathcal{J}[1].$$

The composition

$$T_h^*(Y) \to T^*(Z) \to \mathcal{J}[1]$$

acquires a canonical null-homotopy by constriction, thereby giving rise to a map $h' : Z' \to Y$.

In order to show that $T^*(Z'/Y)$ lives in $\mathrm{QCoh}(Z')^{\leq-(k+2)}$, consider the fiber sequences

$$\mathcal{I} \to \mathcal{O}_Y \to h_*(\mathcal{O}_Z) \text{ and } \mathcal{I}' \to \mathcal{O}_Y \to h'_*(\mathcal{O}_{Z'})$$

and the diagram

$$
\begin{array}{ccccc}
h_*(\mathcal{J}) & \longrightarrow & h'_*(\mathcal{O}_{Z'}) & \longrightarrow & h_*(\mathcal{O}_Z) \\
\uparrow & & \uparrow & & \uparrow {\scriptstyle\mathrm{Id}} \\
\mathcal{I} & \longrightarrow & \mathcal{O}_Y & \longrightarrow & h_*(\mathcal{O}_Z) \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{I}' & \longrightarrow & \mathcal{I}' & \longrightarrow & 0.
\end{array}
$$

By Lemma 5.4.3, the map $\mathcal{I} \to h_*(\mathcal{J})$ identifies with the truncation map

$$\mathcal{I} \to \tau^{\geq 2-k}(\mathcal{I}).$$

Hence, $\mathcal{I}' \in \mathrm{QCoh}(Y)^{\leq-(k-1)}$. Now, this implies that $T^*(Z'/Y) \in \mathrm{QCoh}(Z')^{\leq-(k+2)}$ again by Lemma 5.4.3.

$\square$

6. Infinitesimal cohesiveness

Infinitesimal cohesiveness is a property of a prestack that allows to describe maps into it from a square-zero extension of an affine scheme S as data involving $\mathrm{QCoh}(S)$.

6.1. Infinitesimal cohesiveness of a prestack. In this subsection we introduce the notion of infinitesimal cohesiveness in terms of compatibility with certain type of push-outs.

6.1.1.   Let $\mathcal{X} \in \mathrm{PreStk}$, and let (S, x) be an object of $(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$. For

$$T^*(S) \overset{\gamma}{\to} \mathcal{F} \in \left(\mathrm{QCoh}(S)^{\leq-1}\right)_{T^*(S)/}^{\mathrm{op}} = \mathrm{SqZ}(S)$$

and the corresponding

$$(S \hookrightarrow S') := \mathrm{RealSqZ}(T^*(S) \overset{\gamma}{\to} \mathcal{F}) = S \underset{S_{\mathcal{F}}}{\sqcup} S$$

we obtain a canonically defined map

(6.1) $$\mathrm{Maps}_{S/}(S', \mathcal{X}) \to * \underset{\mathrm{Maps}(S_{\mathcal{F}}, \mathcal{X})}{\times} \mathrm{Maps}(S, \mathcal{X}),$$

where $* \to \mathrm{Maps}(S_{\mathcal{F}}, \mathcal{X})$ corresponds to the composition

$$S_{\mathcal{F}} \xrightarrow{pr} S \overset{x}{\to} \mathcal{X}.$$

DEFINITION 6.1.2.   *We shall say that $\mathcal{X}$ is* infinitesimally cohesive *if the map* (6.1) *is an isomorphism for all S, x and*

$$(T^*(S) \overset{\gamma}{\to} \mathcal{F}) \in \left((\mathrm{QCoh}(S)^{\leq-1})_{T^*(S)/}\right)^{\mathrm{op}}$$

as above.

We observe that by Lemma 1.3.3(b), any $\mathcal{X} = X \in \mathrm{Sch}$ is infinitesimally cohesive.

6.1.3. Suppose that $\mathcal{X}$ is convergent (resp., belongs to $\mathrm{PreStk}_{\mathrm{laft}}$). Then as in Lemmas 3.3.4 (resp., 3.5.3), in order to verify the condition of infinitesimal cohesiveness, it is sufficient to consider $S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$ and $\mathcal{F} \in \mathrm{QCoh}(S)^{>-\infty}$ (resp., $S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ and $\mathcal{F} \in \mathrm{Coh}(S)$).

6.1.4. *The relative situation.* The notion of infinitesimal cohesiveness renders automatically to the relative situation.

We note that the analog of Lemma 2.4.5 holds when we replace 'admitting cotangent spaces' by 'infinitesimal cohesiveness'.

We also note that when $\mathcal{X}$ and $\mathcal{X}_0$ are convergent (resp., locally almost of finite type), the analog of Lemma 4.2.5 holds.

6.2. Rewriting the condition of infinitesimal cohesiveness. We will now rewrite the definition of infinitesimal cohesiveness in terms of QCoh.

6.2.1. Note that the space

$$* \underset{\mathrm{Maps}(S_{\mathcal{F}}, \mathcal{X})}{\times} \mathrm{Maps}(S, \mathcal{X})$$

identifies with the space of homotopies between the following two points of $\mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X})$: the first being

$$S_{\mathcal{F}} \xrightarrow{pr} S \xrightarrow{x} \mathcal{X},$$

and the second being

$$S_{\mathcal{F}} \xrightarrow{\gamma} S_{T^*(S)} \xrightarrow{\mathfrak{d}} S \xrightarrow{x} \mathcal{X}.$$

6.2.2. Assume that $\mathcal{X}$ admits a pro-cotangent space at x. We obtain that the space

$$* \underset{\mathrm{Maps}(S_{\mathcal{F}}, \mathcal{X})}{\times} \mathrm{Maps}(S, \mathcal{X})$$

identifies with the space of null-homotopies of the composed map

$$T^*_x(\mathcal{X}) \xrightarrow{(dx)^*} T^*(S) \xrightarrow{\gamma} \mathcal{F}.$$

6.2.3. Thus, we obtain that if $\mathcal{X}$ admits pro-cotangent spaces, the condition of infinitesimal cohesiveness can be formulated as saying that given (S, x), for every

$$(T^*(S) \to \mathcal{F}) \in (\mathrm{QCoh}(S)^{\leq -1})_{T^*(S)/}$$

and the corresponding

$$(S \hookrightarrow S') = \mathrm{RealSqZ}(T^*(S) \to \mathcal{F}) \in \mathrm{Sch}^{\mathrm{aff}}_{S/},$$

the canonical map of spaces

(6.2) $\qquad \mathrm{Maps}_{S/}(S', \mathcal{X}) \simeq \{\text{null homotopies of } T^*_x(\mathcal{X}) \xrightarrow{(dx)^*} T^*(S) \to \mathcal{F}\}$

be an isomorphism.

Equivalently, this can be phrased as saying that for (S, x), the functor

$$(\mathrm{Sch}^{\mathrm{aff}})_{S//\mathcal{X}} \underset{(\mathrm{Sch}^{\mathrm{aff}})_{S/}}{\times} \mathrm{SqZ}(S) \to (\mathrm{QCoh}(S)^{\leq -1})_{\mathrm{coFib}(T^*_x(\mathcal{X}) \xrightarrow{(dx)^*} T^*(S))/}$$

is an equivalence.

6.2.4. Suppose that $\mathcal{X}$ both admits a pro-cotangent complex and is infinitesimally cohesive. Assume also that $\mathcal{X}$ is a sheaf in the Zariski topology.

Let Z be a scheme. From Lemma 4.6.2, we obtain:

COROLLARY 6.2.5. *For* $(T^*(Z) \to \mathcal{F}) \in (\mathrm{QCoh}(Z)^{\leq -1})_{/T^*(Z)}$ *and*

$$(Z \hookrightarrow Z') = \mathrm{RealSqZ}(T^*(Z) \to \mathcal{F}) \in \mathrm{Sch}_{Z/},$$

the map

$$\mathrm{Maps}_{Z/}(Z', \mathcal{X}) \to \{\text{null homotopies of } T_x^*(\mathcal{X}) \overset{(dx)^*}{\to} T^*(Z) \to \mathcal{F}\}$$

is an isomorphism.

6.3. Consequences of infinitesimal cohesiveness. If a prestack is infinitesimally cohesive, one can deduce that it has certain properties from the fact that the underlying reduced prestack has these properties.

6.3.1. First, combining Sect. 6.2.2 and Proposition 5.4.2, we obtain:

LEMMA 6.3.2. *Assume that* $\mathcal{X} \in \mathrm{PreStk}$ *admits* $(-k)$*-connective pro-cotangent spaces and is infinitesimally cohesive. Let* S *be an object of* $^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, *and let* $S_0 \subset {}^{\mathrm{cl}}S$ *be given by a nilpotent ideal. Then the fibers of the map*

$$\mathrm{Maps}(S, \mathcal{X}) \to \mathrm{Maps}(S_0, \mathcal{X})$$

are $(k+n)$*-truncated.*

In particular, we obtain that if $^{\mathrm{cl}}\mathcal{X}$ takes values in $\mathrm{Sets} \subset \mathrm{Spc}$, then for $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, the space $\mathrm{Maps}(S, \mathcal{X})$ is n-truncated.

6.3.3. From Sect. 5.2.1 we obtain:

LEMMA 6.3.4. *Let* $\mathcal{X}$ *be an object of* PreStk, *which both admits a pro-cotangent complex and is infinitesimally cohesive. Then if*

$$S_1' \underset{S_1}{\sqcup} S_2 \to S_2'$$

is a push-out diagram in $\mathrm{Sch}^{\mathrm{aff}}$, *where* $S_1 \hookrightarrow S_1'$ *has a structure of a square-zero extension. Then*

$$\mathrm{Maps}(S_2', \mathcal{X}) \to \mathrm{Maps}(S_1', \mathcal{X}) \underset{\mathrm{Maps}(S_1, \mathcal{X})}{\times} \mathrm{Maps}(S_2, \mathcal{X})$$

is a pullback diagram.

Iterating, from Lemma 6.3.4, we obtain:

COROLLARY 6.3.5. *Let* $\mathcal{X}$ *be an object of* PreStk *which both admits a pro-cotangent complex and is infinitesimally cohesive. Let*

$$S_1' \underset{S_1}{\sqcup} S_2 \to S_2'$$

be a push-out diagram in $\mathrm{Sch}^{\mathrm{aff}}$ *such that* S_1' *can be obtained from* S_1 *as a finite succession of square-zero extensions. Then*

$$\mathrm{Maps}(S_2', \mathcal{X}) \to \mathrm{Maps}(S_1', \mathcal{X}) \underset{\mathrm{Maps}(S_1, \mathcal{X})}{\times} \mathrm{Maps}(S_2, \mathcal{X})$$

is a pullback diagram.

Furthermore, from Corollary 6.2.5, we obtain:

COROLLARY 6.3.6. *Let $\mathcal{X}$ be an object of* PreStk *which both admits a pro-cotangent complex and is infinitesimally cohesive. Assume also that $\mathcal{X}$ is a sheaf in the Zariski topology. Let*

$$Z_1' \underset{Z_1}{\sqcup} Z_2 \to Z_2'$$

be a push-out diagram in Sch, *where the map $Z_1 \to Z_2$ is affine, and Z_1' can be obtained from Z_1 as a finite succession of square-zero extensions. Then the map*

$$\mathrm{Maps}(Z_2', \mathcal{X}) \to \mathrm{Maps}(Z_1', \mathcal{X}) \underset{\mathrm{Maps}(Z_1, \mathcal{X})}{\times} \mathrm{Maps}(Z_2, \mathcal{X})$$

is an isomorphism.

7. Deformation theory

In this section we finally define what it means for a prestack to admit deformation theory, and discuss some basic consequences of this property.

7.1. Prestacks with deformation theory. In this subsection we give the definition of admitting deformation theory.

7.1.1. We now give the following crucial definition:

DEFINITION 7.1.2. *Let $\mathcal{X}$ be a prestack. We shall say that $\mathcal{X}$* admits deformation theory *(resp.,* admits corepresentable deformation theory*) if:*

- *It is convergent;*
- *It admits a pro-cotangent (resp., cotangent) complex;*
- *It is infinitesimally cohesive.*

Note that the last two conditions are of the form that the functor $\mathcal{X}$ should send certain push-outs in $\mathrm{Sch}^{\mathrm{aff}}$ to pullbacks in Spc; see also Sect. 7.2.4.

7.1.3. In what follows we shall denote by

$$\mathrm{PreStk}_{\mathrm{def}} \subset \mathrm{PreStk} \ \text{ and } \ \mathrm{PreStk}_{\mathrm{laft\text{-}def}} \subset \mathrm{PreStk}_{\mathrm{laft}}$$

the full subcategories spanned by objects that admit deformation theory.

It is clear that the above subcategories are closed under finite limits taken in PreStk.

7.1.4. We shall also consider the following variants:

DEFINITION 7.1.5.

(a) *We shall say that $\mathcal{X}$ admits an $(-n)$-connective deformation theory (resp., corepresentable deformation theory) if it admits deformation theory (resp., corepresentable deformation theory) and its cotangent spaces are $(-n)$-connective.*

(b) *We shall say that $\mathcal{X}$ admits a locally eventually connective deformation theory if it admits deformation theory and its pro-cotangent spaces are locally eventually connective.*

(c) *We shall say that $\mathcal{X}$ admits a uniformly eventually connective deformation theory (resp., corepresentable deformation theory) if there exists an integer n such that $\mathcal{X}$ admits a $(-n)$-connective deformation theory (resp., corepresentable deformation theory).*

As was mentioned above, any scheme X admits a connective corepresentable deformation theory.

7.1.6.   The same definitions carry over to the relative situations for $\mathcal{X} \in \mathrm{PreStk}_{/\mathcal{X}_0}$ for some fixed $\mathcal{X}_0 \in \mathrm{PreStk}$.

Let $\pi : \mathcal{X} \to \mathcal{X}_0$ be a morphism in PreStk. Replacing the words 'infinitesimal cohesiveness' by 'admitting deformation theory' we render the contents of Sect. 6.1.4 to the present context.

7.2. Compatibility with push-outs. In this subsection we rewrite the condition of admitting deformation theory in terms of compatibility with certain type of push-outs.

7.2.1.   One of the main properties of prestacks with deformation theory is given by the following proposition:

PROPOSITION 7.2.2. *Assume that $\mathcal{X}$ admits deformation theory, and let $S_1' \underset{S_1}{\sqcup} S_2$ be a push-out diagram in $\mathrm{Sch}^{\mathrm{aff}}$, where the map $S_1 \to S_1'$ is a nilpotent embedding. Then the map*

$$\mathrm{Maps}(S_1' \underset{S_1}{\sqcup} S_2, \mathcal{X}) \to \mathrm{Maps}(S_1', \mathcal{X}) \underset{\mathrm{Maps}(S_1, \mathcal{X})}{\times} \mathrm{Maps}(S_2, \mathcal{X})$$

is an isomorphism.

PROOF OF PROPOSITION 7.2.2. Follows from Corollary 6.3.5 using Proposition 5.5.3.

$\square$

COROLLARY 7.2.3. *Assume that $\mathcal{X}$ admits deformation theory, and is a sheaf in the Zariski topology. Let*

$$Z_1' \underset{Z_1}{\sqcup} Z_2 \to Z_2'$$

be a push-out diagram in Sch, where $Z_1 \to Z_1'$ is a nilpotent embedding. Assume that the map $Z_1 \to Z_2$ is affine. Then the map

$$\mathrm{Maps}(Z_2', \mathcal{X}) \to \mathrm{Maps}(Z_1', \mathcal{X}) \underset{\mathrm{Maps}(Z_1, \mathcal{X})}{\times} \mathrm{Maps}(Z_2, \mathcal{X})$$

is an isomorphism.

7.2.4.   Now, we have that the following converse of Proposition 7.2.2 holds:

PROPOSITION 7.2.5. *Let $\mathcal{X} \in \mathrm{PreStk}$ be convergent. Assume that whenever $S_1' \underset{S_1}{\sqcup} S_2$ is a push-out diagram in $\mathrm{Sch}^{\mathrm{aff}}$, where the map $S_1 \to S_1'$ is a nilpotent embedding, the map*

$$\mathrm{Maps}(S_1' \underset{S_1}{\sqcup} S_2, \mathcal{X}) \to \mathrm{Maps}(S_1', \mathcal{X}) \underset{\mathrm{Maps}(S_1, \mathcal{X})}{\times} \mathrm{Maps}(S_2, \mathcal{X})$$

is an isomorphism. Then $\mathcal{X}$ admits deformation theory.

PROOF. Let us first show that $\mathcal{X}$ admits pro-cotangent spaces. For $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$ and $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0}$, consider the push-out diagram

$$S \underset{S_{\mathcal{F}[1]}}{\sqcup} S \to S_{\mathcal{F}}$$

(with both maps $S_{\mathcal{F}[1]} \to S$ being pr), and the resulting map

(7.1) $$\mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}) \to \Omega(\mathrm{Maps}_{S/}(S_{\mathcal{F}[1]}, \mathcal{X})).$$

Since the map $S_{\mathcal{F}[1]} \to S$ is a nilpotent embedding, by assumption, the map (7.1) is an isomorphism.

Let now $\mathcal{F} \to \mathcal{F}_1 \to \mathcal{F}_2$ be a fiber sequence in $\mathrm{QCoh}(S)$ with all three terms in $\mathrm{QCoh}(S)^{\leq 0}$. Consider the push-out diagrams

$$S \underset{S_{\mathcal{F}_2}}{\sqcup} S_{\mathcal{F}_1} \to S_{\mathcal{F}}$$

and

$$S \underset{S_{\mathcal{F}_2[1]}}{\sqcup} S_{\mathcal{F}_1[1]} \to S_{\mathcal{F}[1]},$$

and the corresponding maps

(7.2) $$\mathrm{Maps}_{S/}(S_{\mathcal{F}}, \mathcal{X}) \to * \underset{\mathrm{Maps}_{S/}(S_{\mathcal{F}_2}, \mathcal{X})}{\times} \mathrm{Maps}_{S/}(S_{\mathcal{F}_1}, \mathcal{X})$$

and

(7.3) $$\mathrm{Maps}_{S/}(S_{\mathcal{F}[1]}, \mathcal{X}) \to * \underset{\mathrm{Maps}_{S/}(S_{\mathcal{F}_2[1]}, \mathcal{X})}{\times} \mathrm{Maps}_{S/}(S_{\mathcal{F}_1[1]}, \mathcal{X}).$$

Since the map $S_{\mathcal{F}_2[1]} \to S$ is a nilpotent embedding, the map (7.3) is an isomorphism. Taking loops and using (7.1) we obtain that (7.2) is also an isomorphism.

Hence, $\mathcal{X}$ admits pro-cotangent spaces. The fact that $\mathcal{X}$ admits a pro-cotangent complex follows from the fact that $\mathcal{X}$ takes push-outs of the form

$$(S_1)_{\mathcal{F}_1} \underset{S_1}{\sqcup} S_2, \quad \mathcal{F}_1 \in \mathrm{QCoh}(S_1)^{\leq 0}$$

to pullbacks; the latter because $S_1 \to (S_1)_{\mathcal{F}_1}$ is a nilpotent embedding.

Finally, $\mathcal{X}$ is infinitesimally cohesive because it takes push-outs of the form

$$S \underset{S_{\mathcal{F}[1]}}{\sqcup} S, \quad \mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0}$$

to pullbacks; the latter because $S_{\mathcal{F}[1]} \to S$ is a nilpotent embedding.

$\square$

Combining with Proposition 5.5.3, we obtain:

COROLLARY 7.2.6. *Let $\mathcal{X} \in \mathrm{PreStk}$ be convergent. Assume that whenever*

$$S_1' \underset{S_1}{\sqcup} S_2$$

is a push-out diagram in $\mathrm{Sch}^{\mathrm{aff}}$, where the map $S_1 \hookrightarrow S_1'$ has a structure of square-zero extension, the map

$$\mathrm{Maps}(S_1' \underset{S_1}{\sqcup} S_2, \mathcal{X}) \to \mathrm{Maps}(S_1', \mathcal{X}) \underset{\mathrm{Maps}(S_1, \mathcal{X})}{\times} \mathrm{Maps}(S_2, \mathcal{X})$$

is an isomorphism. Then $\mathcal{X}$ admits deformation theory.

7.2.7. It is easy to see that in the circumstances of Corollary 7.2.6, it is enough to consider S_1, S_2, S_1' that belong to $^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$. Furthermore, if $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$, it is enough to take S_1, S_2, S_1' that belong to $^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

Hence, we obtain:

COROLLARY 7.2.8. *The subcategory* $\mathrm{PreStk}_{\mathrm{def}} \subset {}^{\mathrm{conv}}\mathrm{PreStk}$ *is closed under* filtered *colimits, and the same is true for* $\mathrm{PreStk}_{\mathrm{laft\text{-}def}} \subset \mathrm{PreStk}_{\mathrm{laft}}$.

PROOF. Follows from the fact that filtered colimits commute with fiber products. $\square$

7.3. Formal smoothness. In this subsection we discuss the notion of *formal smoothness* of a prestack, and rewrite it for prestacks that admit deformation theory.

7.3.1. Let $\mathcal{X}$ be an object of PreStk. We shall say that $\mathcal{X}$ is *formally smooth*, if whenever $S \to S'$ is a nilpotent embedding of affine schemes, the map

$$\mathrm{Maps}(S', \mathcal{X}) \to \mathrm{Maps}(S, \mathcal{X})$$

is surjective on π_0.

7.3.2. We have the following basic result:

PROPOSITION 7.3.3. *Assume that* $\mathcal{X}$ *admits deformation theory. Then the following conditions are equivalent:*

(a) $\mathcal{X}$ *is formally smooth.*

(b) *For any* $n \geq 0$, *the restriction map*

$$\mathrm{Maps}(S, \mathcal{X}) \to \mathrm{Maps}(^{\leq n}S, \mathcal{X}), \quad S \in \mathrm{Sch}^{\mathrm{aff}}$$

induces an isomorphism on π_n *(equivalently, on* $\pi_{n'}$ *for* $n' \leq n$*).*

(b') *The restriction map*

$$\mathrm{Maps}(S, \mathcal{X}) \to \mathrm{Maps}(^{\mathrm{cl}}S, \mathcal{X}), \quad S \in \mathrm{Sch}^{\mathrm{aff}}$$

induces an isomorphism on π_0.

(c) *For any* $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$ *and* $\mathcal{F} \in \mathrm{QCoh}(S)^{\heartsuit}$, *we have*

$$\mathcal{M}aps(T^*_x(\mathcal{X}), \mathcal{F}) \in \mathrm{Vect}^{\leq 0}.$$

(c') *Same as* (c), *but assuming that* S *is classical.*

PROOF. The implications (b) $\Rightarrow$ (b') and (c) $\Rightarrow$ (c') are tautological.

The implication (a) $\Rightarrow$ (c) is immediate: apply the definition to the nilpotent embedding $S_{\mathcal{F}[i]} \to S$. Similarly, (b') implies (c): use the fact that $^{\mathrm{cl}}S \simeq {}^{\mathrm{cl}}S_{\mathcal{F}[i]}$ for $i > 0$.

The implication (c) $\Rightarrow$ (b) follows from Proposition 5.4.2(b). The implication (c) $\Rightarrow$ (a) follows from Proposition 5.5.3.

The implication (c') $\Rightarrow$ (c) follows from the fact that any object of $\mathrm{QCoh}(S)^{\heartsuit}$ is the direct image under $^{\mathrm{cl}}S \to S$.

$\square$

7.3.4.   Now, assume that $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. In this case we have:

PROPOSITION 7.3.5.   *Then the following conditions are equivalent:*

(i) $\mathcal{X}$ *is formally smooth.*

(i') *The condition of formal smoothness is satisfied for nilpotent embeddings $S \to S'$ with $S, S' \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$.*

(ii) *For any $n \geq 0$, the restriction map*

$$\mathrm{Maps}(S, \mathcal{X}) \to \mathrm{Maps}({}^{\leq n}S, \mathcal{X}), \quad S \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$$

induces an isomorphism on π_n (equivalently, on $\pi_{n'}$ for $n' \leq n$).

(ii') *The restriction map*

$$\mathrm{Maps}(S, \mathcal{X}) \to \mathrm{Maps}({}^{\mathrm{cl}}S, \mathcal{X}), \quad S \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$$

induces an isomorphism on π_0.

(iii) *For any $(S, x) \in ({}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{X}}$ and $\mathcal{F} \in \mathrm{Coh}(S)^{\heartsuit}$, we have*

$$\mathcal{M}aps(T_x^*(\mathcal{X}), \mathcal{F}) \in \mathrm{Vect}^{\leq 0}.$$

(iii') *Same as* (iii), *but assuming that S is reduced.*

PROOF.   The implications (i) $\Rightarrow$ (i'), (ii) $\Rightarrow$ (ii') and (iii) $\Rightarrow$ (iii') are tautological. The implications (i') $\Rightarrow$ (iii), (ii') $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) follow in the same way as in Proposition 7.3.3.

The fact that (iii') implies (iii) follows from the fact that any object in $\mathrm{QCoh}(S)^{\heartsuit}$ is a finite extension of ones coming as direct image under ${}^{\mathrm{red}}S \to S$.

It remains to show that (iii) implies (i). We will show that (iii) implies condition (c') from Proposition 7.3.3. Let (S', x') be an object of $({}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$ and $\mathcal{F}' \in \mathrm{QCoh}(S')^{\heartsuit}$. Since $\mathcal{X}$ is locally almost of finite type, we can factor the map $x' : S' \to \mathcal{X}$ as

$$S' \xrightarrow{f} S \xrightarrow{x} \mathcal{X},$$

where $S \in {}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$. Set $\mathcal{F} := f_*(\mathcal{F}')$. Since $\mathcal{X}$ admits a cotangent complex, we have

$$\mathrm{Maps}(T_{x'}^*(\mathcal{X}), \mathcal{F}'[i]) \simeq \mathrm{Maps}(T_x^*(\mathcal{X}), \mathcal{F}[i]).$$

Write $\mathcal{F}$ as a filtered colimit

$$\operatorname*{colim}_{\alpha} \mathcal{F}_\alpha, \quad \mathcal{F}_\alpha \in \mathrm{Coh}(S)^{\heartsuit}.$$

Now, since $T_x^*(\mathcal{X})$ commutes with filtered colimits in $\mathrm{QCoh}(S)^{\heartsuit}$ (by Lemma 3.5.2), we have:

$$\mathrm{Maps}(T_x^*(\mathcal{X}), \mathcal{F}_\alpha[i]) = 0 \Rightarrow \mathrm{Maps}(T_x^*(\mathcal{X}), \mathcal{F}[i]) = 0,$$

as required.

$\square$

7.3.6.   The definition of formal smoothness and Proposition 7.3.3 can be easily extended to a relative situation.

7.4. Artin stacks. In this subsection we show that Artin stacks, defined as in Volume I, Chapter 3, Sect. 4, admit deformation theory.

7.4.1.   We are going to prove:

PROPOSITION 7.4.2.

(a) *Let $\mathcal{X}$ be an n-Artin stack. Then $\mathcal{X}$ admits an $(-n)$-connective corepresentable deformation theory.*

(b) *If $\mathcal{X}$ is smooth over a scheme Z, then for $x : S \in (^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$, the relative cotangent complex $T_x^*(\mathcal{X}/Z)$ lives in $\mathrm{QCoh}(S)^{\geq 0, \leq n}$.*

Arguing by induction on n, the proposition follows from the next lemma:

LEMMA 7.4.3. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a map in* PreStk. *Assume that:*

- *$\mathcal{X}$ satisfies étale descent;*
- *f is étale-locally surjective;*
- *$\mathcal{Y}$ admits deformation theory;*
- *$\mathcal{Y}$ admits deformation theory relative to $\mathcal{X}$;*
- *$\mathcal{Y}$ is formally smooth over $\mathcal{X}$.*

Then $\mathcal{X}$ admits deformation theory.

7.4.4. *Proof of Lemma 7.4.3.* We will show that if

$$S_1' \underset{S_1}{\sqcup} S_2 \to S_2'$$

is a push-out diagram in $\mathrm{Sch}^{\mathrm{aff}}$, where $S_1 \to S_1'$ has a structure of a square-zero extension, then, given a map $S_2 \to \mathcal{X}$, the map

$$\mathrm{Maps}_{S_2/}(S_2', \mathcal{X}) \to \mathrm{Maps}_{S_1/}(S_1', \mathcal{X})$$

is an isomorphism. The other properties are proved similarly.

By étale descent for $\mathcal{X}$, the statement is local in the étale topology on S_2. Hence, we can assume that the given map $S_2 \to \mathcal{X}$ admits a lift to a map $S_2 \to \mathcal{Y}$.

Let $\mathcal{Y}^\bullet/\mathcal{X}$ be the Čech nerve of f. We have a commutative diagram

$$
\begin{array}{ccc}
|\mathrm{Maps}_{S_2/}(S_2', \mathcal{Y}^\bullet/\mathcal{X})| & \longrightarrow & \mathrm{Maps}_{S_2/}(S_2', \mathcal{X}) \\
\downarrow & & \downarrow \\
|\mathrm{Maps}_{S_1/}(S_1', \mathcal{Y}^\bullet/\mathcal{X})| & \longrightarrow & \mathrm{Maps}_{S_1/}(S_1', \mathcal{X}),
\end{array}
$$

where the horizontal arrows are monomorphisms.

We note that the terms of $\mathcal{Y}^\bullet/\mathcal{X}$ admit deformation theory (by the deformation theory analog of Lemma 2.4.5). Hence, the left vertical arrow is an isomorphism.

Hence, it remains to show that the horizontal arrows are surjective. We claim that this follows from the last requirement on f. We claim that for any square-zero extension

$$S \hookrightarrow S',$$

a map $x' : S' \to \mathcal{X}$ and a lift of the composition

$$x : S \to S' \to \mathcal{X}$$

to a map $y : S \to \mathcal{Y}$, there *exists* a lift of x' to a map $y' : S' \to \mathcal{Y}$.

Indeed, if $S \hookrightarrow S'$ is given by a map $T^*(S) \to \mathcal{F}$, the space of lifts as above is the space of null-homotopies of the resulting map

$$T_y^*(\mathcal{Y}/\mathcal{X}) \to \mathcal{F}.$$

However, the above map admits a null-homotopy since $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq -1}$ and the assumption that $\mathcal{Y} \to \mathcal{X}$ is formally smooth.

$\square$

8. Consequences of admitting deformation theory

In this section we discuss further properties of prestacks that admit deformation theory.

8.1. Digression: properties of maps of prestacks. In this subsection we define several classes of morphisms of prestacks.

8.1.1. Let $^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}$ denote the category of (classical) reduced affine schemes. For a prestack

$$\mathcal{Y} : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc},$$

or an object $\mathcal{Y} \in {}^{\mathrm{cl}}\mathrm{PreStk}$, let $^{\mathrm{red}}\mathcal{Y}$ denote its restriction to $^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}$, which we view as a functor

$$(^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}.$$

We give the following definitions:

DEFINITION 8.1.2. *Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a map in $^{\mathrm{cl}}\mathrm{PreStk}$.*

(a) *We shall say that f is a* closed embedding *if its base change by a classical affine scheme yields a closed embedding. I.e., if for $S_2 \in (^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}_2}$, the fiber product $S_1 := S_2 \underset{\mathcal{X}_2}{\times} \mathcal{X}_1$, taken in $^{\mathrm{cl}}\mathrm{PreStk}$, belongs to $^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}$, and the map $S_1 \to S_2$ is a closed embedding.*

(b) *We shall say that f is a* nil-isomorphism *if it induces an isomorphism $^{\mathrm{red}}\mathcal{X}_1 \to {}^{\mathrm{red}}\mathcal{X}_2$. Equivalently, if for every $S_2 \in (^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}_2}$, the map*

$$^{\mathrm{red}}(S_2 \underset{\mathcal{X}_2}{\times} \mathcal{X}_1) \to S_2$$

(the fiber product is taken in $^{\mathrm{cl}}\mathrm{PreStk}$) is an isomorphism.

(c) *We shall say that f is* nil-closed *if for every $S_2 \in (^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}_2}$, the map*

$$^{\mathrm{red}}(S_2 \underset{\mathcal{X}_2}{\times} \mathcal{X}_1) \to {}^{\mathrm{red}}S_2$$

(the fiber product is taken in $^{\mathrm{cl}}\mathrm{PreStk}$) is a closed embedding.

(d) *We shall say that f is a* nilpotent embedding *if its base change by a classical affine scheme yields a nilpotent embedding. I.e., if in the situation of* (a)*, the map $S_1 \to S_2$ is a nilpotent embedding of classical schemes.*

(d') *We shall say that f is a* pseudo-nilpotent embedding *if it is a nil-isomorphism and for every $S_2 \in (^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}_2}$, there exists a commutative diagram*

$$
\begin{array}{ccc}
S_1 & \longrightarrow & \mathcal{X}_1 \\
\downarrow & & \downarrow \\
S_2 & \longrightarrow & \mathcal{X}_2
\end{array}
$$

with $S_1 \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}$ and $S_1 \to S_2$ a nilpotent embedding.

DEFINITION 8.1.3. *Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a map in* PreStk. *We shall say that f is a closed embedding (resp., nil-isomorphism, nil-closed, nilpotent embedding, pseudo-nilpotent embedding), if the corresponding map $^{\mathrm{cl}}\mathcal{X}_1 \to {}^{\mathrm{cl}}\mathcal{X}_2$ has the corresponding property in the classical setting.*

Clearly:

'closed embedding' $\Rightarrow$ 'nil-closed';

'nilpotent embedding' $\Rightarrow$ 'closed embedding';

'nilpotent embedding' $\Rightarrow$ 'nil-isomorphism' and 'pseudo-nilpotent embedding'.

'pseudo-nilpotent embedding' $\Rightarrow$ 'nil-isomorphism'.

8.1.4.   The condition of being a pseudo-nilpotent embedding may appear a little obscure, but it turns out to be useful. We note, however, that due to the next proposition, the difference between 'nil-isomorphism' and 'pseudo-nilpotent embedding' only exists when our stacks are not locally of finite type:

LEMMA 8.1.5. *Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a nil-isomorphism in $^{\mathrm{cl}}$PreStk. Assume that $\mathcal{X}_2 \in {}^{\mathrm{cl}}\mathrm{PreStk}_{\mathrm{lft}}$. Then f is a pseudo-nilpotent embedding.*

PROOF.   Let $S_2 \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}$, and let $S_2 \to \mathcal{X}_2$ be a map. We need to find an object in the category of diagrams

$$
\begin{array}{ccc}
S_1 & \longrightarrow & \mathcal{X}_1 \\
\downarrow & & \downarrow \\
S_2 & \longrightarrow & \mathcal{X}_2,
\end{array}
$$

where $S_1 \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}$ and $S_1 \to S_2$ is a nilpotent embedding.

By the assumption on $\mathcal{X}_2$, we can assume that $S_2 \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. In this case the required data is supplied by taking $S_1 \coloneqq {}^{\mathrm{red}}S_2$. $\square$

8.2. Descent properties. In this subsection we will show that one can deduce Zariski, Nisnevich or étale descent property of a prestack from the corresponding property at the classical level.

8.2.1.   We will prove:

PROPOSITION 8.2.2. *Let $\mathcal{X} \in \mathrm{PreStk}$ admit defomation theory, and let $\mathcal{X}_{0,\mathrm{cl}} \to {}^{\mathrm{cl}}\mathcal{X}$ be a pseudo-nilpotent embedding of classical prestacks.*

(a) *Assume that $\mathcal{X}_{0,\mathrm{cl}}$ satisfies Zariski (resp., Nisnevich) descent. Then $\mathcal{X}$ also has this property.*

(b) *Assume that $\mathcal{X}_{0,\mathrm{cl}}$ satisfies étale descent. Assume also that the pro-cotangent spaces of $\mathcal{X}$ are locally eventually connective. Then $\mathcal{X}$ also satisfies étale descent.*

8.2.3. *Proof of Proposition 8.2.2.*   By convergence and Proposition 5.4.2, it is enough to show that if

$$S \hookrightarrow S'$$

is a map of affine schemes that has a structure square-zero extension, $x : S \to \mathcal{X}$ is a map and $\pi : \overset{\circ}{S} \to S$ is a Zariski (resp., Nisnevich, étale) cover, then the map

$$\mathrm{Maps}_{S/}(S', \mathcal{X}) \to \mathrm{Tot}(\mathrm{Maps}_{\overset{\circ}{S}_\bullet/}(\overset{\circ}{S}{}'^\bullet, \mathcal{X}))$$

is an isomorphism, where $\pi' : \overset{\circ}{S}{}' \to S'$ is the corresponding cover, and $\overset{\circ}{S}{}^\bullet$ (resp., $\overset{\circ}{S}{}'^\bullet$) is the Čech nerve of π, (resp., π').

We rewrite $\mathrm{Maps}_{S/}(S', \mathcal{X})$ and each $\mathrm{Maps}_{\overset{\circ}{S}_\bullet/}(\overset{\circ}{S}{}'^\bullet, \mathcal{X}))$ as in (6.2). So, $\mathrm{Maps}_{S/}(S', \mathcal{X})$ identifies with the space of null-homotopies of a certain map

$$T^*_x(\mathcal{X}) \to \mathcal{F}, \quad \mathcal{F} \in \mathrm{QCoh}(S)^{>-\infty}$$

and $\mathrm{Tot}(\mathrm{Maps}_{\overset{\circ}{S}_\bullet/}(\overset{\circ}{S}{}'^\bullet, \mathcal{X}))$ identifies with the totalization of the cosimplicial space of null-homotopies of the corresponding maps

$$T^*_x(\mathcal{X}) \to \mathcal{F}^\bullet,$$

where $\mathcal{F}^\bullet$ is the Čech resolution of $\mathcal{F}$ corresponding to π.

Note, however, that in the case of Zariski and Nisnevich covers, one can replace the totalization by a limit over a finite category. Now, the required isomorphism follows from the commutation of $\mathcal{M}aps(T^*_x(\mathcal{X}), -)$ with finite limits.

For an étale cover, if $T^*_x(\mathcal{X})$ belongs to $\mathrm{Pro}(\mathrm{QCoh}(S)^{\le n})$ and $\mathcal{F} \in \mathrm{QCoh}(S)^{\ge -k}$, we can replace the totalization by the limit over the $(n+k)$-skeleton. Hence, the required isomorphism again follows from the commutation of $\mathcal{M}aps(T^*_x(\mathcal{X}), -)$ with finite limits.

$\square$

REMARK 8.2.4. A recent result of Akhil Mathew shows that étale descent in Proposition 8.2.2 holds without the assumption of eventual connectivity.

8.3. Isomorphism properties. The property of having deformation theory can be used to show that certain maps between prestacks are isomorphisms.

8.3.1.   We will prove:

PROPOSITION 8.3.2. *Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a map between objects of $\mathrm{PreStk}_{\mathrm{def}}$. Suppose that there exists a commutative diagram*

$$
\begin{array}{ccc}
 & \mathcal{X}_{0,\mathrm{cl}} & \\
{}^{g_1}\swarrow & & \searrow{}^{g_2} \\
\mathcal{X}_1 & \xrightarrow{\quad f \quad} & \mathcal{X}_2,
\end{array}
$$

where g_1 and g_2 are pseudo-nilpotent embeddings, and $\mathcal{X}_{0,\mathrm{cl}} \in {}^{\mathrm{cl}}\mathrm{PreStk}$. Suppose also that for any $S \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}$ and a map $x_0 : S \to \mathcal{X}_{0,\mathrm{cl}}$, for $x_i := g_i \circ x_0$, the induced map

$$T^*_{x_2}(\mathcal{X}_2) \to T^*_{x_1}(\mathcal{X}_1)$$

is an isomorphism. Then f is an isomorphism.

PROOF. By induction and Proposition 5.4.2, we have to show that given $S \in \mathrm{Sch}^{\mathrm{aff}}$ and a map $S \hookrightarrow S'$ that has a structure of square-zero extension, for a map $x_1 : S \to \mathcal{X}_1$, the space of extensions of x_1 to a map $S' \to \mathcal{X}_1$ maps isomorphically to the space of extensions of $x_2 := f \circ x_1$ to a map $S' \to \mathcal{X}_2$.

Deformation theory implies that the spaces in question are the spaces of null-homotopies of the corresponding maps

$$T^*_{x_1}(\mathcal{X}_1) \to \mathcal{F} \text{ and } T^*_{x_2}(\mathcal{X}_2) \to \mathcal{F},$$

respectively. Hence, it is enough to show that the map $T^*_{x_2}(\mathcal{X}_2) \to T^*_{x_1}(\mathcal{X}_1)$ is an isomorphism in $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$.

The assumption of the proposition implies that there exists a nilpotent embedding $g : \widetilde{S} \to S$, such that for $\widetilde{x}_i = x_i \circ g$, the map

$$T^*_{\widetilde{x}_2}(\mathcal{X}_2) \to T^*_{\widetilde{x}_1}(\mathcal{X}_1)$$

is an isomorphism in $\mathrm{Pro}(\mathrm{QCoh}(\widetilde{S})^-)$. Therefore, it suffices to prove the following:

LEMMA 8.3.3. *For a nilpotent embedding* $g : \widetilde{S} \to S$, *the functor*

$$\mathrm{Pro}(g^*) : \mathrm{Pro}(\mathrm{QCoh}(S)^-) \to \mathrm{Pro}(\mathrm{QCoh}(\widetilde{S})^-)$$

is conservative when restricted to $^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^-)$.

$\square$

8.3.4. *Proof of Lemma 8.3.3.* First, we claim that if $\widetilde{S} \to S$ is a square-zero extension, then the functor

$$\mathrm{Pro}(g^*) : \mathrm{Pro}(\mathrm{QCoh}(S)^-) \to \mathrm{Pro}(\mathrm{QCoh}(\widetilde{S})^-)$$

is conservative on all of $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$.

Indeed, we need to show that if $\mathcal{T} \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)$ is such that $\mathrm{Maps}(\mathcal{T}, g_*(\widetilde{\mathcal{F}})) = 0$ for all $\widetilde{\mathcal{F}} \in \mathrm{QCoh}(\widetilde{S})^-$, then $\mathrm{Maps}(\mathcal{T}, \mathcal{F}) = 0$ for all $\mathcal{F} \in \mathrm{QCoh}(S)^-$. However, this is obvious, since every object of $\mathrm{QCoh}(S)^-$ is a two-step extension of objects in the essential image of g_*.

Hence, the functor $\mathrm{Pro}(g^*)$ is conservative if $\widetilde{S} \to S$ can be written as a finite succession of square-zero extensions.

Using Proposition 5.5.3, we can construct a sequence of schemes

$$\widetilde{S} \to S_0 \to S_1 \to ... \to S_k \to ... \to S,$$

such that for every k, the map $\widetilde{S} \to S_k$ is a finite succession of square-zero extensions and the map $g_k : S_k \to S$ induces an isomorphism $^{\leq k}S_k \to {}^{\leq k}S$.

Let $\mathcal{T} \in^{\mathrm{conv}} \mathrm{Pro}(\mathrm{QCoh}(S)^-)$ be in the kernel of $\mathrm{Pro}(g^*)$. By the above, it is then in the kernel of each $\mathrm{Pro}(g_k^*)$. Note that for $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq n}$, the map

$$\mathcal{F} \to (g_k)_* \circ (g_k)^*(\mathcal{F})$$

induces an isomorphism

$$\tau^{\geq n-k}(\mathcal{F}) \to \tau^{\geq n-k}((g_k)_* \circ (g_k)^*(\mathcal{F})).$$

Hence, by convergence, for $\mathcal{F} \in \mathrm{QCoh}(S)^-$,

$$\mathrm{Maps}(\mathcal{T}, \mathcal{F}) \simeq \lim_k \mathrm{Maps}(\mathcal{T}, (g_k)_* \circ (g_k)^*(\mathcal{F})),$$

while each $\mathrm{Maps}(\mathcal{T}, (g_k)_* \circ (g_k)^*(\mathcal{F}))$ vanishes.

$\square$

8.3.5.   From Proposition 8.3.2 we obtain:

COROLLARY 8.3.6. *Let*

$$
\begin{array}{ccc}
\mathcal{X}_1 & \xrightarrow{\ g_1\ } & \mathcal{Y}_1 \\
{\scriptstyle f_X}\downarrow & & \downarrow{\scriptstyle f_Y} \\
\mathcal{X}_2 & \xrightarrow{\ g_2\ } & \mathcal{Y}_2
\end{array}
$$

be a Cartesian square of objects of $\mathrm{PreStk}_{\mathrm{def}}$*, such that the horizontal maps are pseudo-nilpotent embeddings. Suppose that* f_X *is an isomorphism. Then* f_Y *is an isomorphism.*

9. A criterion for being locally almost of finite type

Deformation theory can be used to show that a prestack is locally almost of finite type, see Theorem 9.1.2 below.

9.1. Statement of the result. In this subsection we state Theorem 9.1.2 and make some initial observations.

9.1.1.   The goal of this section is to prove the following:

THEOREM 9.1.2. *Let* $\mathcal{X}$ *be an object of* $\mathrm{PreStk}_{\mathrm{def}}$*. Suppose that there exists a nilpotent embedding* $\mathcal{X}_0 \to {}^{\mathrm{cl}}\mathcal{X}$*, such that:*

- $\mathcal{X}_0 \in {}^{\mathrm{cl}}\mathrm{PreStk}_{\mathrm{lft}}$*;*
- *For any* $S \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ *and* $x : S \to \mathcal{X}_0$*, we have* $T^*_x(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}}$*.*

Then $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$*.*

As an immediate corollary, we obtain:

COROLLARY 9.1.3. *Let* $\mathcal{X}$ *be an object of* $\mathrm{PreStk}_{\mathrm{def}}$*. Suppose that* ${}^{\mathrm{cl}}\mathcal{X} \in {}^{\mathrm{cl}}\mathrm{PreStk}_{\mathrm{lft}}$*, and that for any* $S \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ *and* $x : S \to \mathcal{X}$*, we have*

$$ T^*_x(\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}}. $$

Then $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$*.*

In addition, we will prove:

THEOREM 9.1.4. *Let* $\mathcal{X}$ *be an object of* $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$*. Then the fully faithful embedding functor*

$$ (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{X}} \to (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}} $$

is cofinal.

REMARK 9.1.5. The assertion of Theorem 9.1.4 would be a tautology from the definition of $\mathrm{PreStk}_{\mathrm{laft}}$ if instead of $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}} \subset \mathrm{Sch}^{\mathrm{aff}}$ we used ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \subset {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$.

REMARK 9.1.6. We note that the proof of Theorem 9.1.4 given in Sect. 9.6 will show that a prestack, satisfying the assumption of Corollary 9.1.3 satisfies the conclusion of Theorem 9.1.4. So, one can use the proof of Theorem 9.1.4 as an alternative (and quicker) way to prove Corollary 9.1.3.

REMARK 9.1.7. The proof of Theorem 9.1.4 shows that for a not necessarily affine (but quasi-compact) sscheme Z equipped with a map to $\mathcal{X}$, the category of factorizations of this map as

$$Z \to Z' \to \mathcal{X}, \quad Z' \in \mathrm{Sch}_{\mathrm{aft}}$$

is contractible (in fact, the opposite category is filtered). Moreover, cofinal in this category is the subcategory consisting of those objects for which the map $Z \to Z'$ is affine.

9.1.8. From now until Sect. 9.6, we will be concerned with the proof of Theorem 9.1.2. We begin with the following observation:

Let $\mathcal{X}$ be any prestack, and assume that it is convergent. The condition that $\mathcal{X}$ belongs to $\mathrm{PreStk}_{\mathrm{laft}}$ says that given $n \geq 0$ and an object $(S, x) \in (^{\leq n}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$, the category, denoted $\mathrm{Factor}(x, \mathrm{ft}, \leq n)$, of factorizations of x as

$$S \to U \to \mathcal{X}, \quad U \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$$

is contractible.

Consider also the categories $\mathrm{Factor}(x, \mathrm{ft}, < \infty)$, $\mathrm{Factor}(x, \mathrm{aft})$ of factorizations of x as

$$S \to U \to \mathcal{X},$$

where we instead require that U belong to $^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ and $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, respectively.

We have the fully faithful functors

$$\mathrm{Factor}(x, \mathrm{ft}, \leq n) \hookrightarrow \mathrm{Factor}(x, \mathrm{ft}, < \infty) \hookrightarrow \mathrm{Factor}(x, \mathrm{aft}),$$

and the map of $\mathrm{Factor}(x, \mathrm{ft}, \leq n)$ into both $\mathrm{Factor}(x, \mathrm{ft}, < \infty)$ and $\mathrm{Factor}(x, \mathrm{aft})$ admits a right adjoint, given by $S_0 \mapsto {}^{\leq n}S_0$.

Hence, $\mathrm{Factor}(x, \mathrm{ft}, \leq n)$ is contractible if and only if $\mathrm{Factor}(x, \mathrm{ft}, < \infty)$ is contractible and if and only if $\mathrm{Factor}(x, \mathrm{aft})$ is.

9.2. Step 1.
9.2.1. Suppose we have an object of $(S, x) \in (^{\leq n}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$. We need to show that the category $\mathrm{Factor}(x, \mathrm{ft}, \leq n)$ is contractible.

Set $S_0 := {}^{\mathrm{cl}}(S \underset{\mathcal{X}}{\times} \mathcal{X}_0)$. Let x_0 denote the resulting map $S_0 \to \mathcal{X}_0$. By assumption, the category $\mathrm{Factor}(x_0, \mathrm{ft}, \mathrm{cl})$ is contractible.

We introduce the category $\mathbf{C}$ to be that of diagrams

$$
\begin{array}{ccccc}
S_0 & \longrightarrow & U_0 & \longrightarrow & \mathcal{X}_0 \\
\downarrow & & \downarrow & & \downarrow \\
S & \longrightarrow & U & \longrightarrow & \mathcal{X},
\end{array}
$$

where $U \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ and $U_0 \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, and $U_0 \to U$ is an arbitrary map.

We have the natural forgetful functors

$$\mathrm{Factor}(x, \mathrm{ft}, \leq n) \leftarrow \mathbf{C} \to \mathrm{Factor}(x_0, \mathrm{ft}, \mathrm{cl}).$$

We will show that both these functors are homotopy equivalences. This would imply that $\mathrm{Factor}(x, \mathrm{ft}, \leq n)$ is contractible.

9.2.2.   The functor $\mathbf{C} \to \mathrm{Factor}(x, \mathrm{ft}, \leq n)$ is a co-Cartesian fibration. Hence, in order to prove that it is a homotopy equivalence, it suffices to show that it has contractible fibers.

However, the fiber in question over a given $(S \to U \to \mathcal{X}) \in \mathrm{Factor}(x, \mathrm{ft}, \leq n)$ has a final object, namely, one with

$$U_0 := {}^{\mathrm{cl}}(U \underset{\mathcal{X}}{\times} \mathcal{X}_0).$$

9.2.3.   The functor $\mathbf{C} \to \mathrm{Factor}(x_0, \mathrm{ft}, \mathrm{cl})$ is a Cartesian fibration. Hence, in order to prove that it is a homotopy equivalence, it suffices to show that it has contractible fibers.

We note that the fiber of the above functor over a given $(S_0 \to U_0 \to \mathcal{X}_0) \in \mathrm{Factor}(x_0, \mathrm{ft}, \mathrm{cl})$ can be described as follows.

Set $\widetilde{S} := S \underset{S_0}{\sqcup} U_0$. Since $\mathcal{X}$ admits deformation theory, we have a canonical map $\widetilde{x} : \widetilde{S} \to \mathcal{X}$.

The fiber in question is the category $\mathrm{Factor}(\widetilde{x}, \mathrm{ft}, \leq n)$ of factorizations of $\widetilde{x}$ as

$$\widetilde{S} \to U \to \mathcal{X}, \quad U \in {}^{\leq n}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}.$$

9.3. Resetting the problem.

9.3.1.   By Step 1, it suffices to prove the contractibility of the category $\mathrm{Factor}(x, \mathrm{ft}, \leq n)$ under the additional assumption that there exists a nilpotent embedding

$$S_0 \to S,$$

where $S_0 \in {}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$.

By Proposition 5.4.2, there exists a finite sequence of affine schemes

$$S_0 \to S_1 \to \ldots \to S_{k-1} \to S_k = S, \quad S_i \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}},$$

such that for every i, the map $S_i \hookrightarrow S_{i+1}$ has a structure of square-zero extension.

9.3.2.   Repeating the manipulation of Step 1, by induction, we obtain that it suffices to prove the following: let S be an object of ${}^{\leq n}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$, and let $S \hookrightarrow S'$ be a square-zero extension, where $S' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$.

Suppose we have a map $x : S' \to \mathcal{X}$. We need to show that the category $\mathrm{Factor}(x, \mathrm{ft}, \leq n)$ is contractible.

9.4. Step 2. Let $S \hookrightarrow S'$ be as in Sect. 9.3.2.

9.4.1.   Let $\mathrm{Factor}(x, \mathrm{ft}, \leq n)_{\mathrm{SqZ}}$ be the category of factorizations of the map $x : S' \to \mathcal{X}$ as

$$S' \to \widetilde{S}' \to \mathcal{X},$$

where the composition $S \hookrightarrow \widetilde{S}'$ is equipped with a structure of square-zero extension, $S' \to \widetilde{S}'$ is equipped with a structure of map in $\mathrm{SqZ}(S)$, and where $\widetilde{S}' \in {}^{\leq n}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$.

Consider also the category $\mathbf{D}$ of factorizations of the map $x : S' \to \mathcal{X}$ as

$$S' \to \widetilde{S}' \to U \to \mathcal{X},$$

where the composition $S \to \widetilde{S}'$ is given a structure of square-zero extension, $S' \to \widetilde{S}'$ is given a structure of map in $\mathrm{SqZ}(S)$, and where $\widetilde{S}', U \in {}^{\leq n}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$.

We have the fogetful functors

$$\mathrm{Factor}(x, \mathrm{ft}, \le n)_{\mathrm{SqZ}} \leftarrow \mathbf{D} \to \mathrm{Factor}(x, \mathrm{ft}, \le n).$$

We will show that both these functors are homotopy equivalences, whereas the category $\mathrm{Factor}(x, \mathrm{ft}, \le n)_{\mathrm{SqZ}}$ is contractible. This will imply that $\mathrm{Factor}(x, \mathrm{ft}, \le n)$ is contractible.

9.4.2.   We note that the functor

$$\mathbf{D} \to \mathrm{Factor}(x, \mathrm{ft}, \le n)_{\mathrm{SqZ}}$$

is a Cartesian fibration. Hence, in order to prove that it is a homotopy equivalence, it suffices to show that it has contractible fibers.

However, the fiber in question over a given $(S' \to \widetilde{S}' \to \mathcal{X}) \in \mathrm{Factor}(x, \mathrm{ft}, \le n)_{\mathrm{SqZ}}$ has an initial point, namely, one with $U = \widetilde{S}'$.

9.4.3.   The functor

$$\mathbf{D} \to \mathrm{Factor}(x, \mathrm{ft}, \le n)$$

is a co-Cartesian fibration. Hence, in order to show that it is a homotopy equivalence, it suffices to show that it has contractible fibers.

However, we note that the fiber of $\mathbf{D}$ over an object $(S' \to U \to \mathcal{X}) \in \mathrm{Factor}(x, \mathrm{ft}, \le n)$ is the category $\mathrm{Factor}(u, \mathrm{ft}, \le n)_{\mathrm{SqZ}}$, where u denotes the map $S' \to U$. I.e., this is a category of the same nature as $\mathrm{Factor}(x, \mathrm{ft}, \le n)_{\mathrm{SqZ}}$, but with $\mathcal{X}$ replaced by U.

Thus, it remains to prove the contractibilty of the category $\mathrm{Factor}(x, \mathrm{ft}, \le n)_{\mathrm{SqZ}}$.

9.5. Step 3.
9.5.1.   Let the square-zero extension $(S \hookrightarrow S')$ be given by

$$T^*(S/\mathcal{X}) \xrightarrow{\gamma} \mathcal{F}, \quad \mathcal{F} \in \mathrm{QCoh}(S)^{\ge -n-1, \le -1}.$$

The category $\mathrm{Factor}(x', \mathrm{ft}, \le n)_{\mathrm{SqZ}}$ is that of factorizations of γ as

$$T^*(S/\mathcal{X}) \xrightarrow{\widetilde{\gamma}} \widetilde{\mathcal{F}} \to \mathcal{F},$$

where $\widetilde{\mathcal{F}} \in \mathrm{Coh}(S)^{\ge -n-1, \le -1}$.

9.5.2.   Note that $\mathcal{F}$ is isomorphic to the filtered colimit

$$\underset{\widetilde{\mathcal{F}} \in (\mathrm{Coh}(S)^{\ge -n-1, \le -1})_{/\mathcal{F}}}{\mathrm{colim}} \widetilde{\mathcal{F}}.$$

Hence, in order to prove that $\mathrm{Factor}(x, \mathrm{ft}, \le n)_{\mathrm{SqZ}}$ is contractible, it suffices to show that the functor

$$\mathcal{M}aps(T^*(S/\mathcal{X}), -) : \mathrm{QCoh}(S)^{\ge -n-1, \le -1} \to \mathrm{Vect}$$

commutes with filtered colimits.

9.5.3. We have
$$T^*(S/\mathcal{X}) \simeq \mathrm{coFib}(T^*(S) \to T^*(\mathcal{X})|_S).$$

Since $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, it suffices to show that
$$T^*(\mathcal{X})|_S \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}}.$$

This follows from the assumption on $\mathcal{X}$ and the next lemma:

LEMMA 9.5.4. *If $i : S_0 \to S$ is a nilpotent embedding of objects of $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, and $\mathcal{T}$ is an object of $^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^-)$ is such that*
$$(\mathrm{Pro}(i^*))(\mathcal{T}) \in \mathrm{Pro}(\mathrm{QCoh}(S_0)^-)_{\mathrm{laft}},$$
then $\mathcal{T} \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}}$.

9.5.5. *Proof of Lemma 9.5.4.* We need to show that the functor
$$\mathrm{Maps}(\mathcal{T}, -) : \mathrm{QCoh}(S)^\heartsuit[n] \to \mathrm{Spc}$$
commutes with filtered colimits for any n.

This allows to replace S and S_0 by $^{\mathrm{cl}}S$ and $^{\mathrm{cl}}S_0$, respectively. I.e, we can assume that S and S_0 are classical. Furthermore, by induction, we can assume that S is a classical square-zero extension of S_0. Now the required assertion follows from the fact that any $\mathcal{F} \in \mathrm{QCoh}(S)^\heartsuit$ can be written as an extension
$$0 \to i_*(\mathcal{F}') \to \mathcal{F} \to i_*(\mathcal{F}'') \to 0,$$
where $\mathcal{F}', \mathcal{F}'' \in \mathrm{QCoh}(S_0)^\heartsuit$ depend functorially on $\mathcal{F}$ (in fact, $\mathcal{F}'' := H^0(i^*(\mathcal{F})))$.
$\square$

9.6. Proof of Theorem 9.1.4.

9.6.1. Suppose we have an object $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$. We need to show that the category $\mathrm{Factor}(x, \mathrm{aft})$ (see Sect. 9.1.8) of factorizations
$$S \to U \to \mathcal{X}, \quad U \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{X}}$$
is contractible.

For every $n \geq 0$, consider the corresponding category $\mathrm{Factor}(x|_{\leq^n S}, \mathrm{ft}, \leq n)$ of factorizations
$$^{\leq n}S \to U_n \to \mathcal{X}, \quad U_n \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}.$$

We note that since $\mathcal{X}$ is convergent, we have
$$\mathrm{Factor}(x, \mathrm{aft}) \simeq \lim_n \mathrm{Factor}(x|_{\leq^n S}, \mathrm{ft}, \leq n).$$

We will use the following observation:

LEMMA 9.6.2. *Let*
$$\mathbf{C}_0 \leftarrow \mathbf{C}_1 \leftarrow \mathbf{C}_2 \leftarrow \dots$$
be a sequence of $(\infty, 1)$-categories. Assume that:
(i) *The category $\mathbf{C}_0$ is filtered.*
(ii) *For every n, the category $\mathbf{C}_{n+1}$ is filtered relative to $\mathbf{C}_n$.*

Then the category
$$\mathbf{C} := \lim_n \mathbf{C}_n$$
is also filtered.

Let us recall that given a functor $\mathbf{C}' \to \mathbf{D}$, we say that $\mathbf{C}'$ is filtered *relative* to $\mathbf{D}$ if for every finite $(\infty, 1)$-category $\mathbf{K}$ and every diagram

$$
\begin{array}{ccc}
\mathbf{K} & \longrightarrow & \mathbf{C}' \\
\downarrow & & \downarrow \\
\mathrm{Cone}(\mathbf{K}) & \longrightarrow & \mathbf{D}
\end{array}
$$

has a lifting property. Here $\mathrm{Cone}(\mathbf{K})$ is obtained from $\mathbf{K}$ by adjoining to it a final object. (For $\mathbf{D} = *$, we obtain the usual notion of $\mathbf{C}'$ being filtered.)

We apply the above lemma to

$$
\mathbf{C}_n := (\mathrm{Factor}(x|_{\leq n_S}, \mathrm{ft}, \leq n))^{\mathrm{op}}.
$$

9.6.3. To prove that the category $(\mathrm{Factor}(x|_{\mathrm{cl}_S}, \mathrm{ft}, \mathrm{cl}))^{\mathrm{op}}$ is filtered we use the following lemma:

LEMMA 9.6.4. *Let* $\mathbf{C}' \to \mathbf{D}$ *be a co-Cartesian fibration in groupoids. Suppose that* $\mathbf{D}$ *is filtered and* $\mathbf{C}'$ *is contractible. Then* $\mathbf{C}'$ *is also filtered.*

PROOF. Let $\mathbf{C}' \to \mathbf{D}$ correspond to a functor $F : \mathbf{D} \to \mathrm{Spc}$. Then the assumption that $\mathbf{C}'$ is contractible means that

$$
\operatorname*{colim}_{\mathbf{D}} F \simeq *.
$$

This is easily seen to imply the assertion of the lemma.

$\square$

We apply Lemma 9.6.4 to the functor

$$
\mathrm{Factor}(x|_{\mathrm{cl}_S}, \mathrm{ft}, \mathrm{cl}) \to (^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{cl}_S/}.
$$

Indeed, the category $\mathrm{Factor}(x|_{\mathrm{cl}_S}, \mathrm{ft}, \mathrm{cl})$ is contractible because $^{\mathrm{cl}}\mathcal{X}$ belongs to $^{\mathrm{cl}}\mathrm{PreStk}_{\mathrm{lft}}$. The category (opposite) to $(^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})_{/\mathrm{cl}_S}$ is filtered by Volume I, Chapter 2, Theorem 1.5.3(b).

9.6.5. Let us now show that $(\mathrm{Factor}(x|_{\leq n+1_S}, \mathrm{ft}, \leq n+1))^{\mathrm{op}}$ is filtered relative with respect to its projection $(\mathrm{Factor}(x|_{\leq n_S}, \mathrm{ft}, \leq n))^{\mathrm{op}}$.

Suppose we have a functor

$$
F_{n+1} : \mathbf{K} \to (\mathrm{Factor}(x|_{\leq n+1_S}, \mathrm{ft}, \leq n+1))^{\mathrm{op}}
$$

and its extension to a functor

$$
\overline{F}_n : \mathrm{Cone}(\mathbf{K}) \to (\mathrm{Factor}(x|_{\leq n_S}, \mathrm{ft}, \leq n))^{\mathrm{op}}.
$$

Let us denote by

$$
^{\leq n}S \to U_n \to \mathcal{X}
$$

the object of $\mathrm{Factor}(x|_{\leq n_S}, \mathrm{ft}, \leq n)$ corresponding to the value of $\overline{F}_n$ on the final object $* \in \mathrm{Cone}(\mathbf{K})$. For $\mathbf{k} \in \mathbf{K}$ denote also

$$
U_{n+1}^{\mathbf{k}} = F_{n+1}(\mathbf{k}) \text{ and } U_n^{\mathbf{k}} = F_n(\mathbf{k}).
$$

Set

$$
U_{n+1}' := {}^{\leq n+1}S \underset{\leq n_S}{\sqcup} U_n.
$$

We have $^{\leq n}U'_{n+1} = U_n$ and a **K**-diagram of maps

(9.1) $$U'_{n+1} \to U^{\mathbf{k}}_{n+1} \to \mathcal{X}.$$

We need to show that there exists $U_{n+1} \in {}^{\leq n+1}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ equipped with a map

$$U'_{n+1} \to U_{n+1}$$

that induces an isomorphism on n-truncations, such that the **K**-diagram (9.1) extends to a diagram

(9.2) $$U'_{n+1} \to U_{n+1} \to U^{\mathbf{k}}_{n+1} \to \mathcal{X}$$

that induces at the level of n-truncations the diagram

$$U_n = U_n \to U^{\mathbf{k}}_n \to \mathcal{X}$$

9.6.6. Let $f_{\mathbf{k}}$ denote the map $^{\leq n}S \to U^{\mathbf{k}}_n$. By Proposition 5.4.2(b), the map $U^{\mathbf{k}}_n \to U^{\mathbf{k}}_{n+1}$ has a canonical structure of square-zero extension by means of some $\mathcal{J}^{\mathbf{k}} \in \mathrm{QCoh}(U^{\mathbf{k}}_n)^{\heartsuit}[n+1]$. Similarly, $U_n \to U'_{n+1}$ has a canonical structure of square-zero extension by means of some $\mathcal{I}' \in \mathrm{QCoh}(U_n)^{\heartsuit}[n+1]$.

Then the datum of the diagram (9.1) is equivalent to that of the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Pro}(f^*_{\mathbf{k}})(T^*(U^{\mathbf{k}}_n/\mathcal{X})) & \longrightarrow & T^*(U_n/\mathcal{X}) \\
\downarrow & & \downarrow \\
f^*_{\mathbf{k}}(\mathcal{J}^{\mathbf{k}}) & \longrightarrow & \mathcal{I}'
\end{array}
$$

and its extension to a diagram (9.2) is equivalent to factoring the above commutative diagram as

$$
\begin{array}{ccc}
\mathrm{Pro}(f^*_{\mathbf{k}})(T^*(U^{\mathbf{k}}_n/\mathcal{X})) & \longrightarrow & T^*(U_n/\mathcal{X}) \\
\downarrow & & \downarrow \\
f^*_{\mathbf{k}}(\mathcal{J}^{\mathbf{k}}) & \longrightarrow \quad \mathcal{I} \quad \longrightarrow & \mathcal{I}',
\end{array}
$$

where $\mathcal{I} \in \mathrm{Coh}(U_n)^{\heartsuit}[n+1]$.

The existence of such an extension follows from the combination of the following facts:

(i) The category **K** is finite;

(ii) The objects $\mathrm{Pro}(f^*_{\mathbf{k}})(T^*(U^{\mathbf{k}}_n/\mathcal{X}))$ belong to $\mathrm{Pro}(S^-)_{\mathrm{laft}}$;

(iii) $\mathcal{I}'$ can be written as filtered colimit of $\mathcal{I}$ with $\mathcal{I} \in \mathrm{Coh}(U_n)^{\heartsuit}[n+1]$.

10. Square-zero extensions of prestacks

This section is auxilliary (it will be needed in Chapter 6, Sect. 2.5), and can be skipped on first pass. We define and (attempt to) classify square-zero extensions of a given prestack $\mathcal{X}$ by an object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})^{\leq 0}$.

10.1. The notion of square-zero extension of a prestack. We define the notion of square-zero extension of a prestack via pullback to affine schemes.

10.1.1.   Let $\mathcal{X}$ be a prestack and let $\mathcal{I}$ be an object of $\mathrm{QCoh}(\mathcal{X})^{\leq 0}$ (i.e., $\mathcal{I}$ is an object of $\mathrm{QCoh}(\mathcal{X})$, whose pullback to every affine scheme is connective.)

We define the notion of *square-zero extension of $\mathcal{X}$ by means of $\mathcal{I}$* to be the datum of a schematic affine map of prestacks $\mathcal{X} \hookrightarrow \mathcal{X}'$, and an assignment for every $(S', x') \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}'}$ of a structure on the map

$$S' \underset{\mathcal{X}'}{\times} \mathcal{X} =: S \hookrightarrow S'$$

of square-zero extension of S by means of $x^*(\mathcal{I})$ (where x is the resulting map $S \to \mathcal{X}$), which is functorial in (S', x') in the sense of Proposition 5.3.2.

Square-zero extensions of $\mathcal{X}$ by means of $\mathcal{I}$ form a space that we denote by $\mathrm{SqZ}(\mathcal{X}, \mathcal{I})$.

10.1.2.   Let

$$(\mathrm{PreStk}, \mathrm{QCoh}^{\leq 0}) \to \mathrm{PreStk}$$

denote the Cartesian fibration corresponding to the functor

$$(\mathrm{QCoh}^{\leq 0})^*_{\mathrm{PreStk}} : (\mathrm{PreStk})^{\mathrm{op}} \to 1\text{-}\mathrm{Cat}\,.$$

The construction of Proposition 5.3.2 defines a Cartesian fibration in spaces

$$\mathrm{SqZ}(\mathrm{PreStk}) \to (\mathrm{PreStk}, \mathrm{QCoh}^{\leq 0}),$$

whose fiber over a given $(\mathcal{X}, \mathcal{I}) \in (\mathrm{PreStk}, \mathrm{QCoh}^{\leq 0})$ is $\mathrm{SqZ}(\mathcal{X}, \mathcal{I})$.

In particular, given an index category I, and an I-family

(10.1) $i \mapsto (\mathcal{X}_i, \mathcal{I}_i), \quad I \to (\mathrm{PreStk}, \mathrm{QCoh}^{\leq 0}),$

we have a well-defined notion of an I-family of maps $\mathcal{X}_i \hookrightarrow \mathcal{X}'_i$, equipped with a structure of square-zero extension by means of $\mathcal{I}_i$, covering (10.1).

10.2. From square-zero extensions to maps in QCoh**.** In this subsection we will assume that $\mathcal{X}$ admits a pro-cotangent complex. We will show that a square-zero extension of $\mathcal{X}$ gives rise to a map in $\mathrm{QCoh}(\mathcal{X})$.

10.2.1.   We claim that there is a natural map of spaces

(10.2) $\mathrm{SqZ}(\mathcal{X}, \mathcal{I}) \to \mathrm{Maps}(T^*(\mathcal{X}), \mathcal{I}[1]),$

where we regard $T^*(\mathcal{X})$ and $\mathcal{I}$ as objects of $\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}$, see Sect. 4.3.1.

To construct (10.2), given a map $\mathcal{X} \hookrightarrow \mathcal{X}'$, equipped with a structure of square-zero extension, and $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$ we need to construct the corresponding map

$$T^*_x(\mathcal{X}) \to x^*(\mathcal{I})[1]$$

in $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$, functorially in (S, x).

10.2.2.   We will use the following lemma:

LEMMA 10.2.3. *For a schematic affine map of prestacks $\mathcal{X} \to \mathcal{X}'$, the functor*

$$(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}'} \to (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}, \quad S' \mapsto S' \underset{\mathcal{X}'}{\times} \mathcal{X}$$

is cofinal.

PROOF. The functor in question admits a left adjoint, given by

$$(S \to \mathcal{X}) \mapsto (S \to \mathcal{X} \to \mathcal{X}').$$

$\square$

10.2.4.   Using the lemma, it suffices to construct the map

$$T_x^*(\mathcal{X}) \to x^*(\mathcal{I})[1],$$

for every $(S', x') \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}'}$, where

$$x : S = S' \underset{\mathcal{X}}{\times} \mathcal{X} \to \mathcal{X}.$$

The latter is given as the composition

$$T_x^*(\mathcal{X}) \overset{(dx)^*}{\longrightarrow} T^*(S) \to x^*(\mathcal{I})[1],$$

where the second arrow represents the structure of square-zero extension on $S \hookrightarrow S'$.

10.2.5.   The following assertion results from the definitions:

LEMMA 10.2.6. *Let $\mathcal{Z}$ be a prestack that admits deformation theory, and let $z : \mathcal{X} \to \mathcal{Z}$ be a map. Then for a map $\mathcal{X} \to \mathcal{X}'$ equipped with a structure of square-zero extension by means of $\mathcal{I} \in \mathrm{QCoh}(\mathcal{X})^{\leq 0}$, the space of extensions of z to a map $z' : \mathcal{X}' \to \mathcal{Z}$ is canonically equivalent to that of null-homotopies of the composed map*

$$z^*(T^*(\mathcal{Z})) \overset{(dz)^*}{\longrightarrow} T^*(\mathcal{X}) \to \mathcal{I}[1].$$

10.3. Classifying square-zero extensions. In this subsection we keep the assumption that $\mathcal{X}$ admits deformation theory. We will (try to) classify square-zero extensions of $\mathcal{X}$.

10.3.1.   We would like to address the following general question:

QUESTION 10.3.2. *Is it true that the functor $\mathrm{SqZ}(\mathcal{X}, \mathcal{I}) \to \mathrm{Maps}_{\mathrm{QCoh}(\mathcal{X})}(T^*(\mathcal{X}), \mathcal{I}[1])$ of (10.2) is an isomorphism of spaces?*

Unfortunately, we can't answer this question in general. In this subsection we will consider a certain particular case.

REMARK 10.3.3. In Chapter 8, Sect. 5.5 we will provide a far more satisfying answer under the assumption that $\mathcal{X}$ be locally almost of finite type.

10.3.4. Let Y be an object of $\mathrm{Sch}^{\mathrm{aff}}$, and let $Y \hookrightarrow Y'$ be given a structure of square-zero extension by means of $\mathcal{I}_Y \in \mathrm{QCoh}(Y)^{\leq 0}$. Let

$$\gamma_Y : T^*(Y) \to \mathcal{I}_Y[1]$$

be the corresponding map.

Fix a map $f : \mathcal{X} \to Y$, and denote $\mathcal{I}_X := f^*(\mathcal{I}_Y)$. Consider the space

$$\mathrm{SqZ}(\mathcal{X}, \mathcal{I}_{\mathcal{X}})_{/\mathrm{SqZ}(Y, \mathcal{I}_Y)}$$

that classifies maps $\mathcal{X} \hookrightarrow \mathcal{X}'$ equipped with a structure of square-zero extension by means of $\mathcal{I}_{\mathcal{X}}$, and a commutative diagram

$$
\begin{array}{ccc}
\mathcal{X} & \longrightarrow & \mathcal{X}' \\
f \downarrow & & \downarrow f' \\
Y & \longrightarrow & Y'
\end{array}
$$

(10.3)

equipped with a structure of map of square-zero extensions that corresponds to the tautological map $f^*(\mathcal{I}_Y) \to \mathcal{I}_{\mathcal{X}}$.

Consider the space

$$\mathrm{Maps}_{\mathrm{QCoh}(\mathcal{X})}(T^*(\mathcal{X}), \mathcal{I}_{\mathcal{X}}[1])_{/\gamma_Y}$$

that classifies maps

$$\gamma_{\mathcal{X}} : T^*(\mathcal{X}) \to \mathcal{I}_{\mathcal{X}}[1],$$

together with the data of commutativity of the diagram

$$
\begin{array}{ccc}
T^*(\mathcal{X}) & \xrightarrow{\ \gamma_{\mathcal{X}}\ } & \mathcal{I}_{\mathcal{X}}[1], \\
(df)^* \uparrow & & \uparrow \sim \\
f^*(T^*(Y)) & \xrightarrow{\ \gamma_Y\ } & f^*(\mathcal{I}_Y)[1].
\end{array}
$$

As in Sect. 10.2, we have a canonically defined functor

(10.4) $\mathrm{SqZ}(\mathcal{X}, \mathcal{I}_{\mathcal{X}})_{/\mathrm{SqZ}(Y, \mathcal{I}_Y)} \to \mathrm{Maps}_{\mathrm{QCoh}(\mathcal{X})}(T^*(\mathcal{X}), \mathcal{I}_{\mathcal{X}}[1])_{/\gamma_Y}.$

We claim:

PROPOSITION 10.3.5. *The functor* (10.4) *is an isomorphism of spaces. Furthermore, for every object of* $\mathrm{SqZ}(\mathcal{X}, \mathcal{I}_{\mathcal{X}})_{/\mathrm{SqZ}(Y, \mathcal{I}_Y)}$ *the diagram* (10.3) *is Cartesian.*

The rest of the subsection is devoted to the proof of this proposition.

10.3.6. We construct the map

(10.5) $\mathrm{Maps}_{\mathrm{QCoh}(\mathcal{X})}(T^*(\mathcal{X}), \mathcal{I}_{\mathcal{X}}[1])_{/\gamma_Y} \to \mathrm{SqZ}(\mathcal{X}, \mathcal{I}_{\mathcal{X}})_{/\mathrm{SqZ}(Y, \mathcal{I}_Y)}$

as follows.

Given $\gamma : T^*(\mathcal{X}) \to \mathcal{I}_{\mathcal{X}}[1]$, we construct the prestack $\mathcal{X}'$ by letting for $S' \in \mathrm{Sch}^{\mathrm{aff}}$ the space $\mathrm{Maps}(S', \mathcal{X}')$ consist of the data of:

- $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$;
- A map $S \to S'$;
- A factorization of the map $x^*(\gamma) : T_x^*(\mathcal{X}) \to x^*(\mathcal{I}_{\mathcal{X}})[1]$ as

$$T_x^*(\mathcal{X}) \xrightarrow{(dx)^*} T^*(S) \xrightarrow{\gamma_S} x^*(\mathcal{I}_{\mathcal{X}})[1],$$

- An isomorphism $\mathrm{RealSqZ}(x^*(\mathcal{I}_{\mathcal{X}})[1], \gamma_S) \simeq (S \hookrightarrow S')$ in $\mathrm{Sch}^{\mathrm{aff}}_{S/}$.

Note that the construction of $\mathcal{X}'$ does not appeal to the datum of the map $Y \hookrightarrow Y'$ or a structure on it of square-zero extension.

10.3.7. That above the datum of $\mathrm{Maps}(S', \mathcal{X}')$ can be rewritten as follows:

- A map $y' : S' \to Y'$ (denote $S := S' \underset{Y'}{\times} Y$, $y : S \to Y$ and $\gamma_S : T^*(S) \to y^*(\mathcal{I}_Y)[1]$);
- A factorization of y as $S \xrightarrow{\ x\ } \mathcal{X} \xrightarrow{\ f\ } Y$;
- A datum of homotopy between

$$x^*(T^*(\mathcal{X})) = T^*_x(\mathcal{X}) \xrightarrow{(dx)^*} T^*(S) \xrightarrow{\gamma_S} y^*(\mathcal{I}_Y)[1] \simeq x^*(\mathcal{I}_{\mathcal{X}})[1]$$

and $x^*(\gamma_{\mathcal{X}})$.

10.3.8. The latter description implies that the space consisting of a data of a map $x' : S' \to \mathcal{X}'$ *and* a map $S' \to S$, which is the *left* inverse of the map $S \to S'$ identifies canonically with the space $\mathrm{Maps}(S', \mathcal{X})$. Indeed, given a map $S' \to \mathcal{X}'$, both pieces of additional data amount to that of null-homotopy of the map $x^*(\gamma_{\mathcal{X}})$.

This gives rise to a canonical map $\mathcal{X} \to \mathcal{X}'$, such that for every $x' : S' \to \mathcal{X}$, the corresponding diagram

$$
\begin{array}{ccc}
S & \longrightarrow & S' \\
{\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x'} \\
\mathcal{X} & \longrightarrow & \mathcal{X}'
\end{array}
$$

is Cartesian.

This gives the map $\mathcal{X} \hookrightarrow \mathcal{X}'$ a structure of square-zero extension by means of $\mathcal{I}_X$, thereby providing a map in (10.5). Furthermore, the diagram (10.3) is Cartesian also by construction.

$\square$

10.4. Deformation theory property of square-zero extensions. In this subsection we let $\mathcal{X}$ and $Y \hookrightarrow Y'$ be as in Proposition 10.3.5. We will show that pretacks $\mathcal{X}'$ as in Proposition 10.3.5 themselves admit deformation theory.

10.4.1. Our goal is to show:

PROPOSITION 10.4.2. *For every object of* $\mathrm{SqZ}(\mathcal{X}, \mathcal{I}_{\mathcal{X}})_{/\mathrm{SqZ}(Y, \mathcal{I}_Y)}$ *we have:*

(a) *The prestack $\mathcal{X}'$ admits deformation theory.*

(b) *If $Y, Y' \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$ and $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$, then $\mathcal{X}' \in \mathrm{PreStk}_{\mathrm{laft}}$.*

The rest of this subsection is devoted to the proof of this proposition.

10.4.3. First, we note that point (a) implies point (b):

We apply Theorem 9.1.2 to the nilpotent embedding $\mathcal{X} \hookrightarrow \mathcal{X}'$. It suffices to show that for any $(S, x) \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{X}}$, the pullback of $T^*(\mathcal{X}/\mathcal{X}')$ under x belongs to $\mathrm{Pro}(\mathrm{QCoh}(S)^-)_{\mathrm{laft}}$.

However, this pullback identifies with the pullback of $T^*(\mathcal{X}/\mathcal{X}')$ under $y := f \circ x$ of $T^*(Y/Y')$, and the assertion follows.

10.4.4. *Convergence.*

Using the interpretation of the space $\mathrm{Maps}(S', \mathcal{X}')$ given in Sect. 10.3.7, in order to prove that $\mathcal{X}'$ is convergent, we need to show that the space of homotopies between two fixed maps

$$x^*(T^*(\mathcal{X})) \rightrightarrows y^*(\mathcal{I}_Y)[1]$$

is mapped isomorphically to the inverse limit over n over similar spaces for

$$S_n := ({}^{\leq n}S') \underset{Y'}{\times} Y.$$

Note that for any n, we have:

$$^{\leq n}(S_n) \simeq {}^{\leq n}S.$$

Hence, the required assertion follows from the fact that

$$x^*(T^*(\mathcal{X})) = T_x^*(\mathcal{X}) \in {}^{\mathrm{conv}}\mathrm{Pro}(\mathrm{QCoh}(S)^-)$$

(see Lemma 3.3.3).

10.4.5. *Compatibility with push-outs.* Let $\widetilde{S}_2' := \widetilde{S}_1' \underset{S_1'}{\sqcup} S_2'$ be a push-out in $\mathrm{Sch}^{\mathrm{aff}}$, where $S_1' \to \widetilde{S}_1'$ is a nilpotent embedding. Let us show that the map

$$\mathrm{Maps}(\widetilde{S}_2', \mathcal{X}') \to \mathrm{Maps}(\widetilde{S}_1', \mathcal{X}') \underset{\mathrm{Maps}(S_1', \mathcal{X}')}{\times} \mathrm{Maps}(S_2', \mathcal{X}')$$

is an isomorphism.

It suffices to show that the map in question is an isomorphism over a given point of

$$\mathrm{Maps}(\widetilde{S}_2', Y') \simeq \mathrm{Maps}(\widetilde{S}_1', Y') \underset{\mathrm{Maps}(S_1', Y')}{\times} \mathrm{Maps}(S_2', Y').$$

Set

$$S_1 := S_1' \underset{Y'}{\times} Y, \ S_2 := S_2' \underset{Y'}{\times} Y, \ \widetilde{S}_1 := \widetilde{S}_1' \underset{Y'}{\times} Y, \ \widetilde{S}_2 := \widetilde{S}_2' \underset{Y'}{\times} Y.$$

It is easy to see that the map

$$\widetilde{S}_1 \underset{S_1}{\sqcup} S_2 \to \widetilde{S}_2$$

is an isomorphism.

Using the interpretation of the space $\mathrm{Maps}(-, \mathcal{X}')$ given in Sect. 10.3.7, we obtain that it suffices to show that, given a map $x : \widetilde{S}_2 \to \mathcal{X}$, the space of homotopies between two given maps

$$x^*(T^*(\mathcal{X})) \to y^*(\mathcal{I}_Y)[1]$$

maps isomorphically to the fiber product of the corresponding spaces on $\widetilde{S}_1$ and S_2 over that on S_1.

However, this follows from Proposition 1.4.2.   □

Ind-schemes and inf-schemes

Introduction

0.1. Inf-schemes. As was explained in the Introduction to Part I, inf-schemes are our primary object of interest. In this Chapter we will finally define what they are.

By definition, an inf-scheme is a laft prestack $\mathcal{X}$ such that:

(a) $\mathcal{X}$ admits deformation theory;

(b) $^{\mathrm{red}}\mathcal{X}$ is a (reduced) scheme.

It is quite remarkable that so general a definition produces a very reasonable object. Let us list some of the properties enjoyed by inf-schemes:

(i) Inf-schemes are well-adapted to the category IndCoh, i.e., the latter will extend to a functor of the $(\infty, 2)$-category of correspondences on inf-schemes. This will be realized in Chapter 3.

(ii) Inf-schemes provide a unified language to talk about $\mathcal{O}$-modules and D-modules; in particular, one can talk about *relative* D-modules along the fibers of a morphism between schemes. This will be realized in Chapter 4.

(iii) Inf-schemes are an adequate framework for *formal moduli problems* and the correspondence between group-objects and their Lie algebras. This will be realized in Chapters 6 and 7.

0.1.1. In this Chapter we will only initiate the study of inf-schemes. The main outcome of this Chapter is the following structural result that comes in two parts, Corollary 4.4.6:

Let $\mathcal{X}$ be an in inf-scheme such that $^{\mathrm{red}}\mathcal{X} = X_0$ is a (reduced) affine scheme. Inside the category $(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$ one can single out a subcategory of those

$$S \overset{x}{\to} \mathcal{X},$$

for which $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$ and the map

$$^{\mathrm{red}}x : {}^{\mathrm{red}}S \to {}^{\mathrm{red}}\mathcal{X} = X_0$$

is an isomorphism. The first assertion of Corollary 4.4.6 is that this subcategory is cofinal.

This means that the datum of $\mathcal{X}$, viewed as a functor from the category (opposite to that) of all affine schemes, is completely determined (i.e., is the left Kan extension) from its restriction to the category of pairs (S, x_0), where $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$ and x_0 is an isomorphism $^{\mathrm{red}}S \to X_0$.

In other words, in order to 'know' $\mathcal{X}$ we only need to know what the functor $\mathcal{X}$ gives in nilpotent thickenings of X_0.

0.1.2.   The second part of Corollary 4.4.6 provides a converse to the above restriction process.

Namely, let $\mathcal{X}_{\text{nil-isom}}$ be an arbitrary functor from the category of affine eventually coconnective schemes almost of finite type, whose reduced subscheme is identified with a given (reduced) affine scheme X_0 of finite type. I.e.,

$$\mathcal{X}_{\text{nil-isom}} : ({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{aft}} \underset{\text{red}\,\text{Sch}^{\text{aff}}}{\times} \{X_0\})^{\text{op}} \to \text{Spc}.$$

We impose the condition that the value of $\mathcal{X}_{\text{nil-isom}}$ on X_0 itself be $* \in \text{Spc}$. Let $\mathcal{X}$ be the left Kan extension of $\mathcal{X}_{\text{nil-isom}}$ under the forgetful functor

$$ {}^{<\infty}\text{Sch}^{\text{aff}}_{\text{aft}} \underset{\text{red}\,\text{Sch}^{\text{aff}}}{\times} \{X_0\} \to {}^{<\infty}\text{Sch}^{\text{aff}} .$$

I.e., the value of $\mathcal{X}$ on an (eventually coconnective) scheme affine scheme S is the category of

$$S \to Z \to \mathcal{X}_{\text{nil-isom}}, \quad Z \in {}^{<\infty}\text{Sch}^{\text{aff}}_{\text{aft}}, \ {}^{\text{red}}Z = X_0.$$

Thus, we can view $\mathcal{X}$ as an object of $\text{PreStk}_{\text{laft}}$, and ${}^{\text{red}}\mathcal{X} = X_0$. We would like $\mathcal{X}$ to be an inf-scheme, but we cannot expect that because there is no reason that for an arbitrary $\mathcal{X}_{\text{nil-isom}}$, the prestack $\mathcal{X}$ will admit deformation theory.

However, there is an obvious necessary condition on $\mathcal{X}_{\text{nil-isom}}$ for $\mathcal{X}$ to have a chance to admit deformation theory. Namely, recall that one of the conditions in admitting deformation theory is that it should take pushout diagrams of the form

(0.1)
$$\begin{array}{ccc} S_1 & \longrightarrow & S_2 \\ \downarrow & & \downarrow \\ S_1' & \longrightarrow & S_2'. \end{array}$$

where $S_1 \to S_1'$ has a structure of square-zero extension, to pullback diagrams in Spc. Thus, a necessary condition on $\mathcal{X}_{\text{nil-isom}}$ in order for $\mathcal{X}$ to admit deformation theory is that $\mathcal{X}_{\text{nil-isom}}$ have the same property with respect to the above push-outs when

$$S_1, S_2 \in {}^{<\infty}\text{Sch}^{\text{aff}}_{\text{aft}} \underset{\text{red}\,\text{Sch}^{\text{aff}}}{\times} \{X_0\}.$$

Now, the second part of Corollary 4.4.6 says that the above condition is also sufficient.

0.1.3.   To summarize, Corollary 4.4.6 says that the operation of restriction under

$$ {}^{<\infty}\text{Sch}^{\text{aff}}_{\text{aft}} \underset{\text{red}\,\text{Sch}^{\text{aff}}}{\times} \{X_0\} \to {}^{<\infty}\text{Sch}^{\text{aff}}$$

defines a fully faithful functor from the category of inf-schemes $\mathcal{X}$, whose underling reduced scheme is identified with X_0 and the full subcategory of the category of functors

$$\mathcal{X}_{\text{nil-isom}} : ({}^{<\infty}\text{Sch}^{\text{aff}}_{\text{aft}} \underset{\text{red}\,\text{Sch}^{\text{aff}}}{\times} \{X_0\})^{\text{op}} \to \text{Spc}, \quad \mathcal{X}_{\text{nil-isom}}(X_0) = *,$$

that take push-out (0.1) squares to pullback squares.

0.2. Ind-schemes. Prior to introducing inf-schemes, in Sects. 1 and 2, we study another type of algebro-geometric objects that often comes up in practice: ind-schemes.

0.2.1. Ind-schemes can be defined in any of the following three equivalent ways (but the equivalence is not altogether trivial):

A prestack $\mathcal{X}$ is said to be an ind-scheme if it is convergent and:

Definition (a): $\mathcal{X}$ can be written as a *filtered* colimit (in PreStk) of quasi-compact schemes, where the transition maps are closed embeddings.

Definition (a'): Same as (a) with $\mathcal{X}$ replaced by $^{\leq n}\mathcal{X}$ for any n.

Definition (b): The subcategory of $(\mathrm{Sch}_{qc})_{/\mathcal{X}}$ consisting of *closed embeddings* is cofinal and filtered.

Definition (b'): Same as (a) with $\mathcal{X}$ replaced by $^{\leq n}\mathcal{X}$ for any n.

Definition (c): $\mathcal{X}$ admits connective deformation theory and $^{cl}\mathcal{X}$ is a classical ind-scheme.

0.2.2. Beyond the equivalence of the above definitions, here is a summary of main results pertaining to ind-schemes:

(i) Ind-schemes satisfy flat descent;

(ii) If an ind-scheme $\mathcal{X}$ is laft as a prestack, then the subcategory of $(\mathrm{Sch}_{qc})_{/\mathcal{X}}$ consisting of closed embeddings $Z \to \mathcal{X}$ with $S \in \mathrm{Sch}_{aft}$ is cofinal and filtered.

(iii) Let $\mathcal{X}$ be a laft prestack that admits connective deformation theory, and such that $^{red}\mathcal{X}$ is a (reduced) indscheme. Then $\mathcal{X}$ is an ind-scheme if and only if the following conditions hold:

(a) For any reduced affine scheme S and $x : S \to \mathcal{X}$, the object $H^0(T_x^*(\mathcal{X})) \in \mathrm{Pro}(\mathrm{QCoh}(S)^\heartsuit)$ can be written as a projective system with surjective transition maps;

(b) Either of the following equivalent conditions holds:
  - The map $^{red}\mathcal{X} \to {}^{cl}\mathcal{X}$ is a monomorphism; or
  - For any reduced affine scheme S and $x : S \to {}^{red}\mathcal{X}$, the object

  $$T_x^*({}^{red}\mathcal{X}/\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)$$

  belongs to $\mathrm{Pro}(\mathrm{QCoh}(S)^-)^{\leq 0}$.

0.2.3. *Formal schemes.* Normally, in the literature by a *formal scheme* one means an ind-scheme $\mathcal{X}$ such that $^{red}\mathcal{X}$ is a (reduced) scheme. Since there many other usages of the word 'formal', in this book, we call these objects *nil-schematic ind-schemes*[1].

We show that if $\mathcal{X}$ is nil-schematic ind-scheme with $^{red}\mathcal{X} = X_0$, then the subcategory of $(\mathrm{Sch}_{qc})_{/\mathcal{X}}$ consisting of *nilpotent embeddings* embeddings $S \to \mathcal{X}$ is cofinal and filtered.

Moreover, we show that if $\mathcal{X}$ is laft as a prestack, the subcategory of $(\mathrm{Sch}_{qc})_{/\mathcal{X}}$ consisting of *nilpotent embeddings* embeddings $Z \to \mathcal{X}$ with $Z \in \mathrm{Sch}_{aft}$ is cofinal and filtered.

---

[1]This is in line with our policy that the adjective 'nil-?' for a morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$ means that the corresponding morphism

$$^{red}f : {}^{red}\mathcal{X}_1 \to {}^{red}\mathcal{X}_2$$

has the property ?.

0.2.4.   Let us emphasize the difference between inf-schemes and nil-schematic ind-schemes.

Namely, in Corollary 4.3.3 (from which we deduce the first direction in Corollary 4.4.6 mentioned above) we show that if $\mathcal{X}$ is an *inf-scheme*, the map

$$\operatorname*{colim}_{Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-isom to } \mathcal{X}}} Z \to \mathcal{X},$$

where the colimit is taken in PreStk, is an isomorphism. Moreover, we will show (see Proposition 4.3.6) that the category $(\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-isom to } \mathcal{X}}$ is *sifted*. In particular, $\mathcal{X}$ can be written as a colimit

$$\operatorname*{colim}_{\alpha \in A} Z_\alpha \to \mathcal{X},$$

for some sifted index category A, where $Z_\alpha \in \mathrm{Sch}_{\mathrm{aft}}$ and the transition maps in this family are *nil-isomorphisms*.

However, if $\mathcal{X}$ is a laft nil-schematic ind-scheme, the map

$$\operatorname*{colim}_{Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\text{nilp-emb into } \mathcal{X}}} Z \to \mathcal{X},$$

where the colimit is taken in PreStk, is an isomorphism, and the category $(\mathrm{Sch}_{\mathrm{aft}})_{\text{nilp-emb into } \mathcal{X}}$ is filtered. In particular, $\mathcal{X}$ can be written as a colimit

$$\operatorname*{colim}_{\alpha \in A} Z_\alpha \to \mathcal{X},$$

where $Z_\alpha \in \mathrm{Sch}_{\mathrm{aft}}$ and the transition maps in this family are *nilpotent embeddings*, and where the index category A is filtered.

So, there are two points of difference: one is the 'filtered' vs. 'sifted' condition on the index category A, and the other is 'nilpotent embeddings' vs 'nil-isomorphisms'.

In addition, given $x : Z \to \mathcal{X}$ with Z not necessarily affine, if $\mathcal{X}$ is a laft nil-schematic ind-scheme, the category of factorizations of x as

$$Z \to Z' \overset{x'}{\to} \mathcal{X}, \quad x' \text{ is a nilpotent embedding}$$

is contractible. If, however, Z is an inf-scheme, the category of factorizations of x as

$$Z \to Z' \overset{x'}{\to} \mathcal{X}, \quad x' \text{ is a nil-isomorphism}$$

need not be contractible (it may be empty).

0.3. Other definitions and results.
0.3.1.   In Sect. 3 we define a notion slightly more general than inf-scheme, namely, that of *ind-inf-scheme*. Namely, a laft prestack $\mathcal{X}$ is said to be an ind-inf-scheme if

(a) $\mathcal{X}$ admits deformation theory;

(b) $^{\mathrm{red}}\mathcal{X}$ is a (reduced) ind-scheme.

Thus, the class of ind-inf-schemes contains both inf-schemes and ind-schemes. We give some infinitesimal criteria that allow to determine when an ind-inf-scheme is an ind-scheme.

0.3.2. In turns that the good behavior of IndCoh on inf-schemes extends at no cost to ind-inf-schemes; this will be done in Chapter 3.

We show that ind-inf-schemes satisfy Nisnevich (and with a little more work, also étale) descent.

0.3.3. Finally, we establish an extension of Corollaries 4.4.6 and Corollary 4.3.3 mentioned above, to the case of ind-inf-schemes; this is done in Sect. 4.

1. Ind-schemes

Ind-schemes are an approximation to our main object of interest (the latter being ind-inf-schemes). In this section we will mainly review various facts established in [**GaRo1**].

1.1. The notion of ind-scheme. Ind-schemes are defined in a very simple way: prestacks that can be presented as *filtered* colimits of schemes under closed embeddings. It is not altogether tautological that this is 'the right notion', but we will see that it is in the course of this section.

1.1.1. Let $\mathcal{X}$ be an object of PreStk.

DEFINITION 1.1.2. *We shall say that $\mathcal{X}$ is an ind-scheme if:*
- *$\mathcal{X}$ is convergent;*
- *As an object of* PreStk, *we can write $\mathcal{X}$ as a* filtered *colimit*

$$\colim_{\alpha} X_{\alpha}, \tag{1.1}$$

where $X_{\alpha} \in \mathrm{Sch}_{\mathrm{qc}}$ and the maps $X_{\alpha_1} \to X_{\alpha_2}$ are closed embeddings.

1.1.3. We let

$$\mathrm{indSch} \subset \mathrm{PreStk}$$

denote the full subcategory spanned by ind-schemes. We also denote

$$\mathrm{indSch}_{\mathrm{laft}} := \mathrm{indSch} \cap \mathrm{PreStk}_{\mathrm{laft}}.$$

It is clear that the above subcategories are closed under finite limits taken in PreStk.

1.1.4. In addition:

DEFINITION 1.1.5. *Let $\mathcal{X}$ be an object of $^{\leq n}$PreStk (resp., $^{\mathrm{cl}}$PreStk, $^{\mathrm{red}}$PreStk). We shall say that $\mathcal{X}$ is an n-coconnective (resp., classical, reduced) ind-scheme if as an object of $^{\leq n}$PreStk (resp., $^{\mathrm{cl}}$PreStk, $^{\mathrm{red}}$PreStk), we can write $\mathcal{X}$ as a* filtered *colimit*

$$\colim_{\alpha} X_{\alpha}, \tag{1.2}$$

where $X_{\alpha} \in {}^{\leq n}\mathrm{Sch}_{\mathrm{qc}}$ (resp., $X_{\alpha} \in {}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{qc}}$, $X_{\alpha} \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{qc}}$) and the maps $X_{\alpha_1} \to X_{\alpha_2}$ are closed embeddings.

We let

$$^{\leq n}\mathrm{indSch} \subset {}^{\leq n}\mathrm{PreStk}, \ ^{\leq n}\mathrm{indSch}_{\mathrm{lft}} \subset {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}},$$
$$^{\mathrm{cl}}\mathrm{indSch} \subset {}^{\mathrm{cl}}\mathrm{PreStk}, \ ^{\mathrm{cl}}\mathrm{indSch}_{\mathrm{lft}} \subset {}^{\mathrm{cl}}\mathrm{PreStk}_{\mathrm{lft}}$$

and

$$^{\mathrm{red}}\mathrm{indSch} \subset {}^{\mathrm{red}}\mathrm{PreStk}, \ ^{\mathrm{red}}\mathrm{indSch}_{\mathrm{lft}} \subset {}^{\mathrm{red}}\mathrm{PreStk}_{\mathrm{lft}},$$

denote the corresponding subcategories.

These subcategories are closed under finite limits in $^{\leq n}\mathrm{PreStk}$ (resp., $^{\mathrm{cl}}\mathrm{PreStk}$, $^{\mathrm{red}}\mathrm{PreStk}$).

1.2. Descent for ind-schemes. In this subsection we show that ind-schemes satisfy flat descent.

1.2.1. We are going to prove:

PROPOSITION 1.2.2. *Let $\mathcal{X} \in \mathrm{indSch}$. Then $\mathcal{X}$ satisfies flat descent.*

For the proof of the proposition we will need the following assertion of independent interest:

LEMMA 1.2.3. *Let $\mathcal{X}$ be an ind-scheme. Then for $S \in {}^{\leq n}\mathrm{Sch}_{\mathrm{qc}}$, the ∞-groupoid $\mathrm{Maps}(S, \mathcal{X})$ is n-truncated.*

PROOF. First we take $n = 0$. In this case the assertion follows from the fact that filtered colimits of discrete objects of Spc are discrete. For general n, the assertion follows from Chapter 1, Lemma 6.3.2 and Proposition 1.3.2 below (which is proved independently). □

1.2.4. *Proof of Proposition 1.2.2, Step 1.* Let $\mathcal{X}$ be written as $\operatorname*{colim}_{\alpha \in A} X_\alpha$, where $X_\alpha \in \mathrm{Sch}_{\mathrm{qc}}$, and the category A is filtered.

Let $\overset{\circ}{S} \to S$ be a faithfully flat map in $\mathrm{Sch}^{\mathrm{aff}}$, and let $\overset{\circ}{S}{}^\bullet/S$ be its Čech nerve. We need to show that the map

(1.3) $$\mathrm{Maps}(S, \mathcal{X}) \to \mathrm{Tot}(\mathrm{Maps}(\overset{\circ}{S}{}^\bullet/S, \mathcal{X}))$$

is an isomorphism.

Let us first assume that S is n-coconnective for some n. Since $\overset{\circ}{S} \to S$ is flat, then $\overset{\circ}{S}$ is also n-coconnective, and so are the terms of $\overset{\circ}{S}{}^\bullet/S$.

We have a commutative diagram in Spc:

$$
\begin{array}{ccc}
\operatorname*{colim}_{\alpha} \mathrm{Maps}(S, X_\alpha) & \xrightarrow{\ \sim\ } & \mathrm{Maps}(S, \mathcal{X}) \\
\downarrow & & \downarrow \\
\operatorname*{colim}_{\alpha} \mathrm{Tot}(\mathrm{Maps}(\overset{\circ}{S}{}^\bullet/S, X_\alpha)) & \longrightarrow & \mathrm{Tot}(\mathrm{Maps}(\overset{\circ}{S}{}^\bullet/S, \mathcal{X})) \\
\downarrow & & \downarrow \\
\operatorname*{colim}_{\alpha} \mathrm{Tot}^{\leq n}(\mathrm{Maps}(\overset{\circ}{S}{}^\bullet/S, X_\alpha)) & \longrightarrow & \mathrm{Tot}^{\leq n}(\mathrm{Maps}(\overset{\circ}{S}{}^\bullet/S, \mathcal{X})) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{Tot}^{\leq n}\left(\operatorname*{colim}_{\alpha} \mathrm{Maps}(\overset{\circ}{S}{}^\bullet/S, X_\alpha)\right) & \xrightarrow{\ \sim\ } & \mathrm{Tot}^{\leq n}(\mathrm{Maps}(\overset{\circ}{S}{}^\bullet/S, \mathcal{X})),
\end{array}
$$

where $\mathrm{Tot}^{\leq n}$ denote the limit over the n-skeleton.

We note that the map

$$\operatorname*{colim}_{\alpha} \operatorname{Maps}(S, X_{\alpha}) \to \operatorname*{colim}_{\alpha} \operatorname{Tot}(\operatorname{Maps}(\mathring{S}^{\bullet}/S, X_{\alpha}))$$

is an isomorphism since maps of schemes satisfy flat descent.

Now, Lemma 1.2.3, implies that

$$\operatorname{Tot}(\operatorname{Maps}(\mathring{S}^{\bullet}/S, \mathcal{X})) \to \operatorname{Tot}^{\leq n}(\operatorname{Maps}(\mathring{S}^{\bullet}/S, \mathcal{X}))$$

and

$$\operatorname{Tot}(\operatorname{Maps}(\mathring{S}^{\bullet}/S, X_{\alpha})) \to \operatorname{Tot}^{\leq n}(\operatorname{Maps}(\mathring{S}^{\bullet}/S, X_{\alpha}).$$

are isomorphisms.

Furthermore, the map

$$\operatorname*{colim}_{\alpha} \operatorname{Tot}^{\leq n}(\operatorname{Maps}(\mathring{S}^{\bullet}/S, X_{\alpha})) \to \operatorname{Tot}^{\leq n}\left(\operatorname*{colim}_{\alpha} \operatorname{Maps}(\mathring{S}^{\bullet}/S, X_{\alpha}))\right)$$

is an isomorphism, since finite limits commute with filtered colimits.

This implies that the map (1.3) is an isomorphism as well.

1.2.5. *Proof of Proposition 1.2.2, Step 2.* For an integer n, we consider the n-coconnective truncation $^{\leq n}S$ of S. Note that since $\mathring{S} \to S$ is flat, the map $^{\leq n}\mathring{S} \to {}^{\leq n}S$ is flat, and the simplicial n-coconnective DG scheme $^{\leq n}(\mathring{S}^{\bullet}/S)$ is the Čech nerve of $^{\leq n}\mathring{S} \to {}^{\leq n}S$.

We have a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Maps}(S, \mathcal{X}) & \longrightarrow & \operatorname{Tot}(\operatorname{Maps}(\mathring{S}^{\bullet}/S, \mathcal{X})) \\
\downarrow & & \downarrow \\
\operatorname*{lim}_{n \in \mathbb{N}^{\mathrm{op}}} \operatorname{Maps}(^{\leq n}S, \mathcal{X}) & \longrightarrow & \operatorname*{lim}_{n \in \mathbb{N}^{\mathrm{op}}} \operatorname{Tot}(\operatorname{Maps}(^{\leq n}(\mathring{S}^{\bullet}/S), \mathcal{X}))
\end{array}
.
$$

In this diagram the vertical arrows are isomorphisms, since $\mathcal{X}$ is convergent. The bottom horizontal arrow is an isomorphism by Step 1. Hence, the top horizontal arrow is an isomorphism as well, as desired.

$\square$

1.2.6.   As a corollary we obtain:

COROLLARY 1.2.7. *Let $\mathcal{X} \in \operatorname{indSch}$ be written as in (1.1). Then for $Z \in \operatorname{Sch}_{\mathrm{qc}}$, the map*

$$\operatorname*{colim}_{\alpha} \operatorname{Maps}(Z, X_{\alpha}) \to \operatorname{Maps}(Z, \mathcal{X})$$

is an isomorphism.

PROOF. Follows from the Zariski descent property of $\mathcal{X}$ and the fact that finite limits commute with filtered colimits. $\square$

1.3. Deformation theory of ind-schemes. In this subsection we show that ind-schemes admit (connective) deformation theory, and that, moreover, they can essentially be characterized by this property.

1.3.1.   We observe:

PROPOSITION 1.3.2. *Let $\mathcal{X}$ be an ind-scheme. Then $\mathcal{X}$ admits a connective deformation theory.*

PROOF. Follows from the fact that the formation of finite limits (involved in the definition of admitting connective deformation theory, see Chapter 1, Lemma 3.1.8) commutes with filtered colimits.   □

1.3.3.   Note that Chapter 1, Lemma 2.5.5 gives an explicit expression to the value of pro-cotangent spaces of an ind-scheme:

LEMMA 1.3.4. *For $\mathcal{X} \in \mathrm{indSch}$ written as in (1.1), and $(Z, x) \in (\mathrm{Sch_{qc}})_{/\mathcal{X}}$ we have:*

$$T_x^*(\mathcal{X}) \simeq \lim_{(\alpha, x_\alpha) \in (A_{x/})^{\mathrm{op}}} T_{x_\alpha}^*(X_\alpha),$$

where $A_{x/}$ is the category of factorizations of x as

$$Z \to X_\alpha \to \mathcal{X}, \quad \alpha \in A.$$

In the above lemma, the limit is taken in $\mathrm{Pro}(\mathrm{QCoh}(Z)^-)$, or equivalently $\mathrm{Pro}(\mathrm{QCoh}(Z)^{\leq 0})$, as $A_{x/}$ is filtered.

1.3.5.   We note the following feature of the pro-cotangent spaces of an ind-scheme.

DEFINITION 1.3.6. *Let Z be a scheme and $\mathcal{T}$ an object of $\mathrm{Pro}(\mathrm{QCoh}(Z)^\heartsuit)$. We shall say that $\mathcal{T}$ can be given by a surjective system if $\mathcal{T}$ can be written as a limit*

$$\lim_{\alpha \in A^{\mathrm{op}}} \mathcal{F}_\alpha,$$

where A is a filtered category, $\mathcal{F}_\alpha \in \mathrm{QCoh}(Z)^\heartsuit$ and for $\alpha_1 \to \alpha_2$, the corresponding map $\mathcal{F}_{\alpha_2} \to \mathcal{F}_{\alpha_1}$ is surjective.

We have:

LEMMA 1.3.7. *An object $\mathcal{T} \in \mathrm{Pro}(\mathrm{QCoh}(Z)^\heartsuit)$ is given by a surjective system if and only if in the category*

$$\left((\mathrm{QCoh}(Z)^\heartsuit)_{\mathcal{T}/}\right)^{\mathrm{op}}$$

the full subcategory, spanned by surjections $\mathcal{T} \twoheadrightarrow \mathcal{F}$, is cofinal.

For future reference, we note that

LEMMA 1.3.8. *If $i : \widetilde{Z} \hookrightarrow Z$ is a nilpotent embedding, and $\mathcal{T} \in \mathrm{Pro}(\mathrm{QCoh}(Z)^\heartsuit)$ is such that $\widetilde{\mathcal{T}} := H^0((\mathrm{Pro}(i^*))(\mathcal{T})) \in \mathrm{Pro}(\mathrm{QCoh}(\widetilde{Z})^\heartsuit)$ is given by a surjective system, then $\mathcal{T}$ is also given by a surjective system.*

1.3.9.   From Lemma 1.3.4 we obtain:

LEMMA 1.3.10. *Let $\mathcal{X}$ be an ind-scheme and $x : Z \to \mathcal{X}$ a point, where $Z \in \mathrm{Sch_{qc}}$. Then $H^0(T_x^*(\mathcal{X})) \in \mathrm{Pro}(\mathrm{QCoh}(Z)^\heartsuit)$ is given by a surjective system.*

1.3.11. The next assertion provides a partial converse to Proposition 1.3.2:

THEOREM 1.3.12. *Let $\mathcal{X}$ be an object of* PreStk *that admits connective deformation theory and that for any $(S, x : S \to \mathcal{X}) \in {}^{\mathrm{cl}}\mathrm{Sch}_{/\mathcal{X}}^{\mathrm{aff}}$, the object $H^0(T_x^*(\mathcal{X})) \in$ $\mathrm{Pro}(\mathrm{QCoh}(S)^{\heartsuit})$ is given by a surjective system. Assume that there exists a map $f : \mathcal{X}_0 \to \mathcal{X}$ such that:*

- *$\mathcal{X}_0$ is a classical ind-scheme;*
- *The map $f : \mathcal{X}_0 \to \mathcal{X}$ is a* monomorphism *when evaluated on classical schemes;*
- *The map f is a pseudo-nilpotent embedding[2].*

Then $\mathcal{X}$ is an ind-scheme.

The proof will be given in Sect. 2.

COROLLARY 1.3.13. *Let $\mathcal{X}$ be an object of* PreStk *that admits connective deformation theory, and such that ${}^{\mathrm{cl}}\mathcal{X}$ is a classical ind-scheme. Then $\mathcal{X}$ is an ind-scheme.*

1.4. Indschemes and truncations. In this subsection we compare our present definition of ind-schemes with that of [**GaRo1**].

1.4.1. We have:

PROPOSITION 1.4.2. *Let $\mathcal{X}$ be an object of ${}^{\mathrm{conv}}$PreStk. Then $\mathcal{X} \in \mathrm{indSch}$ if and only if for every n, we have ${}^{\leq n}\mathcal{X} \in {}^{\leq n}\mathrm{indSch}$.*

PROOF. The 'only if' part is evident. Conversely, let $\mathcal{X}$ be convergent and such that for every n, we have ${}^{\leq n}\mathcal{X} \in {}^{\leq n}\mathrm{indSch}$. By repeating the argument of Proposition 1.3.2, it follows from Chapter 1, Sect. 6.1.3 that $\mathcal{X}$ admits a connective deformation theory.

Hence, such $\mathcal{X}$ satisfies the conditions of Corollary 1.3.13, namely, we take $\mathcal{X}_0 = {}^{\mathrm{cl}}\mathcal{X}$.

□

1.4.3. From Proposition 1.4.2 we obtain:

COROLLARY 1.4.4. *Let $\alpha \mapsto X_\alpha$ be a filtered diagram of objects of $\mathrm{Sch}_{\mathrm{qc}}$ with the maps being closed embeddings. Set $\mathcal{X}' := \underset{\alpha}{\mathrm{colim}}\, X_\alpha$. Then ${}^{\mathrm{conv}}(\mathcal{X}')$ is an indscheme.*

Finally, from Corollary 1.4.4, we deduce:

COROLLARY 1.4.5. *Let $\mathcal{X}_n$ (resp., $\mathcal{X}_{\mathrm{cl}}$, $\mathcal{X}_{\mathrm{red}}$) be an n-coconnective (resp., classical, reduced) ind-scheme. Set*

$$\mathcal{X}' := \mathrm{LKE}_{{}^?\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}(\mathcal{X}_?),$$

where $? = \leq n$ (resp., $? = \mathrm{cl}$, $? = \mathrm{red}$). Then ${}^{\mathrm{conv}}(\mathcal{X}')$ is an ind-scheme.

1.5. Closed embeddings into an ind-scheme. In this subsection we show that an ind-scheme $\mathcal{X}$ can be recovered from the category of schemes equipped with a closed embedding into $\mathcal{X}$.

---

[2]See Chapter 1, Definition 8.1.2(d') for what this means.

1.5.1.   We recall (see Chapter 1, Definition 8.1.2) that a map of prestacks $\mathcal{X}_1 \to \mathcal{X}_2$ is a closed embedding if for $S_2 \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{X}_2}$, the map

$$^{\mathrm{cl}}(S_2 \underset{\mathcal{X}_2}{\times} \mathcal{X}_1) \to {}^{\mathrm{cl}}S_2$$

is a closed embedding of classical affine schemes (in particular, the left hand side is a classical affine scheme).

It is easy to see that if $\mathcal{X}_1 \to \mathcal{X}_2 \to \mathcal{X}_3$ are such that $\mathcal{X}_1 \to \mathcal{X}_3$ and $\mathcal{X}_2 \to \mathcal{X}_3$ are closed embeddings, then so is $\mathcal{X}_1 \to \mathcal{X}_2$.

In what follows, for $\mathcal{X} \in \mathrm{PreStk}$ we let

$$\mathrm{PreStk}_{\mathrm{closed\ in}\ \mathcal{X}} \subset \mathrm{PreStk}_{/\mathcal{X}}$$

denote the full subcategory, consisting of those $(\mathcal{X}', f : \mathcal{X}' \to \mathcal{X})$, for which f is a closed embedding. We will use a similar notation for any category that maps to PreStk, e.g.,

$$\mathrm{Sch}_{\mathrm{closed\ in}\ \mathcal{X}} .$$

1.5.2.   Let $\mathcal{X}$ be an ind-scheme, written as in (1.1). It is clear that for $Z \in \mathrm{Sch}_{\mathrm{qc}}$, a map $Z \to \mathcal{X}$ is a closed embedding if and only if for some/any α, for which the above map factors through a map $Z \to X_\alpha$, the latter is a closed embedding.

1.5.3.   We claim:

PROPOSITION 1.5.4.   *An object $\mathcal{X} \in {}^{\mathrm{conv}}\mathrm{PreStk}$ is an ind-scheme if and only if the following two conditions are satisfied:*

- *The functor $\mathrm{Sch}_{\mathrm{closed\ in}\ \mathcal{X}} \to (\mathrm{Sch}_{\mathrm{qc}})_{/\mathcal{X}}$ is cofinal.*
- *The category $\mathrm{Sch}_{\mathrm{closed\ in}\ \mathcal{X}}$ is filtered.*

PROOF.   Clearly, the conditions of the proposition are sufficient for $\mathcal{X}$ to be an ind-scheme.

Assume now that $\mathcal{X}$ is an ind-scheme. The fact that

$$\mathrm{Sch}_{\mathrm{closed\ in}\ \mathcal{X}} \to (\mathrm{Sch}_{\mathrm{qc}})_{/\mathcal{X}}$$

is cofinal follows from Corollary 1.2.7.

To prove that the category $\mathrm{Sch}_{\mathrm{closed\ in}\ \mathcal{X}}$ is filtered, it is enough to show that it contains finite colimits. Let

(1.4) $i \mapsto Z_i, \quad i \in I$

be a finite diagram in $\mathrm{Sch}_{\mathrm{closed\ in}\ \mathcal{X}}$. Write $\mathcal{X}$ as in (1.1). Then by Corollary 1.2.7, and since the category of indices A is filtered, there exists an index $\alpha \in A$ such that (1.4) factors through a diagram in $\mathrm{Sch}_{\mathrm{closed\ in}\ X_\alpha}$.

By Volume I, Chapter 5, Proposition 1.1.3, the resulting diagram admits a colimit, denote it

$$Z_\alpha \in \mathrm{Sch}_{\mathrm{closed\ in}\ X_\alpha} .$$

Furthermore, by Volume I, Chapter 5, Lemma 1.1.5, for an arrow $\alpha \to \alpha'$ in A, the resulting map $Z_\alpha \to Z_{\alpha'}$ is an isomorphism. Since A is filtered, this implies that Z_α maps isomorphically to the sought-for colimit in $\mathrm{Sch}_{\mathrm{closed\ in}\ \mathcal{X}}$.

$\square$

1.6. Topological conditions. In this subsection we introduce several classes of maps between prestacks, imposing the condition that they behave as 'relative ind-schemes'.

1.6.1.   We give the following definitions:

DEFINITION 1.6.2. *We shall say that a reduced ind-scheme $\mathcal{X}$ is ind-affine if it can be written as in (1.2) with $X_\alpha \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{qc}}$ being affine.*

It is easy to see that $\mathcal{X}$ is ind-affine if and only if for any closed embedding $X \to \mathcal{X}$ with $X \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{qc}}$, the scheme X is affine.

DEFINITION 1.6.3. *We shall say that an ind-scheme (resp., n-coconnective ind-scheme) $\mathcal{X}$ is ind-affine if ${}^{\mathrm{red}}\mathcal{X}$ has this property.*

Again, it is easy to see that $\mathcal{X}$ is ind-affine if and only if for any closed embedding $X \to \mathcal{X}$ with $X \in \mathrm{Sch}_{\mathrm{qc}}$ (resp., ${}^{\leq n}\mathrm{Sch}_{\mathrm{qc}}$), the scheme X is affine[3].

It is easy to see that $\mathcal{X}$ is ind-affine if for any/some presentation (1.1) (resp., (1.2)), the schemes X_α are affine.

1.6.4.   We give the following definition:

DEFINITION 1.6.5.
(a) *We shall say that a morphism $\mathcal{X}_1 \to \mathcal{X}_2$ of prestacks (resp., n-coconnective prestacks, classical prestacks, reduced prestacks) is* ind-schematic *if its base change by an affine scheme (resp., n-connective affine scheme, classical scheme, reduced scheme) yields an ind-scheme (resp., n-coconnective, classical, reduced ind-scheme).*
(b) *We shall say that a morphism $\mathcal{X}_1 \to \mathcal{X}_2$ of prestacks (resp., n-coconnective prestacks, classical prestacks, reduced prestacks) is* ind-affine *if its base change by an affine scheme (resp., n-connective affine scheme, classical scheme, reduced scheme) yields an ind-affine ind-scheme (resp., n-coconnective, classical, reduced ind-affine ind-scheme).*

1.6.6.   We also give the following definitions:

DEFINITION 1.6.7.
(a) *We shall say that a map from a classical prestack $\mathcal{X}$ to a classical affine scheme S is an* ind-closed embedding *if $\mathcal{X}$ is a classical ind-scheme and for any closed embedding $X \to \mathcal{X}$, where $X \in {}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{qc}}$, the composed map $X \to S$ is a closed embedding.*
(b) *We shall say that a map of classical prestacks $\mathcal{X}_1 \to \mathcal{X}_2$ is an* ind-closed embedding *if its base change by a classical affine scheme yields a map which is an ind-closed embedding.*
(c) *We shall say that a map of prestacks $\mathcal{X}_1 \to \mathcal{X}_2$ is is an* ind-closed embedding *if the corresponding map ${}^{\mathrm{cl}}\mathcal{X}_1 \to {}^{\mathrm{cl}}\mathcal{X}_2$ is.*

REMARK 1.6.8. Note that 'closed embedding' is stronger than 'ind-closed embedding'. E.g.,
$$\mathrm{Spf}(k[[t]]) \to \mathrm{Spec}(k[t])$$

---

[3]Here we use the fact that if a classical scheme X is such that ${}^{\mathrm{red}}X$ is affine, then X itself is affine.

is an an ind-closed embedding, but not a closed embedding. And similarly, for

$$\underset{I}{\sqcup} \, \mathrm{pt} \to \mathbb{A}^1,$$

where I is an arbitrary infinite set of distinct k-points in $\mathbb{A}^1$.

1.6.9.   Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a map of (classical) ind-schemes. It is easy to see that it is an ind-closed embedding (resp., ind-affine) if and only if the following is satisfied:

If

$$\mathcal{X}_1 := \underset{\alpha \in A}{\mathrm{colim}} \, X_{1,\alpha} \text{ and } \mathcal{X}_2 := \underset{\beta \in A}{\mathrm{colim}} \, X_{2,\beta},$$

then for every index α, and every/some index β for which $X_{1,\alpha} \to \mathcal{X}_1 \to \mathcal{X}_2$ factors as

$$X_{1,\alpha} \to X_{2,\beta} \to \mathcal{X}_2,$$

the map $X_{1,\alpha} \to X_{2,\beta}$ is a closed embedding (resp., an affine morphism between schemes).

In addition, f is an ind-closed embedding if and only if for every closed embedding $X_1 \to \mathcal{X}_1$, the composition $X_1 \to \mathcal{X}_1 \to \mathcal{X}_2$ is a closed embedding.

1.6.10.   For future reference we also give the following definitions:

DEFINITION 1.6.11.

(a) *We shall say that a map from a reduced prestack $\mathcal{X}$ to a reduced affine scheme S is* ind-proper *(resp.,* ind-finite*) if $\mathcal{X}$ is a reduced ind-scheme and for any closed embedding $X \to \mathcal{X}$, where $X \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{qc}}$, the composite map $X \to S$ is proper (resp., finite).*

(b) *We shall say that a map of reduced prestacks $\mathcal{X}_1 \to \mathcal{X}_2$ is* (ind)-proper *(resp., (ind)-finite) if its base change by a reduced affine scheme yields a map which is (ind)-proper (resp., (ind)-finite).*

(c) *We shall say that a map of prestacks $\mathcal{X}_1 \to \mathcal{X}_2$ is* (ind)-proper *(resp., (ind)-finite) if the corresponding map ${}^{\mathrm{red}}\mathcal{X}_1 \to {}^{\mathrm{red}}\mathcal{X}_2$ is (ind)-proper (resp., (ind)-finite).*

1.6.12.   Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a map of reduced ind-schemes. It is easy to see that it is ind-proper (resp., ind-finite) if and only if the following is satisfied:

If

$$\mathcal{X}_1 := \underset{\alpha \in A}{\mathrm{colim}} \, X_{1,\alpha} \text{ and } \mathcal{X}_2 := \underset{\beta \in A}{\mathrm{colim}} \, X_{2,\beta},$$

then for every index α, and every/some index β for which $X_{1,\alpha} \to \mathcal{X}_1 \to \mathcal{X}_2$ factors as

$$X_{1,\alpha} \to X_{2,\beta} \to \mathcal{X}_2,$$

the map $X_{1,\alpha} \to X_{2,\beta}$ is proper (resp., finite).

1.7. Indschemes and the finite type condition. In this subsection we show that ind-schemes are nicely compatible with the 'locally almost of finite type' condition.

1.7.1.   We have the following assertion:

PROPOSITION 1.7.2. *Let $\mathcal{X}$ be an object of* $\mathrm{indSch}_{\mathrm{laft}}$. *Then the category* $(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed\ in}\ \mathcal{X}}$ *is filtered and the functor*

$$(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed\ in}\ \mathcal{X}} \to \mathrm{Sch}_{\mathrm{closed\ in}\ \mathcal{X}}$$

is cofinal.

The proof will be given in Sect. 2.7.

1.7.3.   From Proposition 1.7.2 we obtain:

COROLLARY 1.7.4. *An object* $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ *is an ind-scheme if and only if the following two conditions are satisfied:*

- *The functor* $(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed\ in}\ \mathcal{X}} \to (\mathrm{Sch}_{\mathrm{qc}})_{/\mathcal{X}}$ *is cofinal.*
- *The category* $(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed\ in}\ \mathcal{X}}$ *is filtered.*

PROOF. It is clear that the conditions of the proposition are sufficient for $\mathcal{X}$ to be an ind-scheme. The converse implication follows by combining Propositions 1.5.4 and 1.7.2, and the fact that a category cofinal in a filtered category is filtered. $\square$

Now, from Corollary 1.7.4 we obtain:

COROLLARY 1.7.5. *Let $\mathcal{X}$ be an object of* $\mathrm{indSch}_{\mathrm{laft}}$.
(a) *As an object of* PreStk, *it can be written as a* filtered *colimit*

$$(1.5) \qquad\qquad\qquad \mathcal{X} \simeq \mathrm{colim}_{\alpha} X_{\alpha},$$

where $X_{\alpha} \in \mathrm{Sch}_{\mathrm{aft}}$ and the maps $X_{\alpha_1} \to X_{\alpha_2}$ are closed embeddings.
(a') *The map*

$$\mathrm{colim}_{Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed\ in}\ \mathcal{X}}} Z \to \mathcal{X},$$

where the colimit is taken in PreStk, *is an isomorphism.*
(b) *The functors*

$$(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed\ in}\ \mathcal{X}} \to (\mathrm{Sch}_{\mathrm{aft}})_{/\mathcal{X}}$$

and

$$(\mathrm{Sch}_{\mathrm{aft}})_{/\mathcal{X}} \to (\mathrm{Sch}_{\mathrm{qc}})_{/\mathcal{X}}$$

are cofinal.

1.7.6.   The following theorem is a variant of Theorem 1.3.12 in the locally of finite type case.

THEOREM 1.7.7. *Let $\mathcal{X}$ be an object of* $\mathrm{PreStk}_{\mathrm{laft}}$, *which admits a connective deformation theory, and such that:*

- *For any $(S, x : S \to \mathcal{X}) \in \left(^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}\right)_{/\mathcal{X}}$, the object $H^0(T_x^*(\mathcal{X})) \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\heartsuit})$ is given by a surjective system.*
- *$^{\mathrm{red}}\mathcal{X}$ is a reduced ind-scheme.*

Then the following conditions are equivalent:
(a) *$\mathcal{X}$ is an ind-scheme;*
(b) *$^{\mathrm{cl}}\mathcal{X}$ is a classical ind-scheme;*
(c) *The map* $\mathrm{LKE}_{(^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to (^{cl}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}}(^{\mathrm{red}}\mathcal{X}) \to {}^{\mathrm{cl}}\mathcal{X}$, *is a* monomorphism.

(d) *For any $S \in {}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}$ and a map $x : S \to \mathcal{X}$, the object*

$$T_x^*({}^{\mathrm{red}}\mathcal{X}/\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^-)$$

belongs to $\mathrm{Pro}(\mathrm{QCoh}(S)^{\leq -1})$. *(Here by a slight abuse of notation, we denote by* ${}^{\mathrm{red}}\mathcal{X}$ *the ind-scheme obtained by the procedure of Corollary 1.4.5.)*

The proof will be given in Sect. 2.6.

1.8. Nil-schematic ind-schemes. In this subsection we study ind-schemes, whose underlying reduced ind-scheme is a scheme[4].

1.8.1.   We give the following definition:

DEFINITION 1.8.2.   *We shall say that an ind-scheme $\mathcal{X}$ is nil-schematic if the reduced ind-scheme* ${}^{\mathrm{red}}\mathcal{X}$ *is a scheme.*

1.8.3.   For $\mathcal{X} \in \mathrm{PreStk}$ let

$$\mathrm{PreStk}_{\text{nilp-emb into } \mathcal{X}} \subset \mathrm{PreStk}_{/\mathcal{X}}$$

be the full subcategory spanned by objects $f : \mathcal{X}' \to \mathcal{X}$ for which f is a nilpotent embedding.

We will use a similar notation for full subcategories of PreStk, e.g.,

$$\mathrm{Sch}_{\text{nilp-emb to } \mathcal{X}} \subset \mathrm{Sch}_{/\mathcal{X}},$$

etc.

1.8.4.   In this subsection we will prove the following:

PROPOSITION 1.8.5.   *Let $\mathcal{X}$ be a nil-schematic ind-scheme locally almost of finite type. Then the category* $(\mathrm{Sch}_{\mathrm{aft}})_{\text{nilp-emb into } \mathcal{X}}$ *is filtered, and the functor*

$$(\mathrm{Sch}_{\mathrm{aft}})_{\text{nilp-emb into } \mathcal{X}} \to (\mathrm{Sch}_{\mathrm{aft}})_{\text{closed in } \mathcal{X}}$$

is cofinal.

As a formal consequence we obtain:

COROLLARY 1.8.6.   *Let $\mathcal{X}$ be a nil-schematic ind-scheme locally almost of finite type.*

(a) *As an object of* PreStk*, it can be written as a* filtered *colimit*

(1.6) $$\mathcal{X} \simeq \operatornamewithlimits{colim}_{\alpha} X_{\alpha},$$

where $X_{\alpha} \in \mathrm{Sch}_{\mathrm{aft}}$ and the maps $X_{\alpha_1} \to X_{\alpha_2}$ are nilpotent embeddings.

(a') *The map*

$$\operatornamewithlimits{colim}_{Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\text{nilp-emb into } \mathcal{X}}} Z \to \mathcal{X},$$

where the colimit is taken in PreStk*, is an isomorphism.*

(a'') *The category* $(\mathrm{Sch}_{\mathrm{aft}})_{\text{nilp-emb into } \mathcal{X}}$ *is filtered.*

(b) *The functor*

$$(\mathrm{Sch}_{\mathrm{aft}})_{\text{nilp-emb into } \mathcal{X}} \to (\mathrm{Sch}_{\mathrm{qc}})_{/\mathcal{X}}$$

is cofinal.

---

[4]Elsewhere in the literature, such ind-schemes are called 'formal schemes'. We do not use this terminology to avoid clashing with other usages of the word 'formal'.

1.8.7. *Proof of Proposition 1.8.5.* Since the category $(\mathrm{Sch}_{\mathrm{aft}})_{\text{closed in } \mathcal{X}}$ is filtered, it suffices to show that for any object

$$(Z \to \mathcal{X}) \in (\mathrm{Sch}_{\mathrm{aft}})_{\text{closed in } \mathcal{X}},$$

there exists a factorization

$$Z \to W \to \mathcal{X},$$

where $(W \to \mathcal{X}) \in (\mathrm{Sch}_{\mathrm{aft}})_{\text{nilp-emb into } \mathcal{X}}$.

Note that since Z is almost of finite type, the map $^{\mathrm{red}}Z \to Z$ is a nilpotent embedding. The sought-for scheme W is constructed as

$$Z \underset{^{\mathrm{red}}Z}{\sqcup} {}^{\mathrm{red}}\mathcal{X},$$

using Chapter 1, Corollary 7.2.3.

$\square$

2. Proofs of results concerning ind-schemes

2.1. Proof of Theorem 1.3.12, Plan. Let $\mathcal{X}$ and $\mathcal{X}_0$ be as in Theorem 1.3.12.

2.1.1. We consider the following full subcategory of $(\mathrm{Sch}_{\mathrm{qc}})_{/\mathcal{X}}$, to be denoted A. Its objects are those

$$Z \to \mathcal{X},$$

for which there *exists* a commutative diagram

$$
\begin{array}{ccc}
Z_0 & \longrightarrow & \mathcal{X}_0 \\
\downarrow & & \downarrow \\
Z & \longrightarrow & \mathcal{X},
\end{array}
$$

where $Z_0 \in {}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{qc}}$, the map $Z_0 \to Z$ is a nilpotent embedding, and $Z_0 \to \mathcal{X}_0$ is a closed embedding.

Note that this condition implies that the map $Z \to \mathcal{X}$ is nil-closed. In particular, any map $Z_1 \to Z_2$ in A is nil-closed, and hence affine.

2.1.2. Let B be a full subcategory of A, consisting of those

$$x : Z \to \mathcal{X}$$

that satisfy the following condition:

The map $(dx)^* : T_x^*(\mathcal{X}) \to T^*(Z)$ induces a surjection $H^0(T_x^*(\mathcal{X})) \twoheadrightarrow H^0(T^*(Z))$.

2.1.3. We will prove Theorem 1.3.12 by establishing the following facts:

(1) Any map $Z_1 \to Z_2$ in B is a closed embedding;
(2) The category B is filtered;
(3) The map $\underset{(Z \to \mathcal{X}) \in B}{\mathrm{colim}} Z \to \mathcal{X}$ is an isomorphism.

2.2. Step 1: proof that the maps are closed embeddings. We will prove a slightly stronger assertion: any map $Z_1 \to Z_2$, where $Z_1 \in B$ and $Z_2 \in A$, is a closed embedding.

2.2.1.   Let $Z_1 \to Z_2$ be a map in A. Consider the corresponding nilpotent embeddings $Z_{0,i} \hookrightarrow Z_i$, $Z_{0,i} \in {}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{qc}}$, $i = 1, 2$.

Let Z_0 be the intersection of the closed subschemes $Z_{0,1}$ and ${}^{\mathrm{cl}}Z_1 \underset{{}^{\mathrm{cl}}Z_2}{\times} Z_{0,2}$ in ${}^{\mathrm{cl}}Z_1$.

The map $Z_0 \to Z_{0,1}$ is a nilpotent embedding. The map $Z_0 \to Z_{0,2}$ is a closed embedding, because the composition $Z_0 \to Z_{0,2} \to \mathcal{X}_0$ is.

2.2.2.   The assertion of Step 1 follows now from the following general statement:

Let

$$
\begin{array}{ccc}
 & Z_0 & \\
g_1 \swarrow & & \searrow g_2 \\
Z_1 & \xrightarrow{\quad f \quad} & Z_2,
\end{array}
$$

be a diagram in $\mathrm{Sch}_{\mathrm{qc}}$, where g_2 is a closed embedding, and g_1 a nilpotent embedding. We have:

PROPOSITION 2.2.3.   *The following conditions are equivalent:*

(a) *f is a closed embedding;*

(b) *f is a monomorphism when evaluated on classical affine schemes;*

(c) *The map $T_{g_2}^*(Z_2) \to T_{g_1}^*(Z_1)$, induced by $(df)^*$, gives rise to a surjection*

$$H^0(T_{g_2}^*(Z_2)) \to H^0(T_{g_1}^*(Z_1)).$$

The rest of this subsection is devoted to the proof of Proposition 2.2.3.

2.2.4.   Clearly, (a) implies (b) and (b) implies (c). Let us show that (c) implies (a). Clearly, the statement reduces to one about classical schemes. So, we can assume that Z_0, Z_1 and Z_2 are classical.

Let Z_0 be given in Z_1 by an ideal that vanishes to the power n. We will argue by induction on n, starting with $n = 2$.

2.2.5.   For $n = 2$, the map $Z_0 \to Z_1$ is a square-zero extension, say by $\mathcal{I}_1 \in \mathrm{QCoh}(Z_0)^\heartsuit$. Replacing Z_2 by the classical 1st infinitesimal neighborhood of Z_0, we can assume that $Z_0 \to Z_2$ is also a square-zero extension, say by $\mathcal{I}_2 \in \mathrm{QCoh}(Z_0)^\heartsuit$. We have:

$$H^{-1}(T^*(Z_0/Z_i)) \simeq \mathcal{I}_i.$$

We have a map of exact sequences

$$
\begin{array}{ccccccc}
H^{-1}(T^*(Z_0)) & \longrightarrow & H^{-1}(T^*(Z_0/Z_2)) & \longrightarrow & H^0(T_{g_2}^*(Z_2)) & \longrightarrow & H^0(T^*(Z_0)) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
H^{-1}(T^*(Z_0)) & \longrightarrow & H^{-1}(T^*(Z_0/Z_1)) & \longrightarrow & H^0(T_{g_1}^*(Z_1)) & \longrightarrow & H^0(T^*(Z_0)),
\end{array}
$$

hence assumption (c) implies that $\mathcal{I}_2 \to \mathcal{I}_1$ is surjective, as required.

2.2.6. To carry out the induction step, let $Z_{1/2}$ be a closed subscheme of Z_1 such that

$$Z_0 \subset Z_{1/2} \subset Z_1,$$

and such that the ideal of Z_0 in $Z_{1/2}$ and the ideal of $Z_{1/2}$ in Z_1 vanish to a smaller power.

Note that the assumption of (c) holds for the map $Z_{1/2} \to Z_2$. Hence, by induction hypothesis applied to

the map $Z_{1/2} \to Z_2$ is a closed embedding.

We now apply the induction hypothesis to

and deduce that $Z_1 \to Z_2$ is a closed embedding.

$\square$

2.3. Step 2: construction of a left adjoint. In order to proceed with the proof of Theorem 1.3.12 we will now show that the inclusion

$$B \hookrightarrow A$$

admits a left adjoint.

2.3.1. Thus, given an object $Z \to \mathcal{X}$ of A, we need to show that the category $\mathbf{D}(Z)$ of factorizations

$$Z \to Z' \to \mathcal{X},$$

where $(Z' \to \mathcal{X}) \in B$, admits an initial object.

2.3.2. We first reduce to the case of classical schemes. Indeed, let

$$^{\mathrm{cl}}Z \to Z'_{\mathrm{cl}} \to \mathcal{X}$$

be the initial object in the category $\mathbf{D}(^{\mathrm{cl}}Z)$ (in this case Z'_{cl} is automatically classical). Then the object

$$Z \underset{^{\mathrm{cl}}Z}{\sqcup} Z'_{\mathrm{cl}}$$

is initial in $\mathbf{D}(Z)$.

2.3.3.   From now, until the end of Step 2, all schemes will be classical, and we shall
sometimes omit "cl" from the notation.

By assumption, there exists a diagram

$$
\begin{array}{ccc}
Z_0 & \longrightarrow & \mathcal{X}_0 \\
\downarrow & & \downarrow \\
Z & \overset{x}{\longrightarrow} & \mathcal{X},
\end{array}
$$

where $Z_0 \to Z$ is a nilpotent embedding, and $Z_0 \to \mathcal{X}_0$ is a closed embedding (and
in particular, a monomorphism when evaluated on classical schemes).

Note that for any $(Z \to Z' \to \mathcal{X}) \in \mathbf{D}(Z)$, the composed map $Z_0 \to Z'$ is a
closed embedding (e.g., by Step 1).

Let $Z_0 \hookrightarrow Z$ be given by the ideal that vanishes to the power n. We will argue
by induction on n, starting with $n = 2$.

2.3.4.   Thus, we first assume that $Z_0 \hookrightarrow Z$ is a square-zero extension. Let

$$
\mathbf{D}(Z)_{\mathrm{SqZ}} \subset \mathbf{D}(Z)
$$

be the full subcategory spanned by those objects, for which $Z_0 \to Z'$ is a square-
zero extension. (Note that since we are working with classical schemes, being a
square-zero extension is a property and not an extra structure.)

Note that the embedding $\mathbf{D}(Z)_{\mathrm{SqZ}} \hookrightarrow \mathbf{D}(Z)$ admits a right adjoint, which sends
Z' to the classical 1st infinitesimal neighborhood of Z_0 in Z'. Hence, it is enough
to show that $\mathbf{D}(Z)_{\mathrm{SqZ}}$ admits an initial object.

2.3.5.   Denote $x_0 = x|_{Z_0}$. The data of a square-zero extension $Z_0 \hookrightarrow Z$ and a map
x, extending x_0 is given by the data of $\mathcal{I} \in \mathrm{QCoh}(Z_0)^\heartsuit$ and a map

$$
(2.1) \qquad\qquad \mathrm{coFib}(T^*_{x_0}(\mathcal{X}) \to T^*(Z_0))[-1] \to \mathcal{I}.
$$

By assumption, x_0 is a monomorphism. This implies that

$$
(dx_0)^* : T^*_{x_0}(\mathcal{X}) \to T^*(Z_0)
$$

induces a surjection

$$
(2.2) \qquad\qquad H^0(T^*_{x_0}(\mathcal{X})) \twoheadrightarrow H^0(T^*(Z_0)).
$$

Hence,

$$
T^*(Z_0/\mathcal{X})[-1] = \mathrm{coFib}(T^*_{x_0}(\mathcal{X}) \to T^*(Z_0))[-1] \in \mathrm{Pro}(\mathrm{QCoh}(Z_0)^{\geq 0}).
$$

Now, the assumption on $T^*(\mathcal{X})$ implies that

$$
H^0\left(\mathrm{coFib}(T^*_{x_0}(\mathcal{X}) \to T^*(Z_0))[-1]\right)
$$

is also given by a surjective family.

Hence, the map (2.1) canonically factors as

$$
\mathrm{coFib}(T^*_{x_0}(\mathcal{X}) \to T^*(Z_0))[-1] \to \mathcal{I}' \to \mathcal{I},
$$

where

$$
H^0\left(T^*(Z_0/\mathcal{X})[-1]\right) = H^0\left(\mathrm{coFib}(T^*_{x_0}(\mathcal{X}) \to T^*(Z_0))[-1]\right) \to \mathcal{I}'
$$

is surjective.

Let Z' be the square-zero extension of Z_0 that corresponds to

$$\mathrm{coFib}(T^*_{x_0}(\mathcal{X}) \to T^*(Z_0))[-1] \to \mathcal{I}'.$$

It is easy to see that Z' is the initial object in $\mathbf{D}(Z)_{\mathrm{SqZ}}$.

2.3.6. We are now ready to carry out the induction step. Choose a classical sub-scheme $Z_{1/2}$

$$Z_0 \hookrightarrow Z_{1/2} \hookrightarrow Z,$$

such that

$$Z_{1/2} \hookrightarrow Z$$

is a square-zero extension, and the ideal of Z_0 in $Z_{1/2}$ vanishes to a smaller power.

By the induction hypothesis, the category $\mathbf{D}(Z_{1/2})$ admits an initial object, denote it by $Z'_{1/2}$. Denote

$$\widetilde{Z} := Z \underset{Z_{1/2}}{\sqcup} Z'_{1/2},$$

and let

$$\widetilde{x} : \widetilde{Z} \to \mathcal{X}$$

denote the resulting map.

Note that

$$Z'_{1/2} \hookrightarrow \widetilde{Z}$$

is a square-zero extension. Hence, by Sects. 2.3.4 and 2.3.5, the category $\mathbf{D}(\widetilde{Z})$ admits an initial object. Indeed, the proof only used the fact that (2.2) was surjective, which is satisfied for the map $Z'_{1/2} \to \mathcal{X}$ by construction.

Let

$$\widetilde{Z} \to Z' \to \mathcal{X}$$

an initial object of $\mathbf{D}(\widetilde{Z})$. It is easy to see that the resulting object

$$Z \to Z' \to \mathcal{X}$$

is the initial one in $\mathbf{D}(Z)$.

2.4. Step 3: proof of filteredness.
2.4.1. We consider two auxilliary categories. We let B' be the category

$$(^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{qc}})_{\text{closed in } \mathcal{X}_0}.$$

We let B'' be the category of commutative diagrams

$$
\begin{array}{ccc}
Z_0 & \longrightarrow & \mathcal{X}_0 \\
\downarrow & & \downarrow \\
Z & \longrightarrow & \mathcal{X},
\end{array}
$$

where $(Z \to \mathcal{X}) \in B$, $(Z_0 \to \mathcal{X}_0) \in B'$, and $Z_0 \to Z$ is a nilpotent embedding.

We have the naturally defined forgetful functors:

$$B \leftarrow B'' \to B'.$$

2.4.2.   Note that the category B' is filtered by the assumption on $\mathcal{X}_0$. We will now show that the category B'' is filtered as well.

We will use the following general assertion:

LEMMA 2.4.3. *Let $F : \mathbf{C} \to \mathbf{D}$ be a co-Cartesian fibration. Assume that $\mathbf{D}$ is filtered and that the fibers of F are also filtered. Then $\mathbf{C}$ is filtered.*

We claim that the above lemma is applicable to the above functor

$$B'' \to B'.$$

This would imply that B'' is filtered.

2.4.4.   Let us show that $B'' \to B'$ is a co-Cartesian fibration. Given a diagram

$$
\begin{array}{ccc}
Z_0^1 & \longrightarrow & \mathcal{X}_0 \\
\downarrow & & \downarrow \\
Z^1 & \longrightarrow & \mathcal{X},
\end{array}
$$

and a map $Z_0^1 \to Z_0^2$ we construct the sought-for object

$$
\begin{array}{ccc}
Z_0^2 & \longrightarrow & \mathcal{X}_0 \\
\downarrow & & \downarrow \\
Z^2 & \longrightarrow & \mathcal{X}
\end{array}
$$

as follows. First, we set

$$\widetilde{Z}_2 := Z^1 \underset{Z_0^1}{\sqcup} Z_0^2,$$

which is equipped with a canonical map to $\mathcal{X}$.

We have $(\widetilde{Z}_2 \to \mathcal{X}) \in A$ and the required object $(Z_2 \to \mathcal{X}) \in B$ is obtained by applying the left adjoint $A \to B$ constructed in Step 2.

2.4.5.   Let us now show that the fiber of B'' over a given object $(Z_0 \to \mathcal{X}_0) \in B'$ is filtered. We claim that the fiber in question admits coproducts and push-outs.

For products, given two objects Z^1 and Z^2, we take

$$\widetilde{Z} := Z^1 \underset{Z_0}{\sqcup} Z^2,$$

which is equipped with a canonical map to $\mathcal{X}$.

We have $(\widetilde{Z} \to \mathcal{X}) \in A$ and the sought-for coproduct in B is obtained by applying the left adjoint $A \to B$.

The proof for push-outs is similar (using the fact that all maps in B are closed embeddings).

2.4.6. Thus, we have shown that B'' is filtered. To prove that B is filtered, we will use the following general statement:

LEMMA 2.4.7. *Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between $(\infty, 1)$-categories. Assume that F is cofinal and $\mathbf{C}$ is filtered. Then $\mathbf{D}$ is filtered.*

We claim that the above lemma is applicable to the functor $B'' \to B$. This would imply that B is filtered.

We have the following general statement:

LEMMA 2.4.8. *Let $F : \mathbf{C} \to \mathbf{D}$ be a Cartesian fibration. Then F is cofinal if and only if it has contractible fibers.*

Hence, it is enough to show that $B'' \to B$ is a Cartesian fibration and that it has contractible fibers.

The fact that $B'' \to B$ is a Cartesian fibration is obvious via the formation of fiber products (again, using the fact that any map in B is a closed embedding).

The fact that the fibers of $B'' \to B$ are contractible is proved in Sect. 2.5.4 below.

2.5. Step 4: proof of the isomorphism. We will now show that the map

$$\operatorname*{colim}_{(Z \to \mathcal{X}) \in B} Z \to \mathcal{X}$$

is an isomorphism, thereby proving Theorem 1.3.12.

2.5.1. We need to show that for $S \in \mathrm{Sch}^{\mathrm{aff}}$ and a map $S \to \mathcal{X}$, the category $\mathbf{C}$ of factorizations

$$S \to Z \to \mathcal{X},$$

with $(Z \to \mathcal{X}) \in B$ is contractible.

We introduce several auxilliary categories.

2.5.2. We let $\mathbf{C}'$ be the category of diagrams

(2.3)
$$
\begin{array}{ccc}
Z_0 & \longrightarrow & \mathcal{X}_0 \\
\downarrow & & \downarrow \\
S \longrightarrow Z & \longrightarrow & \mathcal{X},
\end{array}
$$

where $(Z \to \mathcal{X}) \in B$, $Z_0 \in {}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{qc}}$, the map $Z_0 \to Z$ is a nilpotent embedding, and $Z_0 \to \mathcal{X}_0$ is a closed embedding.

We let $\mathbf{C}''$ be the category of diagrams

$$
\begin{array}{ccccc}
S_0 & \longrightarrow & Z_0 & \longrightarrow & \mathcal{X}_0 \\
\downarrow & & \downarrow & & \downarrow \\
S & \longrightarrow & Z & \longrightarrow & \mathcal{X},
\end{array}
$$

where $Z \to \mathcal{X}$, $Z_0 \to Z$, $Z_0 \to \mathcal{X}_0$ are as above, $S_0 \in {}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{qc}}$, and $S_0 \to S$ is a nilpotent embedding.

We let $\mathbf{C}'''$ denote the category of diagrams

(2.4)
$$
\begin{array}{ccccc}
S_0 & \longrightarrow & Z_0 & \longrightarrow & \mathcal{X}_0 \\
\downarrow & & & & \downarrow \\
S & \longrightarrow & & & \mathcal{X}
\end{array}
$$

where $S_0 \to S$ and $Z_0 \to \mathcal{X}_0$ are as above.

We let $\mathbf{C}''''$ denote the category of diagrams

(2.5)
$$
\begin{array}{ccc}
S_0 & \longrightarrow & \mathcal{X}_0 \\
\downarrow & & \downarrow \\
S & \longrightarrow & \mathcal{X},
\end{array}
$$

where $S_0 \to S$ is as above.

2.5.3.   We have the forgetful functors

$$\mathbf{C} \leftarrow \mathbf{C}' \leftarrow \mathbf{C}'' \to \mathbf{C}''' \to \mathbf{C}''''.$$

We claim that all of the above functors are homotopy equivalences and that $\mathbf{C}''''$ is contractible. This will imply that that $\mathbf{C}$ is contracible.

2.5.4.   The functor $\mathbf{C}' \to \mathbf{C}$ is a Cartesian fibration (via the formation of fiber products). Hence, in order to show that it is a homotopy equivalence, it suffices to show that it has contractible fibers.

The fiber of $\mathbf{C}' \to \mathbf{C}$ over an object $(S \to Z \to \mathcal{X}) \in \mathbf{C}$ is the category of ways to complete

$$
\begin{array}{ccc}
& & \mathcal{X}_0 \\
& & \downarrow \\
Z & \longrightarrow & \mathcal{X}
\end{array}
$$

to a commutative diagram

$$
\begin{array}{ccc}
Z_0 & \longrightarrow & \mathcal{X}_0 \\
\downarrow & & \downarrow \\
Z & \longrightarrow & \mathcal{X},
\end{array}
$$

where $Z_0 \to Z$ is a nilpotent embedding, and $Z_0 \to \mathcal{X}_0$ is a closed embedding.

The assumption that $(Z \to \mathcal{X})$ belongs to A means that the above category is non-empty. To prove that this category is contractible, it is sufficient to show that it contains products. These are given by intersecting the corresponding closed subschemes inside ^{cl}Z (here we use the fact that $\mathcal{X}_0 \to \mathcal{X}$ is a monomorphism of classical prestacks).

2.5.5. The functor $\mathbf{C}'' \to \mathbf{C}'$ is a Cartesian fibration. Hence, in order to show that it is a homotopy equivalence, it suffices to show that it has contractible fibers.

The fiber of $\mathbf{C}'' \to \mathbf{C}'$ over an object (2.3) is the category of fillings of

$$
\begin{array}{ccc}
& & Z_0 \\
& & \downarrow \\
S & \longrightarrow & Z
\end{array}
$$

to a commutative diagram

$$
\begin{array}{ccc}
S_0 & \longrightarrow & Z_0 \\
\downarrow & & \downarrow \\
S & \longrightarrow & Z,
\end{array}
$$

where $S_0 \to S$ is a nilpotent embedding. This category is contractible, because it contains the final object, namely, $S_0 := S \underset{Z}{\times} Z_0$.

2.5.6. Coniser the functor $\mathbf{C}'' \to \mathbf{C}'''$. We claim that it is a co-Cartesian fibration. Indeed, given a map from a diagram

$$
\begin{array}{ccccc}
S_{0,1} & \longrightarrow & Z_{0,1} & \longrightarrow & \mathcal{X}_0 \\
\downarrow & & & & \downarrow \\
S & & \longrightarrow & & \mathcal{X}
\end{array}
$$

to a diagram

$$
\begin{array}{ccccc}
S_{0,2} & \longrightarrow & Z_{0,2} & \longrightarrow & \mathcal{X}_0 \\
\downarrow & & & & \downarrow \\
S & & \longrightarrow & & \mathcal{X},
\end{array}
$$

and a diagram

$$
\begin{array}{ccccc}
S_{0,1} & \longrightarrow & Z_{0,1} & \longrightarrow & \mathcal{X}_0 \\
\downarrow & & \downarrow & & \downarrow \\
S & \longrightarrow & Z_1 & \longrightarrow & \mathcal{X},
\end{array}
$$

we construct the corresponding diagram

$$
\begin{array}{ccccc}
S_{0,2} & \longrightarrow & Z_{0,2} & \longrightarrow & \mathcal{X}_0 \\
\downarrow & & \downarrow & & \downarrow \\
S & \longrightarrow & Z_2 & \longrightarrow & \mathcal{X}
\end{array}
$$

as follows.

Set $\widetilde{Z}_2 := Z_1 \underset{Z_{0,1}}{\sqcup} Z_{0,2}$. We have $(Z_{0,2} \to \widetilde{Z}_2) \in A$. The sought-for object $(Z_{0,2} \to Z_2) \in B$ is obtained from $Z_{0,2} \to \widetilde{Z}_2$ by applying the left adjoint functor to $B \hookrightarrow A$ from Step 2.

2.5.7.   Hence, in order to show that $\mathbf{C}'' \to \mathbf{C}'''$ is a homotopy equivalence, it suffices to show that it has contractible fibers.

The fiber of $\mathbf{C}'' \to \mathbf{C}'''$ over an object (2.4) is the category of factorizations of the map

$$S \underset{S_0}{\sqcup} Z_0 \to \mathcal{X}$$

as

$$S \underset{S_0}{\sqcup} Z_0 \to Z \to \mathcal{X},$$

where the composition

$$Z_0 \to S \underset{S_0}{\sqcup} Z_0 \to Z$$

is a nilpotent embedding, and $(Z \to \mathcal{X}) \in B$.

We claim that the above category of factorizations contains an initial object. Indeed, set $\widetilde{Z} := S \underset{S_0}{\sqcup} Z_0$, where the formation of the push-out is well-behaved because the map $S_0 \to Z_0$ is affine (recall that all our schemes were assumed separated).

We have $(\widetilde{Z} \to \mathcal{X}) \in A$. Now, the sought-for initial object is obtained by applying to $\widetilde{Z}$ the left adjoint to $B \hookrightarrow A$.

2.5.8.   The functor $\mathbf{C}''' \to \mathbf{C}''''$ is a Cartesian fibration. Hence, in order to show that it is a homotopy equivalence, it suffices to show that it has contractible fibers.

The fiber of $\mathbf{C}''' \to \mathbf{C}''''$ over an object (2.5) is the category of factorizations of $S_0 \to \mathcal{X}_0$ via a closed embedding

$$S_0 \to Z_0 \to \mathcal{X}.$$

The latter category contractible by the assumption that $\mathcal{X}_0$ is a classical ind-scheme.

2.5.9.   Finally, we claim that $\mathbf{C}''''$ is contractible. Indeed, it is non-empty by the assumption that $\mathcal{X}_0 \to \mathcal{X}$ is a pseudo-nilpotent embedding. Furthermore, it contains finite products: these are obtained by intersecting the corresponding closed subschemes in $^{\mathrm{cl}}S$ as in Sect. 2.5.4.

$$\square \text{(Theorem 1.3.12)}$$

2.6. Proof of Theorem 1.7.7.

2.6.1.   The implication (a) $\Rightarrow$ (b) is tautological.

Note also that at this point we also know that (b) implies (a): this follows from Corollary 1.3.13.

2.6.2.   The implication (b) $\Rightarrow$ (c) is easy: we need to show that for $S \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}$, the map

$$\mathrm{Maps}(S, {}^{\mathrm{red}}\mathcal{X}) \to \mathrm{Maps}(S, {}^{\mathrm{cl}}\mathcal{X})$$

is a monomorphism of groupoids.

Writing $^{\mathrm{cl}}\mathcal{X} = \underset{\alpha \in A}{\mathrm{colim}}\, X_\alpha$ with $X_\alpha \in {}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{qc}}$ and A filtered, the above map becomes

$$\underset{\alpha \in A}{\mathrm{colim}}\, \mathrm{Maps}(S, {}^{\mathrm{red}}X_\alpha) \to \mathrm{Maps}(S, X_\alpha),$$

which is an injection (of sets), since A is filtered.

2.6.3. We now claim that (c) implies (a). Indeed, we apply Theorem 1.3.12 to $\mathcal{X}_0 = {}^{\mathrm{red}}\mathcal{X}$. Thus, we only have to show that ${}^{\mathrm{red}}\mathcal{X} \to \mathcal{X}$ is a pseudo-nilpotent embedding. However, this follows from Chapter 1, Lemma 8.1.5.

2.6.4. The implication (c) $\Rightarrow$ (d) is tautological from the definition of pro-cotangent spaces. Hence, it remains to show that (d) implies (c).

By Volume I, Chapter 2, Lemma 1.6.8, the functor

$$\mathrm{LKE}_{({}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \hookrightarrow ({}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}}$$

commutes with finite limits, and in particular, preserves monomorphisms. Hence, it is sufficient to show that for $S \in {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, the map

$$\mathrm{Maps}(S, {}^{\mathrm{red}}\mathcal{X}) \to \mathrm{Maps}(S, \mathcal{X})$$

is a monomorphism of groupoids.

The map

$$\mathrm{Maps}(S_0, {}^{\mathrm{red}}\mathcal{X}) \to \mathrm{Maps}(S_0, \mathcal{X})$$

is a monomorphism (in fact, an isomorphism) for $S_0 = {}^{\mathrm{red}}S$.

Since S is of finite type, there exists a finite sequence of square-zero extensions

$${}^{\mathrm{red}}S = S_0 \hookrightarrow S_1 \hookrightarrow \dots \hookrightarrow S_n = S.$$

We will show by induction that the maps

$$\mathrm{Maps}(S_i, {}^{\mathrm{red}}\mathcal{X}) \to \mathrm{Maps}(S_i, \mathcal{X})$$

are monomorphisms.

2.6.5. The case $i = 0$ has been considered above. To carry out the induction step, we need to show that for any $x_i : S_i \to {}^{\mathrm{red}}\mathcal{X}$, the map

$$\{x_i\} \underset{\mathrm{Maps}(S_i, {}^{\mathrm{red}}\mathcal{X})}{\times} \mathrm{Maps}(S_{i+1}, {}^{\mathrm{red}}\mathcal{X}) \to \{x_i\} \underset{\mathrm{Maps}(S_i, \mathcal{X})}{\times} \mathrm{Maps}(S_{i+1}, \mathcal{X})$$

is a monomorphism.

Let the square-zero extension $S_i \hookrightarrow S_{i+1}$ be given by an object

$$\mathcal{I} \in \mathrm{QCoh}(S_i)^{\heartsuit}_{T^*(S_i)[-1]/}.$$

Then the groupoid

$$\{x_i\} \underset{\mathrm{Maps}(S_i, {}^{\mathrm{red}}\mathcal{X})}{\times} \mathrm{Maps}(S_{i+1}, {}^{\mathrm{red}}\mathcal{X})$$

identifies with that of null-homotopies of the composition

$$T^*_{x_i}({}^{\mathrm{red}}\mathcal{X}) \to T^*(S_i) \to \mathcal{I}[1],$$

while the groupoid

$$\{x_i\} \underset{\mathrm{Maps}(S_i, \mathcal{X})}{\times} \mathrm{Maps}(S_{i+1}, \mathcal{X})$$

identifies with that of null-homotopies of the composition

$$T^*_{x_i}(\mathcal{X}) \to T^*(S_i) \to \mathcal{I}[1].$$

Hence, the required monomorphism property follows from the fact that

$$T^*_{x_i}({}^{\mathrm{red}}\mathcal{X}/\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S_i)^{\leq -1}),$$

which in turn follows from condition (d) and the fact that $S_0 \to S_i$ is a nilpotent embedding.

$$\square\text{(Theorem 1.7.7)}$$

2.7. Proof of Proposition 1.7.2.

2.7.1. *Step 0.* Since the category $\mathrm{Sch}_{\text{closed in } \mathcal{X}}$ is filtered, in order to prove the proposition, it is sufficient to show that every closed embedding $Z \to \mathcal{X}$ can be factored as

$$Z \to \widetilde{Z} \to \mathcal{X},$$

where $\widetilde{Z} \in \mathrm{Sch}_{\text{aft}}$ and $\widetilde{Z} \to \mathcal{X}$ is a closed embedding.

Given a closed embedding $Z \to \mathcal{X}$, we will construct a *compatible* system of factorizations

$$^{\leq n}Z \to \widetilde{Z}_n \to \mathcal{X}, \quad \widetilde{Z}_n \in \left(^{\leq n}\mathrm{Sch}_{\text{ft}}\right)_{\text{closed in } \mathcal{X}}, \quad {}^{\leq n-1}\widetilde{Z}_n \simeq \widetilde{Z}_{n-1}.$$

We shall proceed by induction on n, starting from $n = 0$.

2.7.2. *Step 1.* We claim that $^{\text{cl}}Z$ is already of finite type. I.e., we claim that the functor

$$\left(^{\text{cl}}\mathrm{Sch}_{\text{aft}}\right)_{\text{closed in } \mathcal{X}} \to \left(^{\text{cl}}\mathrm{Sch}_{\text{qc}}\right)_{\text{closed in } \mathcal{X}}$$

is an equivalence. Indeed, write

$$^{\text{cl}}\mathcal{X} \simeq \operatorname*{colim}_{\alpha} X_\alpha, \quad X_\alpha \in {}^{\text{cl}}\mathrm{Sch}_{\text{qc}}.$$

Since a closed classical subscheme of a classical scheme of finite type is itself of finite type, it suffices to show that all X_α are of finite type.

Recall the following characterization of classical schemes of finite type: $X \in {}^{\text{cl}}\mathrm{Sch}_{\text{qc}}$ is of finite type if and only of for a classical commutative k-algebra R and a filtered family of *subalgebras* R_i with $\underset{i}{\cup} R_i = R$, the map

$$\operatorname*{colim}_{i} \mathrm{Maps}(\mathrm{Spec}(R_i), X) \to \mathrm{Maps}(\mathrm{Spec}(R), X)$$

is an equivalence.

For a R and R_i as above, and any index α, the diagram

$$
\begin{array}{ccc}
\operatorname*{colim}_{i} \mathrm{Maps}(\mathrm{Spec}(R_i), X_\alpha) & \longrightarrow & \operatorname*{colim}_{i} \mathrm{Maps}(\mathrm{Spec}(R_i), \mathcal{X}) \\
\downarrow & & \downarrow \\
\mathrm{Maps}(\mathrm{Spec}(R), X_\alpha) & \longrightarrow & \mathrm{Maps}(\mathrm{Spec}(R), \mathcal{X})
\end{array}
$$

is a pullback square. Hence, the fact that the right vertical arrow is an isomorphism implies that the right vertical arrow is an isomorphism, as required.

2.7.3. *Step 2.* We shall now carry out the induction step. Assume that

$$^{\leq n}Z \to \widetilde{Z}_n \to \mathcal{X}$$

has been constructed.

Set

$$\widetilde{Z}'_{n+1} := \widetilde{Z}_n \underset{\leq n Z}{\sqcup} {}^{\leq n+1}Z.$$

By Chapter 1, Proposition 5.4.2, the morphism $^{\leq n}Z \to {}^{\leq n+1}Z$ has a (canonical) structure of square-zero extension. Hence, the morphism

$$\widetilde{Z}_n \to \widetilde{Z}'_{n+1}$$

also has a structure of square-zero extension, by an ideal $\mathcal{I}' \in \mathrm{QCoh}(\widetilde{Z}_n)^\heartsuit[n+1]$.

Since the morphism $^{\leq n}Z \to \widetilde{Z}_n$ is affine, we have a canonical map

$$\widetilde{Z}'_{n+1} \to \mathcal{X}.$$

We need to factor the latter morphism as

$$\widetilde{Z}'_{n+1} \to \widetilde{Z}_{n+1} \to \mathcal{X},$$

where $\widetilde{Z}_{n+1} \in {}^{\leq n+1}\mathrm{Sch}_{\mathrm{ft}}$, and $\widetilde{Z}_n \to {}^{\leq n}\widetilde{Z}_{n+1}$ is an isomorphism.

We claim that we can find such a $\widetilde{Z}_{n+1}$ so that $\widetilde{Z}_n \to {}^{\leq n}\widetilde{Z}_{n+1}$ is a square-zero extension by an ideal $\mathcal{I} \in \mathrm{Coh}(\widetilde{Z}_n)^\heartsuit[n+1]$.

This follows by the argument in Step 3 of the proof of Chapter 1, Theorem 9.1.2

$$\square\text{(Proposition 1.7.2)}$$

3. (Ind)-inf-schemes

Ind-inf-schemes are our primary object of study. These are the algebro-geometric spaces on which the category IndCoh is defined along with the operations of !-pullback and *-pushforward; in this respect they behave much in the same way as schemes (the main difference is the absense of t-structure); we will develop this in Chapter 3. In addition, it turns out that ind-inf-schemes are well-adapted to a lot of formal differential geometry, as we shall see in Chapter 8 and Chapter 9.

What is surprising is that the class of ind-inf-schemes is quite large. In this section we define ind-inf-schemes and discuss some basic properties.

3.1. The notion of (ind)-inf-scheme. We will only define the notion of (ind)-inf-scheme, under the 'laft' hypothesis. One can give a definition in general, but it is more technical and currently we do not see sufficient applications for it.

3.1.1. Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\mathrm{laft}}$.

DEFINITION 3.1.2. *We shall say that $\mathcal{X}$ is an* inf-scheme *(resp.,* ind-inf-scheme*) if:*

- *$\mathcal{X}$ admits deformation theory;*
- *The reduced prestack $^{\mathrm{red}}\mathcal{X}$ is a reduced quasi-compact scheme (resp., ind-scheme).*

We let $\mathrm{indinfSch_{laft}}$ (resp., $\mathrm{infSch_{laft}}$) denote the full subcategory of $\mathrm{PreStk_{laft}}$ spanned by ind-inf-schemes (resp., inf-schemes). It is clear that both subcategories are closed under finite limits.

3.1.3. *Examples.*

(i) By Proposition 1.3.2, any object of indSch is an ind-inf-scheme.

(ii) Let $\mathcal{Z}$ be an object of $\mathrm{PreStk_{laft}}$. Consider the de Rham prestack $\mathcal{Z}_{\mathrm{dR}}$:

$$\mathrm{Maps}(S, \mathcal{Z}_{\mathrm{dR}}) := \mathrm{Maps}(^{\mathrm{red}}S, \mathcal{Z}).$$

If $^{\mathrm{red}}\mathcal{Z}$ is a reduced ind-scheme (resp., scheme), then $\mathcal{Z}_{\mathrm{dR}}$ is an ind-inf-scheme (resp., inf-scheme). Indeed,

$$^{\mathrm{red}}\mathcal{Z}_{\mathrm{dR}} = {}^{\mathrm{red}}\mathcal{Z},$$

while the cotangent complex of $\mathcal{Z}_{\mathrm{dR}}$ is zero.

However, $\mathcal{Z}_{\mathrm{dR}}$ is *not* an ind-scheme. For example, it violates condition (d) of Theorem 1.7.7.

(iii) Let $\mathcal{Y} \to \mathcal{X}$ be a map in $\mathrm{PreStk_{laft}}$. We definite the formal completion *of $\mathcal{X}$ along $\mathcal{Y}$* (or *of $\mathcal{Y}$ in $\mathcal{X}$*), denoted $\mathcal{X}_{\mathcal{Y}}^{\wedge}$ to be the prestack

$$\mathcal{X} \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{Y}_{\mathrm{dR}}.$$

Note that $^{\mathrm{red}}\mathcal{X}_{\mathcal{Y}}^{\wedge} \simeq {}^{\mathrm{red}}\mathcal{Y}$.

Hence, if $^{\mathrm{red}}\mathcal{Y}$ is a reduced ind-scheme (resp., scheme), and $\mathcal{X}$ admits deformation theory, then $\mathcal{X}_{\mathcal{Y}}^{\wedge}$ is an ind-inf-scheme (resp., inf-scheme).

3.1.4.   We give the following definition:

DEFINITION 3.1.5. *Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a morphism in $\mathrm{PreStk_{laft}}$. We shall say that f is* (ind)-inf-schematic *if its base change by an affine scheme (almost of finite type) yields an (ind)-inf-scheme.*

3.2. Properties of (ind)-inf-schemes.
3.2.1.   By Chapter 1, Proposition 8.2.2(a) we have:

COROLLARY 3.2.2. *Let $\mathcal{X} \in \mathrm{indinfSch}$. Then $\mathcal{X}$ satisfies Nisnevich descent.*

REMARK 3.2.3. According to Chapter 1, Remark 8.2.3, any object of indinfSch satisfies étale descent.

3.2.4.   We now claim:

LEMMA 3.2.5. *Any ind-inf-scheme $\mathcal{X}$ can be exhibited as a filtered colimit in PreStk*

$$\operatorname*{colim}_{\alpha} \mathcal{X}_{\alpha},$$

where $\mathcal{X}_{\alpha} \in \mathrm{infSch_{laft}}$ and the maps $X_{\alpha_1} \to X_{\alpha_2}$ are ind-closed embeddings.

PROOF. Write $^{\mathrm{red}}\mathcal{X}$ as a filtered colimit in $^{\mathrm{red}}\mathrm{PreStk}$

$$\operatorname*{colim}_{\alpha} X_{\alpha}, \quad X_{\alpha} \in {}^{\mathrm{red}}\mathrm{Sch_{ft}}.$$

Let $\mathcal{X}_{\alpha}$ be the formal completion of X_{α} in $\mathcal{X}$, i.e.,

$$\mathcal{X}_{\alpha} = (X_{\alpha})_{\mathrm{dR}} \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{X}.$$

This gives the desired presentation.   □

3.3. Ind-inf-schemes vs. ind-schemes. In this subsection we discuss various conditions that guarantee that a given (ind)-inf-scheme is in fact an (ind)-scheme.

3.3.1.   First we observe:

LEMMA 3.3.2. *Let* $\mathcal{X}' \to \mathcal{X}$ *be a map in* $\mathrm{PreStk}_{\mathrm{laft}}$ *with* $\mathcal{X}$ *an ind-inf-scheme (resp., ind-scheme). Then* $\mathcal{X}'$ *is an ind-inf-scheme (resp., ind-scheme) if and only if for every* $S \in (^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{X}}$, *the base change* $S \underset{\mathcal{X}}{\times} \mathcal{X}'$ *is an ind-inf-scheme (resp., ind-scheme).*

3.3.3. *When is an ind-inf-scheme an ind-scheme?* A partial answer to this question is provided by Theorem 1.3.12. Here is a more algorithmic answer:

COROLLARY 3.3.4. *An object* $\mathcal{X} \in \mathrm{indinfSch}_{\mathrm{laft}}$ *belongs to* $\mathrm{indSch}_{\mathrm{laft}}$ *if and only if:*

- *For any* $(S, x : S \to \mathcal{X}) \in (^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{X}}$, *we have* $T_x^*(^{\mathrm{red}}\mathcal{X}/\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S))^{\le -1}$;
- *For any* $(S, x : S \to \mathcal{X}) \in (^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{X}}$, *the object* $H^0(T_x^*(\mathcal{X})) \in \mathrm{Pro}(\mathrm{QCoh}(S)^\heartsuit)$ *is given by a surjective system.*

PROOF. This is a restatement of Theorem 1.7.7.   □

The above assertion has a number of corollaries that will be useful in the sequel:

COROLLARY 3.3.5. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a nil-isomorphism, where* $\mathcal{X} \in \mathrm{indinfSch}_{\mathrm{laft}}$ *and* $\mathcal{Y} \in \mathrm{indSch}_{\mathrm{laft}}$. *Assume that for every* $(S, x : S \to \mathcal{X}) \in (^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{X}}$ *we have:*

- $T_x^*(\mathcal{X}/\mathcal{Y}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\le 0})$;
- *The object* $H^0(T_x^*(\mathcal{X}/\mathcal{Y})) \in \mathrm{Pro}(\mathrm{QCoh}(S)^\heartsuit)$ *is given by a surjective system.*

Then $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$.

PROOF. We claim that $\mathcal{X}$ satisfies the conditions of Corollary 3.3.4. We need to show that for every $(S, x : S \to \mathcal{X}) \in (^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{X}}$ we have:

- $T_x^*(^{\mathrm{red}}\mathcal{X}/\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\le -1})$;
- $H^0(T_x^*(\mathcal{X})) \in \mathrm{Pro}(\mathrm{QCoh}(S)^\heartsuit)$ is given by a surjective system.

Consider the fiber sequence

$$T_x^*(\mathcal{X}/\mathcal{Y}) \to T_x^*(^{\mathrm{red}}\mathcal{X}/\mathcal{Y}) \to T_x^*(^{\mathrm{red}}\mathcal{X}/\mathcal{X})$$

in $\mathrm{Pro}(\mathrm{QCoh}(S)^-)$.

Since $\mathcal{Y}$ is an ind-scheme and $^{\mathrm{red}}\mathcal{X} \simeq {}^{\mathrm{red}}\mathcal{Y}$, we have

$$T_x^*(^{\mathrm{red}}\mathcal{X}/\mathcal{Y}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\le -1}).$$

Hence, $T_x^*(^{\mathrm{red}}\mathcal{X}/\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\le -1})$, as desired.

Consider now the fiber sequence

$$T_{f\circ x}^*(\mathcal{Y}) \to T_x^*(\mathcal{X}) \to T_x^*(\mathcal{X}/\mathcal{Y}),$$

and the corresponding exact sequence

$$H^0(T_{f\circ x}^*(\mathcal{Y})) \to H^0(T_x^*(\mathcal{X})) \to H^0(T_x^*(\mathcal{X}/\mathcal{Y})) \to 0.$$

From here we obtain that $H^0(T_x^*(\mathcal{X})) \in \mathrm{Pro}(\mathrm{QCoh}(S)^\heartsuit)$ is given by a surjective system, since both $H^0(T_{f\circ x}^*(\mathcal{Y}))$ and $H^0(T_x^*(\mathcal{X}/\mathcal{Y}))$ are.

□

4. (Ind)-inf-schemes and nil-closed embeddings

The results of this section are of crucial technical importance. We prove two types of results. One is about approximating (ind)-inf-schemes by schemes; this is needed for the development of IndCoh on (ind)-inf-schemes. The other is about recovering an (ind)-inf-scheme as a prestack (i.e., a presheaf on the category of affine schemes) from its restriction to a much smaller subcategory of test schemes; this is needed for the study of formal moduli problems in Chapter 5.

4.1. Exhibiting ind-inf-schemes as colimits. In this subsection we show that an (ind)-inf-scheme $\mathcal{X}$ is isomorphic to the colimit of schemes equipped with a nil-closed map into $\mathcal{X}$.

4.1.1.   For $\mathcal{X} \in \mathrm{PreStk}$ let

$$\mathrm{PreStk}_{\text{nil-closed in }\mathcal{X}} \subset \mathrm{PreStk}_{/\mathcal{X}}$$

be the full subcategory spanned by objects $f : \mathcal{X}' \to \mathcal{X}$ for which f is nil-closed.

We will use a similar notation for full subcategories of PreStk, e.g.,

$$\mathrm{Sch}_{\text{nil-closed in }\mathcal{X}} \subset \mathrm{Sch}_{/\mathcal{X}},$$

etc.

4.1.2.   We have the following assertion (cf. Corollary 1.7.5(a') in the case of ind-schemes):

THEOREM 4.1.3. *Let $\mathcal{X}$ be an object of* $\mathrm{indinfSch}_{\mathrm{laft}}$. *Then the map*

$$\operatorname*{colim}_{Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-closed in }\mathcal{X}}} Z \to \mathcal{X},$$

where the colimit is taken in PreStk, *is an isomorphism.*

Evaluating the two sides in Theorem 4.1.3 on $^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$, we obtain:

COROLLARY 4.1.4. *Let $\mathcal{X}$ be an object of* $\mathrm{indinfSch}_{\mathrm{laft}}$. *Then the map*

$$\operatorname*{colim}_{Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-closed in }\mathcal{X}}} Z \to \mathcal{X},$$

where the colimit is taken in $\mathrm{PreStk}_{\mathrm{laft}}$, *is an isomorphism.*

With future applications in mind, let us state the following particular case of Corollary 4.1.4 separately:

COROLLARY 4.1.5. *Let $\mathcal{X} \in \mathrm{indinfSch}_{\mathrm{laft}}$ be such that* $^{\mathrm{red}}\mathcal{X} = \mathcal{X}_0$ *is ind-affine. Then the functor*

$$\left(^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}\right)_{/\mathcal{X}} \underset{(^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{X}_0}}{\times} \left(^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}\right)_{\text{closed in }\mathcal{X}_0} \hookrightarrow \left(^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}\right)_{/\mathcal{X}}$$

is cofinal.

REMARK 4.1.6. We note that the analog of Corollary 1.7.5(b) *fails* for ind-inf-schemes. I.e., it is *not* true that the inclusion

$$(\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-closed in }\mathcal{X}} \hookrightarrow (\mathrm{Sch}_{\mathrm{aft}})_{/\mathcal{X}}$$

is cofinal.

The rest of this subsection is devoted to the proof of Theorem 4.1.3.

4.1.7. *Step 0.* For $(S, x) \in ({}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$ consider the category $\mathrm{Factor}(x, \mathrm{nil\text{-}closed}, \mathrm{ft}, \mathrm{cl})$ of factorizations

$$S \to Z \to \mathcal{X},$$

where $Z \in ({}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{ft}})_{\mathrm{nil\text{-}closed\ in\ } \mathcal{X}}$. In Steps 1-6 we will show that this category is contractible.

Since ${}^{\mathrm{cl}}\mathcal{X} \in \mathrm{PreStk}_{\mathrm{lft}}$, it is easy to see that we can assume that $S \in {}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$.

4.1.8. *Step 1.* Denote $S_0 := {}^{\mathrm{red}}S$ and $x_0 := x|_{S_0}$. Consider the category

$$\mathrm{Factor}(x_0, \mathrm{nil\text{-}closed}, \mathrm{red})$$

of factorizations

$$S_0 \to Z_0 \to \mathcal{X},$$

where $Z_0 \in ({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}})_{\mathrm{closed\ in\ } {}^{\mathrm{red}}\mathcal{X}}$.

The category $\mathrm{Factor}(x_0, \mathrm{nil\text{-}closed}, \mathrm{red})$ is contractible, since ${}^{\mathrm{red}}\mathcal{X}$ is a (reduced) ind-scheme locally of finite type.

We have a functor

(4.1) $\mathrm{Factor}(x, \mathrm{nil\text{-}closed}, \mathrm{ft}, \mathrm{cl}) \to \mathrm{Factor}(x_0, \mathrm{nil\text{-}closed}, \mathrm{red}), \quad Z \mapsto {}^{\mathrm{red}}Z,$

and it is enough to show that (4.1) is a homotopy equivalence.

We note that (4.1) is a co-Cartesian fibration via the formation of push-outs. Hence, it is enough to show that the fibers of (4.1) are contractible.

4.1.9. *Step 2.* For an object $S_0 \to Z_0 \to \mathcal{X}$ of $\mathrm{Factor}(x_0, \mathrm{nil\text{-}closed}, \mathrm{red})$. The fiber of (4.1) over this object is described as follows.

Let Z' denote the push-out

$$S \underset{S_0}{\sqcup} Z_0.$$

(Note, however, that Z' is *not necessarily* of finite type.)

Since S was assumed of finite type, the map $S_0 \to S$ is a nilpotent embedding (in fact, a finite succession of square-zero extensions). Since the morphism $S_0 \to Z_0$ is affine, by Chapter 1, Corollary 7.2.3, we obtain a canonically defined map $Z' \to \mathcal{X}$.

The sought-for fiber is the category of factorizations of the above map $Z' \to \mathcal{X}$ as

$$Z' \to Z \to \mathcal{X},$$

where $Z \in {}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{ft}}$ and ${}^{\mathrm{red}}Z = Z_0$.

4.1.10. *Step 3.* We will prove the following general assertion. Suppose that $Z_0' \to Z'$ can be written as a finite succession of square-zero extensions, with $Z_0', Z' \in ({}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{qc}})_{/\mathcal{X}}$ and $Z_0' \in {}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{ft}}$.

Let $\mathbf{C}(x)$ denote the category of factorizations of the map $x : Z' \to \mathcal{X}$ as

$$Z' \to Z \to \mathcal{X},$$

where $Z \in {}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{ft}}$ and $Z_0' \to Z$ a nil-isomorphism.

We claim that $\mathbf{C}(x)$ is contractible.

4.1.11. *Step 4.* Suppose that $Z_0' \to Z'$ can be written as a succession of m square-zero extensions. We will argue by induction on m.

If $m = 1$, the required assertion is proved by repeating Chapter 1, Proof of Theorem 9.1.2, Steps 2-3.

Let us carry out the induction step. Choose an intermediate extension

$$Z_0' \to Z_{\frac{1}{2}}' \to Z',$$

and let $x_{\frac{1}{2}}$ denote the map $Z_{\frac{1}{2}}' \to \mathcal{X}$.

Let $\mathbf{C}(x_{\frac{1}{2}})$ denote the corresponding category of factorizations of $x_{\frac{1}{2}}$. By the induction hypothesis, we can assume that $\mathbf{C}(x_{\frac{1}{2}})$ is contractible.

Let $\mathbf{D}$ denote the category of commutative diagrams

$$
\begin{array}{ccc}
Z' & \longrightarrow & Z \\
\uparrow & & \uparrow \\
Z_{\frac{1}{2}}' & \longrightarrow & Z_{\frac{1}{2}}
\end{array}
$$

in $({}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{qc}})_{Z_0'//\mathcal{X}}$ with $Z, Z_{\frac{1}{2}} \in {}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{ft}}$, and where all the maps are nil-isomorphisms.

We have the forgetful functors

$$\mathbf{C}(x_{\frac{1}{2}}) \leftarrow \mathbf{D} \to \mathbf{C}(x).$$

We will show that both these functors are homotopy equivalences. This will prove that $\mathbf{C}(x)$ is contractible.

4.1.12. *Step 5.* The functor $\mathbf{D} \to \mathbf{C}(x)$ is a Cartesian fibration. Hence, in order to show that it is a homotopy equivalence, it is enough to show that it has contractible fibers.

However, for an object $(Z' \to Z \to \mathcal{X}) \in \mathbf{C}(x)$, the fiber in question has a final point: take $Z_{\frac{1}{2}} = Z$.

4.1.13. *Step 6.* The functor $\mathbf{D} \to \mathbf{C}(x_{\frac{1}{2}})$ is a co-Cartesian fibration via the formation of push-outs. Hence, in order to show that it is a homotopy equivalence, it is enough to show that it has contractible fibers.

For an object

$$(Z_{\frac{1}{2}}' \to Z_{\frac{1}{2}} \to \mathcal{X}) \in \mathbf{C}(x_{\frac{1}{2}}),$$

set

$$\widetilde{Z}' := Z' \underset{Z_{\frac{1}{2}}'}{\sqcup} Z_{\frac{1}{2}}.$$

Let $\widetilde{x}$ denote the resulting map $\widetilde{Z}' \to \mathcal{X}$. The fiber in question is the category $\mathbf{C}(\widetilde{x})$. This category is contractible by the induction hypothesis, applied to the nil-isomorphism

$$Z_{\frac{1}{2}} \to \widetilde{Z}'.$$

4.1.14. *Step 7.* Now, let (S, x) be an arbitrary object of $(\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$. Let us show that the category $\mathrm{Factor}(x, \text{nil-closed}, \text{aft})$ of factorizations

$$S \to Z \to \mathcal{X},$$

where $Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-closed in } \mathcal{X}}$ is contractible.

Denote $S_0 = {}^{\mathrm{cl}}S$ and $x_0 = x|_{S_0}$. Consider the functor

$$\mathrm{Factor}(x, \text{nil-closed}, \text{aft}) \to \mathrm{Factor}(x_0, \text{nil-closed}, \text{ft}, \text{cl}), \quad Z \mapsto Z_0 := {}^{\mathrm{cl}}Z.$$

This functor is a coCartesian fibration via the formation of push-outs. Since we already know that $\mathrm{Factor}(x_0, \text{nil-closed}, \text{cl})$ is contractible, it suffices to show that the fibers of the above functor are contractible. The latter is established by repeating the argument of Chapter 1, Theorem 9.1.4.

4.2. A construction of ind-inf-schemes. Our current goal is to prove a partial converse to Theorem 4.1.3, which will give rise to a procedure for explicitly constructing ind-inf-schemes.

4.2.1. We start with an object $\mathcal{X}_0 \in {}^{\mathrm{red}}\mathrm{indSch}_{\mathrm{lft}}$. In this subsection we will assume that $\mathcal{X}_0$ is ind-affine.

Let $\mathcal{X}_{\text{nil-closed}}$ be a presheaf on the category

$${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \underset{{}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\times} ({}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{closed in } \mathcal{X}_0},$$

where the functor ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \to {}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ is $S \mapsto {}^{\mathrm{red}}S$.

4.2.2. We impose the following assumptions on $\mathcal{X}_{\text{nil-closed}}$:

- The restriction of $\mathcal{X}_{\text{nil-closed}}$ to the full subcategory

$$({}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{closed in } \mathcal{X}_0} = {}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \underset{{}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\times} ({}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{closed in } \mathcal{X}_0} \subset$$

$$\subset {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \underset{{}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\times} ({}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{closed in } \mathcal{X}_0}$$

  takes the value $* \in \mathrm{Spc}$.
- For a push-out diagram

$$Z_1 \underset{Z}{\sqcup} Z'$$

  in ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \underset{{}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\times} ({}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{closed in } \mathcal{X}_0}$, where $Z \hookrightarrow Z'$ has a structure of square-zero extension, the resulting map

$$\mathcal{X}_{\text{nil-closed}}(Z_1 \underset{Z}{\sqcup} Z') \to \mathcal{X}_{\text{nil-closed}}(Z_1) \underset{\mathcal{X}_{\text{nil-closed}}(Z)}{\times} \mathcal{X}_{\text{nil-closed}}(Z')$$

  is an isomorphism (cf. characterization of deformation theory in Chapter 1, Corollary 6.3.6).

4.2.3. Let $\mathcal{X}$ denote the left Kan extension of $\mathcal{X}_{\text{nil-closed}}$ under the fully faithful embedding

$$\left({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \underset{{}_{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\times} ({}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{closed in }\mathcal{X}_0}\right)^{\mathrm{op}} \hookrightarrow \left({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}} \underset{{}_{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}}{\times} ({}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}_0}\right)^{\mathrm{op}}.$$

Note that

$${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}} \underset{{}_{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}}{\times} ({}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}_0} = ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{/(\mathcal{X}_0)_{\mathrm{dR}}}.$$

Thus, we can view $\mathcal{X}$ as an object of ${}^{\mathrm{conv}}\mathrm{PreStk}$ mapping to $(\mathcal{X}_0)_{\mathrm{dR}}$. By construction, $\mathcal{X}$ belongs to

$$\mathrm{PreStk}_{\mathrm{laft}} \subset {}^{\mathrm{conv}}\mathrm{PreStk},$$

and ${}^{\mathrm{red}}\mathcal{X}$ is canonically isomorphic to $\mathcal{X}_0$.

4.2.4. We are going to prove:

THEOREM 4.2.5. *Under the above circumstances $\mathcal{X} \in \mathrm{indinfSch}_{\mathrm{laft}}$.*

By combining with Corollary 4.1.5, we obtain:

COROLLARY 4.2.6. *The assignements*

$$\mathcal{X}_{\text{nil-closed}} \rightsquigarrow \mathcal{X}$$

and

$$\mathcal{X} \rightsquigarrow \mathcal{X}|_{{}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \underset{{}_{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\times} ({}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{closed in }\mathcal{X}_0}}$$

define mutually inverse equivalences between

$$(\mathrm{indinfSch}_{\mathrm{laft}})_{/(\mathcal{X}_0)_{\mathrm{dR}}} \underset{({}^{\mathrm{red}}\mathrm{indSch}_{\mathrm{lft}})_{/\mathcal{X}_0}}{\times} *$$

and the category of presheaves on ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \underset{{}_{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\times} ({}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{closed in }\mathcal{X}_0}$, *satisfying the assumptions of Sect. 4.2.2.*

The rest of the subsection is devoted to the proof of Theorem 4.2.5.

4.2.7. *Step 1.* We only have to show that $\mathcal{X}$ admits deformation theory. Since $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$, by Chapter 1, Corollary 7.2.6, it suffices to check the following:

Let $S \to Z$ be a map in $({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$, where ${}^{\mathrm{red}}Z \to \mathcal{X}_0$ is a closed embedding and $Z \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. We need to show that for a map $S \hookrightarrow S'$, equipped with a structure of square-zero extension, and such that $S' \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$, the map

$$\mathrm{Maps}_{Z/}(Z', \mathcal{X}) \to \mathrm{Maps}_{S/}(S', \mathcal{X})$$

is an isomorphism, where $Z' := Z \underset{S}{\sqcup} S'$.

Fix a point $x' \in \mathrm{Maps}_{S/}(S', \mathcal{X})$. We need to show that the groupoid

$$(4.2) \qquad\qquad \mathrm{Maps}_{Z/}(Z', \mathcal{X}) \underset{\mathrm{Maps}_{S/}(S', \mathcal{X})}{\times} \{x'\}$$

is contractible.

4.2.8. *Step 2.* Let $\mathbf{C}$ be the category of factorizations of the given map $S \to \mathcal{X}$ as

$$S \to Z_1 \to \mathcal{X},$$

where $^{\mathrm{red}}Z_1 \to \mathcal{X}_0$ is a closed embedding, and $Z_1 \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. By the construction of $\mathcal{X}$ as a left Kan extension, the category $\mathbf{C}$ is contractible.

Let $\mathbf{C}'$ be the category whose objects are commutative diagrams

$$
\begin{array}{ccccc}
S' & \longrightarrow & Z'_1 := Z_1 \underset{S}{\sqcup} S' & \longrightarrow & \mathcal{X} \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle\mathrm{id}} \\
S & \longrightarrow & Z_1 & \longrightarrow & \mathcal{X},
\end{array}
$$

where the bottom row is an object of $\mathbf{C}$.

We have a natural forgetful functor $\mathbf{C}' \to \mathbf{C}$. We claim that this functor is a co-Cartesian fibration in groupoids, such that every edge in $\mathbf{C}$ induces a homotopy equivalence between the fibers. The claim will be proved in Step 6.

Since the category $\mathbf{C}$ is contractible, we obtain that for any $(S \to Z_1 \to \mathcal{X}) \in \mathbf{C}$ the map

$$\mathbf{C}' \underset{\mathbf{C}}{\times} \{(S \to Z_1 \to \mathcal{X})\} \to \mathbf{C}'$$

is a homotopy equivalence. In particular, we can take $(S \to Z_1 \to \mathcal{X})$ to be the orginal map $(S \to Z \to \mathcal{X})$.

(Note that

$$\mathbf{C}' \underset{\mathbf{C}}{\times} \{(S \to Z \to \mathcal{X})\}$$

identifies with the groupoid (4.2), whose contractibilty we want to establish.)

4.2.9. *Step 3.* Let $\mathrm{Factor}(x')$ denote the category of factorizations of x' as

$$S' \to W \to \mathcal{X},$$

with $W \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ and $^{\mathrm{red}}W \to \mathcal{X}_0$ being a closed embedding. This category is contractible by definition.

We claim that there is a canonical functor

$$\mathrm{Factor}(x') \to \mathbf{C}'.$$

Indeed, for an object of $\mathrm{Factor}(x')$ as above, consider the composed map

$$S \to S' \to W,$$

and set $W' := W \underset{S}{\sqcup} S'$. The extension $W \hookrightarrow W'$ splits by construction.

We regard

$$S \to W \to \mathcal{X}$$

as an object of $\mathbf{C}$. And we regard the composition

$$W' \to W \to \mathcal{X}$$

as an object of $\mathbf{C}'$ over it.

4.2.10. *Step 4.* Let $\mathbf{D}$ denote the category where an object is given by the following data:

- A square-zero extension $Z \hookrightarrow \widetilde{Z}'$ with $\widetilde{Z}' \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$;
- A map $Z' \to \widetilde{Z}'$ in the category of square-zero extensions of Z.
- A map $\widetilde{x}' : \widetilde{Z}' \to \mathcal{X}$, extending x' and compatible with the restriction to Z.

We have a natural functor $\mathbf{D} \to \mathrm{Maps}_{Z/}(Z', \mathcal{X}) \underset{\mathrm{Maps}_{S/}(S', \mathcal{Y})}{\times} \{x'\}$, and we claim that this functor is a homotopy equivalence.

Indeed, note that the scheme Z' can be written as a filtered limit of the $\widetilde{Z}'$'s, taken over the category of square-zero extensions

$$Z \hookrightarrow \widetilde{Z}', \quad \widetilde{Z}' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}},$$

where n is such that $Z, Z' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$. Hence, our assertion follows from the fact that $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$, and hence takes filtered limits in ${}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ to colimits.

Note also that we have a naturally defined functor

$$\mathbf{D} \to \mathrm{Factor}(x')$$

that sends $\widetilde{Z}'$ to W.

4.2.11. *Step 5.* We have a *non-commuting* diagram of categories:

(4.3)
$$
\begin{array}{ccc}
\mathrm{Maps}_{Z/}(Z', \mathcal{X}) \underset{\mathrm{Maps}_{S/}(S', \mathcal{X})}{\times} \{x'\} & \longrightarrow & \mathbf{C}' \\
\uparrow & & \uparrow \\
\mathbf{D} & \longrightarrow & \mathrm{Factor}(x').
\end{array}
$$

However, we claim that the two resulting maps

$$\mathbf{D} \rightrightarrows \mathbf{C}'$$

are homotopic. Indeed, the two functors send an object of $\mathbf{D}$ as above to

$$
\begin{array}{ccc}
S' & \longrightarrow Z' \longrightarrow & \mathcal{X} \\
\uparrow & \uparrow & \uparrow {\scriptstyle \mathrm{id}} \\
S & \longrightarrow Z \longrightarrow & \mathcal{X},
\end{array}
$$

(for the clockwise circuit)

$$
\begin{array}{ccc}
S' & \longrightarrow \widetilde{Z}' \underset{S}{\sqcup} S' \longrightarrow & \mathcal{X} \\
\uparrow & \uparrow & \uparrow {\scriptstyle \mathrm{id}} \\
S & \longrightarrow \widetilde{Z}' \xrightarrow{\ \widetilde{x}'\ } & \mathcal{X},
\end{array}
$$

(for the anti-clockwise circuit), respectively. The required homotopy is provided by the map $Z \to \widetilde{Z}'$.

Note that the clockwise circuit in (4.3) is a homotopy equivalence. Hence, we obtain that $\mathrm{Maps}_{Z/}(Z', \mathcal{X}) \underset{\mathrm{Maps}_{S/}(S', \mathcal{Y})}{\times} \{x'\}$ is a retract of $\mathrm{Factor}(x')$. Therefore, it is contractible, as required.

4.2.12. *Step 6.* It suffices to show that whenever $Z_1 \to Z_2$ is a map in

$$^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \underset{^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\times} (^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{closed\ in}\ \mathcal{X}_0},$$

and $Z_1 \hookrightarrow Z_1'$ is a square-zero extension with $Z_1' \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}$, then for

$$Z_2' := Z_2 \underset{Z_1}{\sqcup} Z_1',$$

the map

$$\mathrm{Maps}(Z_2', \mathcal{X}) \to \mathrm{Maps}(Z_2, \mathcal{X}) \underset{\mathrm{Maps}(Z_1, \mathcal{X})}{\times} \mathrm{Maps}(Z_1', \mathcal{X})$$

is an isomorphism.

As in Step 3, we write

$$Z_1' \simeq \lim_{\alpha \in A} Z_{1,\alpha}',$$

where A is a filtered category, and $Z_1 \hookrightarrow Z_{1,\alpha}'$ are square-zero extensions with $Z_{1,\alpha}' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$ for some n. Then

$$Z_2' \simeq \lim_{\alpha \in A} Z_{2,\alpha}',$$

where

$$Z_{2,\alpha}' := Z_2 \underset{Z_1}{\sqcup} Z_{1,\alpha}'.$$

The required assertion now follows from the fact $\mathcal{X}$ belongs to $\mathrm{PreStk}_{\mathrm{laft}}$, and hence takes filtered limits in $^{\leq n}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$ to colimits.

4.3. Exhibiting inf-schemes as colimits. In this subsection we will adapt Theorem 4.1.3 to the case of inf-schemes. Namely, we will show that in this case we can replace the word 'nil-closed' by 'nil-isomorphism'.

4.3.1. For $\mathcal{X} \in \mathrm{PreStk}$ let

$$\mathrm{PreStk}_{\mathrm{nil\text{-}isom\ to}\ \mathcal{X}} \subset \mathrm{PreStk}_{/\mathcal{X}}$$

be the full subcategory spanned by objects $f : \mathcal{X}' \to \mathcal{X}$ for which f is a nil-isomorphism.

We will use a similar notation for full subcategories of PreStk, e.g.,

$$\mathrm{Sch}_{\mathrm{nil\text{-}isom\ to}\ \mathcal{X}} \subset \mathrm{Sch}_{/\mathcal{X}},$$

etc.

We claim (compare with Proposition 1.8.5 in the case of nil-schematic ind-schemes):

PROPOSITION 4.3.2. *Let $\mathcal{X}$ be an object of* $\mathrm{infSch}_{\mathrm{laft}}$. *Then the inclusion*

$$(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nil\text{-}isom\ to}\ \mathcal{X}} \hookrightarrow (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nil\text{-}closed\ in}\ \mathcal{X}}$$

is cofinal.

PROOF. It is enough to show that the embedding in question admits a left adjoint. Given an object

$$(Z \to \mathcal{X}) \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{\mathrm{nil\text{-}closed\ in}\ \mathcal{X}},$$

we note that since $Z \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, the map $^{\mathrm{red}}Z \to Z$ is a nilpotent embedding.

Now, the value of the left adjoint in question is given by sending

$$(Z \to \mathcal{X}) \in (\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}})_{\text{nil-closed in } \mathcal{X}}$$

to

$$Z \underset{^{\mathrm{red}}Z}{\sqcup} {}^{\mathrm{red}}\mathcal{X},$$

which maps to $\mathcal{X}$ using Chapter 1, Corollary 7.2.3.

$\square$

Combining with Theorem 4.1.3, we obtain (compare with Corollary 1.8.6(a') in the case of nil-schematic ind-schemes):

COROLLARY 4.3.3. *Let $\mathcal{X}$ be an object of* $\mathrm{infSch}_{\mathrm{laft}}$. *Then the map*

$$\underset{Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-isom to } \mathcal{X}}}{\mathrm{colim}} Z \to \mathcal{X},$$

induces an isomorphism, when the colimit is taken in either PreStk *or* $\mathrm{PreStk}_{\mathrm{laft}}$.

Note that Corollary 4.3.3 admits the following corollary:

COROLLARY 4.3.4. *Let $\mathcal{X} \in \mathrm{infSch}_{\mathrm{laft}}$ be such that* ${}^{\mathrm{red}}\mathcal{X} = X_0 \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$. *Then the functor*

$$({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{X}} \underset{({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/X_0}}{\times} \{X_0\} \hookrightarrow ({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}$$

is cofinal.

4.3.5. We now claim (compare with Corollary 1.8.6(a") in the case of nil-schematic ind-schemes):

PROPOSITION 4.3.6. *For $\mathcal{X} \in \mathrm{infSch}_{\mathrm{laft}}$ the category* $(\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-isom to } \mathcal{X}}$ *is sifted.*

PROOF. We need to show that for a pair of nilpotent embeddings $f_1 : Z_1 \to \mathcal{X}$ and $f_2 : Z_2 \to \mathcal{X}$, the category of

$$(Z_1 \overset{g_1}{\to} Z, \; Z_2 \overset{g_2}{\to} Z, \; Z \overset{f}{\to} \mathcal{X}, \; f_1 \sim f \circ g_1, \; f_2 \sim f \circ g_2)$$

is contractible.

We claim, however, that the category in question admits an initial object, namely

$$Z := Z_1 \underset{^{\mathrm{red}}\mathcal{X}}{\sqcup} Z_2,$$

see Chapter 1, Corollary 7.2.3.

$\square$

4.4. A construction of inf-schemes. In this subsection we will consider a version of Proposition 4.4.5 for inf-schemes. This version will be crucial for our study of formal moduli problems in Chapter 5.

4.4.1. We start with an object $X_0 \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$, and let $\mathcal{X}_{\text{nil-isom}}$ be a presheaf on the category

$$^{<\infty}\mathrm{Sch}_{\mathrm{aft}} \underset{^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}}{\times} \{X_0\},$$

where the functor ${}^{<\infty}\mathrm{Sch}_{\mathrm{aft}} \to {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}$ is $S \mapsto {}^{\mathrm{red}}S$.

4.4.2.   We impose the following two conditions:

- $\mathcal{X}_{\text{nil-isom}}(X_0) = *$.

- For a push-out diagram

$$Z_1 \underset{Z}{\sqcup} Z'$$

in $^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}} \underset{^{\text{red}}\text{Sch}^{\text{aff}}_{\text{ft}}}{\times} \{X_0\}$, where $Z \hookrightarrow Z'$ has a structure of square-zero extension, the resulting map

$$\mathcal{X}_{\text{nil-isom}}(Z_1 \underset{Z}{\sqcup} Z') \to \mathcal{X}_{\text{nil-isom}}(Z_1) \underset{\mathcal{X}_{\text{nil-isom}}(Z)}{\times} \mathcal{X}_{\text{nil-isom}}(Z')$$

is an isomorphism (cf. remark following Chapter 1, Definition 6.1.2).

4.4.3.   Let $\mathcal{X}$ denote the left Kan extension of $\mathcal{X}_{\text{nil-isom}}$ under the fully faithful embedding

$$\left(^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}} \underset{^{\text{red}}\text{Sch}^{\text{aff}}_{\text{ft}}}{\times} \{X_0\}\right)^{\text{op}} \hookrightarrow \left(^{<\infty}\text{Sch}^{\text{aff}} \underset{^{\text{red}}\text{Sch}^{\text{aff}}}{\times} (^{\text{red}}\text{Sch}^{\text{aff}})_{/X_0}\right)^{\text{op}}.$$

Thus, we can view $\mathcal{X}$ as an object of $^{\text{conv}}\text{PreStk}$ mapping to $(X_0)_{\text{dR}}$. By construction $\mathcal{X}$ belongs to

$$\text{PreStk}_{\text{laft}} \subset {}^{\text{conv}}\text{PreStk},$$

and $^{\text{red}}\mathcal{X}$ is canonically isomorphic to X_0.

4.4.4.   We are going to prove:

PROPOSITION 4.4.5.   *Under the above circumstances* $\mathcal{X} \in \text{infSch}_{\text{laft}}$.

Combining with Corollary 4.3.4, we obtain:

COROLLARY 4.4.6.   *The assignements*

$$\mathcal{X}_{\text{nil-isom}} \rightsquigarrow \mathcal{X} \ \textit{and} \ \mathcal{X} \rightsquigarrow \mathcal{X}|_{^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}} \underset{^{\text{red}}\text{Sch}^{\text{aff}}_{\text{ft}}}{\times} \{X_0\}}$$

define mutually inverse equivalences between

$$(\text{infSch}_{\text{laft}})_{/(X_0)_{\text{dR}}} \underset{(^{\text{red}}\text{indSch}_{\text{lft}})_{/X_0}}{\times} *$$

and the category of presheaves on $^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}} \underset{^{\text{red}}\text{Sch}^{\text{aff}}_{\text{ft}}}{\times} \{X_0\}$*, satisfying the two assumptions of Sect. 4.4.2.*

4.4.7.   *Proof of Proposition 4.4.5.* Let $\mathcal{X}_{\text{ind-closed}}$ denote the presheaf on the category

$$^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}} \underset{^{\text{red}}\text{Sch}^{\text{aff}}_{\text{ft}}}{\times} (^{\text{red}}\text{Sch}^{\text{aff}}_{\text{ft}})_{\text{closed in } X_0}$$

equal to the left Kan extension of $\mathcal{X}_{\text{nil-isom}}$ under the fully faithful embedding

$$\left(^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}} \underset{^{\text{red}}\text{Sch}^{\text{aff}}_{\text{ft}}}{\times} \{X_0\}\right)^{\text{op}} \hookrightarrow \left(^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}} \underset{^{\text{red}}\text{Sch}^{\text{aff}}_{\text{ft}}}{\times} (^{\text{red}}\text{Sch}^{\text{aff}}_{\text{ft}})_{\text{closed in } X_0}\right)^{\text{op}}.$$

Now, by Theorem 4.2.5, it is sufficient to show that $\mathcal{X}_{\text{ind-closed}}$ satisfies the conditions of Sect. 4.2.2.

Note, however, that the functor

$$^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \underset{^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\times} \{X_0\} \hookrightarrow {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}} \underset{^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\times} ({}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{closed\ in\ } X_0}$$

admits a left adjoint, given by

$$Z \mapsto Z \underset{^{\mathrm{red}}Z}{\sqcup} X_0.$$

Hence, the value of $\mathcal{X}_{\mathrm{ind\text{-}closed}}$ on $Z \to \mathcal{X}$ can be calculated as

$$\mathcal{X}_{\mathrm{nil\text{-}isom}}(Z \underset{^{\mathrm{red}}Z}{\sqcup} X_0).$$

This implies the required condition on $\mathcal{X}_{\mathrm{ind\text{-}closed}}$, since the above left adjoint preserves push-outs.

Ind-coherent sheaves on ind-inf-schemes

Introduction

In this Chapter we will perform a construction central to this book: we will extend the assignment

$$X \rightsquigarrow \text{IndCoh}(X),$$

viewed as a functor out of the $(\infty,2)$-category of correspondences on schemes, to a functor out of the $(\infty,2)$-category of correspondences on ind-inf-schemes.

0.1. Why does everything work so nicely? Let us explain the mechanism of why IndCoh works out so well on ind-inf-schemes (there is not much difference between ind-inf-schemes and inf-schemes, the former being just a little more general).

0.1.1. The key assertion here is Chapter 2, Theorem 4.1.3. It says that an ind-inf-scheme can be written as

$$\mathcal{X} = \operatorname*{colim}_{a \in A} X_a,$$

where the colimit is taken in $\text{PreStk}_{\text{laft}}$, and where the maps $X_a \xrightarrow{i_{a,b}} X_b$ are nil-closed embeddings of schemes; in particular, they are proper.

We can (tautologically) write $\text{IndCoh}(\mathcal{X})$ as the limit

$$\lim_{a \in A^{\text{op}}} \text{IndCoh}(X_a)$$

under the functors $i_{a,b}^!$.

Hence, by Volume I, Chapter 1, Proposition 2.5.7 we have:

$$\text{IndCoh}(\mathcal{X}) \simeq \operatorname*{colim}_{a \in A} \text{IndCoh}(X_a),$$

where the functors $\text{IndCoh}(X_a) \to \text{IndCoh}(X_b)$ are $(i_{a,b})_*^{\text{IndCoh}}$.

0.1.2. The latter presentation implies that a functor out of $\text{IndCoh}(\mathcal{X})$ amounts to a compatible family of functors out of the categories $\text{IndCoh}(X_a)$.

This readily implies that if $f : \mathcal{X} \to \mathcal{Y}$ is an ind-inf-schematic ind-proper map between laft prestacks, then the functor $f^!$ admits a left adjoint that satisfies base change against !-pullbacks. The latter, in turn, entails the descent property of IndCoh for point-wise surjective ind-inf-schematic ind-proper maps.

In addition, we obtain that $\text{IndCoh}(\mathcal{X})$ is compactly generated, and has a t-structure with reasonable properties.

0.2. Direct image for IndCoh on ind-inf-schemes. The first step in making IndCoh into a functor out of the category of correspondences is the construction the direct image part of this functor.

0.2.1. Since ind-inf-schemes are laft prestacks, we know what $\mathrm{IndCoh}(\mathcal{X})$ is for $\mathcal{X} \in \mathrm{indinfSch_{laft}}$. We also know how to form the !-pullback for a morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$. What we do not yet know is how to form push-forwards.

The construction of push-forwards will be given as a result of the combination of Corollary 4.3.5 and Theorem 4.3.3. It amounts to the following.

As we have already mentioned, if $i : \mathcal{Y} \to \mathcal{X}$ is an ind-proper map between ind-inf-schemes, then the functor $i^! : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{X})$ admits a left adjoint, to be denoted i_*^{IndCoh}. In particular, we can take i to be nil-closed map from a scheme.

Now, the claim is that for a map $f : \mathcal{X}_1 \to \mathcal{X}_2$ there is a uniquely defined functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}_1) \to \mathrm{IndCoh}(\mathcal{X}_2)$$

such that for every commutative diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\ i_1\ } & \mathcal{X}_1 \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
X_2 & \xrightarrow{\ i_2\ } & \mathcal{X}_2,
\end{array}
$$

where X_1, X_2 are schemes and i_1, i_2 are nil-closed maps, we have

$$f_*^{\mathrm{IndCoh}} \circ (i_1)_*^{\mathrm{IndCoh}} \simeq (i_2)_*^{\mathrm{IndCoh}} \circ g_*^{\mathrm{IndCoh}}.$$

Moreover, if f is itself nil-closed, then f_*^{IndCoh} identifies with the left adjoint of $f^!$.

Essentially, the existence and uniqueness of f_*^{IndCoh} follows from the description of functors out of $\mathrm{IndCoh}(\mathcal{X})$ (in this case $\mathcal{X} = \mathcal{X}_1$) in Sect. 0.1.2. What this amounts to technically will be reviewed in Sect. 0.2.3.

0.2.2. Having defined the functor f_*^{IndCoh} for any morphism $\mathcal{X} \to \mathcal{Y}$, we can in particular take $\mathcal{Y} = \mathrm{pt}$. In this way we obtain the functor of global sections

$$\Gamma(\mathcal{X}, -)^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{Vect}.$$

Here is an example of what this functor does. As we will see in Chapter IV.3, the category of inf-schemes whose reduced scheme is pt is canonically equivalent to the category of Lie algebras in Vect. For a given Lie algebra $\mathfrak{g}$, the category IndCoh on the corresponding inf-scheme $B(\mathfrak{g})$ identifies canonically with the category $\mathfrak{g}\text{-mod}$ of modules over $\mathfrak{g}$.

Under this identification, the functor of global sections $\Gamma(B(\mathfrak{g}), -)^{\mathrm{IndCoh}}$ corresponds to the functor of $\mathfrak{g}$-coinvariants. (Moreover, the forgetful functor $\mathfrak{g}\text{-mod} \to \mathrm{Vect}$ is the pullback under the map $\mathrm{pt} \to B(\mathfrak{g})$.)

0.2.3. A more precise description of the construction in Sect. 0.2.1 is as follows.

We consider the category $\mathrm{indinfSch_{laft}}$ and its full subcategory $\mathrm{Sch_{aft}}$. We now consider the categories

$$(\mathrm{Sch_{aft}})_{\text{nil-closed}} \subset (\mathrm{indinfSch_{laft}})_{\text{nil-closed}},$$

where we restrict morphisms to be nil-closed.

Consider the functor

$$\mathrm{IndCoh_{Sch_{aft}}} : \mathrm{Sch_{aft}} \to \mathrm{DGCat_{cont}}, \quad X \mapsto \mathrm{IndCoh}(X), \quad (X_1 \xrightarrow{g} X_2) \mapsto g_*^{\mathrm{IndCoh}}.$$

We consider the operation of *left Kan extension*

$$\mathrm{LKE}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nil\text{-}closed}} \hookrightarrow (\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}closed}}}(\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}|_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nil\text{-}closed}}});$$

this a functor

$$(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}closed}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

One shows (Proposition 4.1.3 and Lemma 1.4.4) that the value of the above functor on a given $\mathcal{X} \in \mathrm{indinfSch}_{\mathrm{laft}}$ identifies canonically with $\mathrm{IndCoh}(\mathcal{X})$.

Now, the key assertion is Theorem 4.3.3 that says that the natural transformation

$$\mathrm{LKE}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nil\text{-}closed}} \hookrightarrow (\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}closed}}}(\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}|_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nil\text{-}closed}}}) \to$$

$$\to \mathrm{LKE}_{\mathrm{Sch}_{\mathrm{aft}} \hookrightarrow \mathrm{indinfSch}_{\mathrm{laft}}}(\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}})|_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}closed}}})$$

is an isomorphism. This theorem ensures that the functor

$$\mathrm{LKE}_{\mathrm{Sch}_{\mathrm{aft}} \hookrightarrow \mathrm{indinfSch}_{\mathrm{laft}}}(\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}) : \mathrm{indinfSch}_{\mathrm{laft}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

takes the value $\mathrm{IndCoh}(\mathcal{X})$ on a given $\mathcal{X} \in \mathrm{indinfSch}_{\mathrm{laft}}$. Being a functor, it gives rise to the sought-for functoriality of IndCoh:

$$\mathcal{X}_1 \xrightarrow{f} \mathcal{X}_2 \quad \mapsto \quad f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}_1) \to \mathrm{IndCoh}(\mathcal{X}_2).$$

0.3. Extending to correspondences.

0.3.1.   In order to extend the functor

$$\mathrm{IndCoh}_{\mathrm{indinfSch}_{\mathrm{aft}}} := \mathrm{LKE}_{\mathrm{Sch}_{\mathrm{aft}} \hookrightarrow \mathrm{indinfSch}_{\mathrm{laft}}}(\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}})$$

to a functor out of the category of correspondences, we apply the machinery of Volume I, Chapter 8, Sect. 1. The only thing to check is that the functors

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}_1) \to \mathrm{IndCoh}(\mathcal{X}_2)$$

thus constructed satisfy base change against the !-pullback functors under ind-proper maps. I.e., given a Cartesian diagram of objects of $\mathrm{indinfSch}_{\mathrm{laft}}$

$$
\begin{array}{ccc}
\mathcal{X}_1' & \xrightarrow{\ g_1\ } & \mathcal{X}_1 \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
\mathcal{X}_2' & \xrightarrow{\ g_2\ } & \mathcal{X}_2
\end{array}
$$

where g_2 (and hence g_1) is ind-proper, we have, by adjunction, natural transformations

$$(f')_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\mathrm{IndCoh}}$$

and

$$(g_1)_*^{\mathrm{IndCoh}} \circ (f')^! \to f^! \circ (g_2)_*^{\mathrm{IndCoh}}.$$

We show that these natural transformations are isomorphisms. Once this is done, by Volume I, Chapter 8, Theorem 1.1.9, we obtain the desired functor

$$\mathrm{IndCoh}_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{all;all}}^{\mathrm{ind\text{-}proper}}} : \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{all;all}}^{\mathrm{ind\text{-}proper}} \to \mathrm{DGCat}_{\mathrm{cont}}^{2\text{-Cat}}.$$

This is a functor from the $(\infty, 2)$-category of correspondences, whose objects are ind-inf-schemes, horizontal and vertical morphisms are arbitrary maps, and 2-morphisms are given by ind-proper maps.

0.3.2.   Finally, we apply Volume I, Chapter 8, Theorem 6.1.5 and extend the latter functor to a functor

$$\text{IndCoh}_{\text{Corr}(\text{PreStk}_{\text{laft}})^{\text{indinfsch \& ind-proper}}_{\text{indinfsch;all}}} : \text{Corr}(\text{PreStk}_{\text{laft}})^{\text{indinfsch \& ind-proper}}_{\text{indinfsch;all}} \to \text{DGCat}^{2\text{-Cat}}_{\text{cont}} .$$

I.e., it is a functor from the $(\infty, 2)$-category of correspondences, whose objects are all laft prestacks, horizontal morphisms are arbitrary maps, vertical morphisms are ind-inf-schematic maps, and 2-morphisms are ind-inf-schematic and ind-proper maps.

This is the furthest point that we can imagine that the theory of IndCoh can be extended to.

0.4. What else is done in this Chapter?

0.4.1.   In Sect. 1 we analyze the behavior of the category IndCoh on *ind-schemes*. Some of the assertions concerning ind-schemes (such as base change) are redundant: they will be reproved for ind-inf-schemes in greater generality. We have included them in order to compare the statements (and methods of their proofs) for ind-schemes and ind-inf-schemes.

We show that for an ind-scheme $\mathcal{X}$, the category $\text{IndCoh}(\mathcal{X})$ is compactly generated and that its compact objects are of the form $i_*^{\text{IndCoh}}(\mathcal{F})$, where $i : X \to \mathcal{X}$ is a closed embedding with $X \in \text{Sch}_{\text{aft}}$ and $\mathcal{F} \in \text{Coh}(X)$. In the above formula, $i_*^{\text{IndCoh}}(\mathcal{F})$ is the left adjoint to the functor $i^!$.

We show that the category $\text{IndCoh}(\mathcal{X})$ has a unique t-structure, for which the above functors $i_*^{\text{IndCoh}}(\mathcal{F})$ are t-exact.

We apply a left Kan extension to the functor

$$\text{IndCoh}_{\text{Sch}_{\text{aft}}} : \text{Sch}_{\text{aft}} \to \text{DGCat}_{\text{cont}}$$

and obtain a functor

$$\text{IndCoh}_{\text{indSch}_{\text{laft}}} : \text{indSch}_{\text{laft}} \to \text{DGCat}_{\text{cont}} .$$

We show that its value on a given $\mathcal{X} \in \text{indSch}_{\text{laft}}$ identifies canonically with $\text{IndCoh}(\mathcal{X})$. This is quite a bit easier than for ind-inf-scheme because of Chapter 2, Corollary 1.7.5(b)), the analog of which *fails* for ind-inf-schemes.

The construction of $\text{IndCoh}_{\text{indSch}_{\text{laft}}}$ has the property that for an ind-proper map $f : \mathcal{X} \to \mathcal{Y}$, the corresponding functor $f_*^{\text{IndCoh}} : \text{IndCoh}(\mathcal{X}) \to \text{IndCoh}(\mathcal{Y})$ is the left adjoint of $f^!$.

We show that a morphism $f : \mathcal{X} \to \mathcal{Y}$ between ind-schemes, the functor

$$f_*^{\text{IndCoh}} : \text{IndCoh}(\mathcal{X}) \to \text{IndCoh}(\mathcal{Y})$$

is left t-exact, and if f is ind-affine, then it is t-exact.

0.4.2. In Sect. 2 we establish the base change property for ind-schematic ind-proper morphisms. Namely, let

$$
\begin{array}{ccc}
\mathcal{X}_1' & \xrightarrow{g_1} & \mathcal{X}_1 \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
\mathcal{X}_2' & \xrightarrow{g_2} & \mathcal{X}_2
\end{array}
$$

be a Cartesian diagram of laft prestacks.

Suppose that the vertical arrows are ind-schematic ind-proper. In this case, by adjunction we obtain a natural transformation

$$(f')_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\mathrm{IndCoh}}$$

and we show that it is an isomorphism.

Now suppose that in the above diagram all prestacks are ind-schemes the horizontal arrows are ind-proper. Then, again by adjunction, we have a natural transformation

$$(f')_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\mathrm{IndCoh}},$$

and we also show that it is an isomorphism.

0.4.3. In Sect. 3 we initiate the study of IndCoh on ind-inf-schemes. The key statement is Proposition 3.1.2. It says that for an ind-inf-schematic nil-isomorphism $f : \mathcal{X} \to \mathcal{Y}$, the functor $f^!$ is:

(i) conservative;

(ii) admits a left adjoint;

(iii) the left adjoint of $f^!$ satisfies base change against !-pullbacks.

The proof of Proposition 3.1.2 relies on Chapter 2, Corollary 4.3.3, which is deduced from Chapter 2, Theorem 4.1.3 and says that an ind-inf-scheme whose underlying reduced ind-scheme is an affine scheme, can be written as a colimit of affine schemes under nil-closed maps.

From Proposition 3.1.2 we deduce the various favorable properties of IndCoh on ind-inf-schemes mentioned in Sect. 0.1.

In particular, we establish ind-proper descent for IndCoh. The statement here is that if $\mathcal{X} \to \mathcal{Y}$ is ind-inf-schematic ind-proper and point-wise surjective map between laft prestacks, then the pullback functor

$$\mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{Tot}(\mathrm{IndCoh}(\mathcal{X}^\bullet))$$

is an equivalence, where $\mathcal{X}^\bullet$ is the co-simplicial prestack given by the Čech nerve of $\mathcal{X} \to \mathcal{Y}$.

0.4.4.   In Sect. 4 we carry out the construction of the functoriality of IndCoh with respect to the operation of direct image, described in Sect. 0.2 above.

In Sect. 5 we carry out the construction of IndCoh as a functor out of the category of correspondences, already explained in Sect. 0.3.

In Sect. 6 we show that for in ind-inf-scheme $\mathcal{X}$, the category $\mathrm{IndCoh}(\mathcal{X})$ is canonically self-dual. I.e., there is a canonically defined identification

$$\mathbb{D}_{\mathcal{X}}^{\mathrm{Serre}} : \mathrm{IndCoh}(\mathcal{X})^{\vee} \to \mathrm{IndCoh}(\mathcal{X}),$$

or equivalently

$$\mathbf{D}_{\mathcal{X}}^{\mathrm{Serre}} : (\mathrm{IndCoh}(\mathcal{X})^c)^{\mathrm{op}} \to \mathrm{IndCoh}(\mathcal{X})^c.$$

Under this identification, for a morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$, the functor *dual* to $f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}_1) \to \mathrm{IndCoh}(\mathcal{X}_2)$ is

$$f^! : \mathrm{IndCoh}(\mathcal{X}_2) \to \mathrm{IndCoh}(\mathcal{X}_1).$$

It follows formally that if f is ind-proper, then f_*^{IndCoh} sends $\mathrm{IndCoh}(\mathcal{X}_1)^c$ to $\mathrm{IndCoh}(\mathcal{X}_2)^c$ and

$$\mathbf{D}_{\mathcal{X}_2}^{\mathrm{Serre}} \circ (f_*^{\mathrm{IndCoh}})^{\mathrm{op}} \circ \mathbf{D}_{\mathcal{X}_1}^{\mathrm{Serre}} \simeq f_*^{\mathrm{IndCoh}}$$

as functors $\mathrm{IndCoh}(\mathcal{X}_1)^c \to \mathrm{IndCoh}(\mathcal{X}_2)^c$.

1. Ind-coherent sheaves on ind-schemes

In order to develop the theory of IndCoh on ind-inf-schemes, we first need to do this for ind-schemes. The latter theory follows rather easily from the one on schemes.

In this section we will mainly review the results from [**GaRo1**, Sect. 2].

1.1. Basic properties. In this subsection we will express the category IndCoh on an ind-scheme $\mathcal{X}$ in terms of that on schemes equipped with a closed embedding into $\mathcal{X}$.

1.1.1.   Let $\mathrm{IndCoh}_{\mathrm{indSch}_{\mathrm{laft}}}^!$ denote the restriction of the functor $\mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}}^!$ to the full subcategory

$$(\mathrm{indSch}_{\mathrm{laft}})^{\mathrm{op}} \hookrightarrow (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}.$$

In particular, for $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$ we have a well-defined category $\mathrm{IndCoh}(\mathcal{X})$.

1.1.2.   Suppose $\mathcal{X}$ has been written as

$$(1.1) \qquad\qquad \mathcal{X} \simeq {}^{\mathrm{conv}}\mathcal{X}' \quad \mathcal{X}' \simeq \operatorname*{colim}_{a \in A} X_a,$$

where $X_a \in \mathrm{Sch}_{\mathrm{aft}}$ with the maps $i_{a,b} : X_a \to X_b$ being closed embeddings. In this case we have:

PROPOSITION 1.1.3.   *Under the above circumstances, !-restriction defines an equivalence*

$$\mathrm{IndCoh}(\mathcal{X}) \to \lim_{a \in A^{\mathrm{op}}} \mathrm{IndCoh}(X^a),$$

where for $a \to b$, the corresponding functor $\mathrm{IndCoh}(X_b) \to \mathrm{IndCoh}(X_a)$ is $i_{a,b}^!$.

PROOF.   This follows from the convergence property of the functor $\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}}}^!$, see Volume I, Chapter 5, Lemma 3.2.4 and Sect. 3.4.1.

$\square$

REMARK 1.1.4. The reason we exhibit an ind-scheme $\mathcal{X}$ as $^{\mathrm{conv}}(\mathrm{colim}_{a\in A} X_a)$ rather than just as $\mathrm{colim}_{a\in A} X_a$ is that the former presentation comes up in practice more often: many ind-schemes are given in this form. The fact that the resulting prestack is indeed an ind-scheme (i.e., can be written as a colimit of schemes under closed embeddings) is Chapter 2, Corollary 1.4.4 and is somewhat non-trivial.

1.1.5.   Combining the above proposition with Volume I, Chapter 1, Proposition 2.5.7, we obtain:

COROLLARY 1.1.6. *For $\mathcal{X}$ written as in* (1.1), *we have*
$$\mathrm{IndCoh}(\mathcal{X}) \simeq \mathrm{colim}_{a\in A} \mathrm{IndCoh}(X^a),$$
where for $a \to b$, the corresponding functor $\mathrm{IndCoh}(X_a) \to \mathrm{IndCoh}(X_b)$ *is* $(i_{a,b})_*^{\mathrm{IndCoh}}$.

COROLLARY 1.1.7. *For $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$ and a closed embedding $i : X \to \mathcal{X}$ from $X \in \mathrm{Sch}_{\mathrm{aft}}$, the functor*
$$i_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(\mathcal{X}),$$
left adjoint to $i^!$, is well-defined.

For $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$, let $\mathrm{Coh}(\mathcal{X})$ denote the full subcategory of $\mathrm{IndCoh}(\mathcal{X})$ spanned by objects
$$i_*^{\mathrm{IndCoh}}(\mathcal{F}), \quad i : X \to \mathcal{X} \text{ is a closed embedding and } \mathcal{F} \in \mathrm{Coh}(X).$$

From Corollary 1.1.6 we obtain:

COROLLARY 1.1.8. *For an ind-scheme $\mathcal{X}$, the category $\mathrm{IndCoh}(\mathcal{X})$ is compactly generated and*
$$\mathrm{IndCoh}(\mathcal{X})^c = \mathrm{Coh}(\mathcal{X}).$$

PROOF. Follows from [**DrGa1**, Corollary 1.9.4 and Lemma 1.9.5.]. $\square$

1.1.9.   Here is another convenient fact about the category $\mathrm{IndCoh}(\mathcal{X})$, where $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$. Let
$$X' \xrightarrow{i'} \mathcal{X} \xleftarrow{i''} X''$$
be closed embeddings.

We would like to calculate the composite
$$(i')^! \circ (i'')_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X'') \to \mathrm{IndCoh}(X').$$

Let A denote the category $(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed\ in}\ \mathcal{X}}$, so that X' and X'' correspond to indices a' and a'', respectively. Let $i_{a'}$ and $i_{a''}$ denote the corresponding closed embeddings, i.e., the maps i' and i'', respectively. Let B be any category cofinal in
$$A_{a' \sqcup a''/} := A_{a'/} \underset{A}{\times} A_{a''/}.$$
For $b \in B$, let
$$X' = X_{a'} \xrightarrow{i_{a',b}} X_b \xleftarrow{i_{a'',b}} X_{a''} = X''$$
denote the corresponding maps.

The next assertion results from [**Ga4**, Lemma 1.3.6]:

LEMMA 1.1.10. *Under the above circumstances, we have a canonical isomorphism*

$$(i')^! \circ (i'')_*^{\mathrm{IndCoh}} \simeq \operatorname*{colim}_{b \in B} (i_{a',b})^! \circ (i_{a'',b})_*^{\mathrm{IndCoh}}.$$

1.2. t-structure. In this subsection we will study the naturally defined t-structure on IndCoh of an ind-scheme.

1.2.1. For $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$ we introduce a t-structure on the category $\mathrm{IndCoh}(\mathcal{X})$ as follows:

An object $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ belongs to $\mathrm{IndCoh}(\mathcal{X})^{\geq 0}$ if and only if for every closed embedding $i : X \to \mathcal{X}$, where $X \in \mathrm{Sch}_{\mathrm{aft}}$, we have $i^!(\mathcal{F}) \in \mathrm{IndCoh}(X)^{\geq 0}$.

By construction, this t-structure is compatible with filtered colimits, which by definition means that $\mathrm{IndCoh}(\mathcal{X})^{\geq 0}$ is preserved by filtered colimits.

1.2.2. We can describe this t-structure and the category $\mathrm{IndCoh}(\mathcal{X})^{\leq 0}$ more explicitly. Write

$$^{\mathrm{cl}}\mathcal{X} \simeq \operatorname*{colim}_{a \in A} X_a,$$

where $X_a \in (^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed\ in}\ \mathcal{X}}$.

For each a, let i_a denote the corresponding map (automatically, a closed embedding) $X_a \to \mathcal{X}$. By Corollary 1.1.7, we have a pair of adjoint functors

$$(i_a)_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X_a) \rightleftarrows \mathrm{IndCoh}(\mathcal{X}) : i_a^!.$$

LEMMA 1.2.3. *Under the above circumstances we have:*

(a) *An object $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ belongs to $\mathrm{IndCoh}^{\geq 0}(\mathcal{X})$ if and only if for every a, the object $i_a^!(\mathcal{F}) \in \mathrm{IndCoh}(X_a)$ belongs to $\mathrm{IndCoh}(X_a)^{\geq 0}$.*

(b) *The category $\mathrm{IndCoh}(\mathcal{X})^{\leq 0}$ is generated under colimits by the essential images of the functors $(i_a)_*^{\mathrm{IndCoh}} (\mathrm{Coh}(X_a)^{\leq 0})$.*

PROOF. It is easy to see that for a quasi-compact DG scheme X, the category $\mathrm{IndCoh}(X)^{\leq 0}$ is generated under colimits by $\mathrm{Coh}(^{\mathrm{cl}}X)^{\leq 0}$. In particular, by adjunction, an object $\mathcal{F} \in \mathrm{IndCoh}(X)$ is coconnective if and only if its restriction to $^{\mathrm{cl}}X$ is coconnective.

Hence, in the definition of $\mathrm{IndCoh}(\mathcal{X})^{\geq 0}$, instead of all closed embeddings $X \to \mathcal{X}$, it suffices to use only those with X a classical scheme.

Note that the category A is cofinal in $(^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed\ in}\ \mathcal{X}}$. This implies point (a) of the lemma. Point (b) follows formally from point (a).

$\square$

1.2.4. Suppose $i : X \to \mathcal{X}$ is a closed embedding of a scheme into an ind-scheme. By Corollary 1.1.7, we have a well-defined functor

$$i_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(\mathcal{X}),$$

which is the right adjoint to $i^!$. Since $i^!$ is left t-exact, the functor i_*^{IndCoh} is right t-exact. However, we claim:

LEMMA 1.2.5. *The functor i_*^{IndCoh} is t-exact.*

PROOF. We need to show that for $\mathcal{F} \in \mathrm{IndCoh}(X)^{\geq 0}$, and a closed embedding $i' : X' \to \mathcal{X}$, we have

$$(i')^! \circ i_*^{\mathrm{IndCoh}}(\mathcal{F}) \in \mathrm{IndCoh}(X')^{\geq 0}.$$

This follows from Lemma 1.1.10: in the notations of *loc.cit.*, each of the functors $(i_{a'',b})_*^{\mathrm{IndCoh}}$ is t-exact (because $i_{a'',b}$ is a closed embedding), each of the functors $(i_{a'',b})^!$ is left t-exact (because $i_{a',b}$ is a closed embedding), and the category B is filtered.

$\square$

COROLLARY 1.2.6. *The subcategory*

$$\mathrm{Coh}(\mathcal{X}) = \mathrm{IndCoh}(\mathcal{X})^c$$

is preserved by the truncation functors.

PROOF. Follows from Lemma 1.2.5 and the corresponding fact for schemes. $\square$

COROLLARY 1.2.7. *The t-structure on $\mathrm{IndCoh}(\mathcal{X})$ is obtained from the t-structure on $\mathrm{Coh}(\mathcal{X})$ by the procedure of Volume I, Chapter 4, Lemma 1.2.4.*

1.3. Recovering IndCoh from ind-proper maps. The contents of this subsection are rather formal: we show that the functor IndCoh on ind-schemes can be recovered from the corresponding functor on schemes, where we restrict 1-morphisms to be proper, or even closed embeddings. This is not surprising, given the definition of ind-schemes.

1.3.1. Recall what it means for a map in PreStk to be *ind-proper* (resp., *ind-closed embedding*), see Chapter 2, Definitions 1.6.7 and 1.6.11.

1.3.2. Consider the corresponding 1-full subactegories

$$(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}closed}} \subset (\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}$$

and the corresponding categories

$$(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}} \subset (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}.$$

Consider the corresponding fully faithful embeddings

$$(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}} \hookrightarrow (\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}closed}},$$

and

$$(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}} \hookrightarrow (\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}.$$

Let $\mathrm{IndCoh}^!_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}}$ denote the functor

$$\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{aft}}}|_{((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}})^{\mathrm{op}}} : ((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

and similarly, for 'proper' replaced by 'closed'.

Let $\mathrm{IndCoh}^!_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}}$ denote the functor

$$\mathrm{IndCoh}^!_{\mathrm{indSch}_{\mathrm{laft}}}|_{((\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}})^{\mathrm{op}}} : ((\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

and similarly, for 'ind-proper' replaced by 'ind-closed'.

1.3.3. We claim:

PROPOSITION 1.3.4. *For* $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$, *the functors*

(1.2)
$$(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed \, in} \, \mathcal{X}} \simeq (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}} \underset{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}closed}}}{\times} ((\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}closed}})_{/\mathcal{X}} \to (\mathrm{Sch}_{\mathrm{aft}})_{/\mathcal{X}}$$

and

(1.3)
$$(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}/\mathcal{X}} \simeq (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}} \underset{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}}{\times} ((\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}})_{/\mathcal{X}} \to (\mathrm{Sch}_{\mathrm{aft}})_{/\mathcal{X}}$$

are cofinal.

PROOF. The cofinality of (1.2) is given by Chapter 2, Corollary 1.7.5(b). Since (1.3) is fully faithful, we have that the functor

$$(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed \, in} \, \mathcal{X}} \to (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}} \underset{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}}{\times} ((\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}})_{/\mathcal{X}} \, ,$$

and hence (1.3), is also cofinal.   □

COROLLARY 1.3.5. *The naturally defined functors*

$$\mathrm{IndCoh}^{!}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}} \to \mathrm{RKE}_{((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}})^{\mathrm{op}} \to ((\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}})^{\mathrm{op}}} \left(\mathrm{IndCoh}^{!}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}} \right),$$

and

$$\mathrm{IndCoh}^{!}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}closed}}} \to \mathrm{RKE}_{((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}})^{\mathrm{op}} \to ((\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}closed}})^{\mathrm{op}}} \left(\mathrm{IndCoh}^{!}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}}} \right)$$

are isomorphisms.

PROOF. The cofinality of (1.2) implies that the functor

$$\mathrm{IndCoh}^{!}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}closed}}} \to \mathrm{RKE}_{((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}})^{\mathrm{op}} \to ((\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}closed}})^{\mathrm{op}}} \left(\mathrm{IndCoh}^{!}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}}} \right)$$

is an isomorphism.

The cofinality of (1.3) implies that the functor

$$\mathrm{IndCoh}^{!}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}} \to \mathrm{RKE}_{((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}})^{\mathrm{op}} \to ((\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}})^{\mathrm{op}}} \left(\mathrm{IndCoh}^{!}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}} \right)$$

is an isomorphism.

                                                                                □

1.4. Direct image for IndCoh **on ind-schemes.** In this subsection we show how to construct the functor of direct image on IndCoh for morphisms between ind-schemes.

1.4.1. Consider the functor

$$\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}} : \mathrm{Sch}_{\mathrm{aft}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

where for a morphism $f : X_1 \to X_2$ in $\mathrm{Sch}_{\mathrm{aft}}$, the functor

$$\mathrm{IndCoh}(X_1) \to \mathrm{IndCoh}(X_2)$$

is f_{*}^{IndCoh}, see Volume I, Chapter 4, Sect. 2.2.

Recall the notation

$$\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}} = \mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}|_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}} : (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

and consider also the corresponding functor

$$\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}}} : (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}} \to \mathrm{DGCat}_{\mathrm{cont}} \, .$$

Denote

$$\mathrm{IndCoh}_{\mathrm{indSch}_{\mathrm{laft}}} \coloneqq \mathrm{LKE}_{(\mathrm{Sch}_{\mathrm{aft}}) \hookrightarrow (\mathrm{indSch}_{\mathrm{laft}})}(\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})}),$$

and let

$$\mathrm{IndCoh}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}}, \ \mathrm{IndCoh}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}closed}}}$$

denote its restriction to the corresponding 1-full subcategories.

The same proof as that of Corollary 1.3.5 gives:

PROPOSITION 1.4.2. *The natural maps*

$$\mathrm{LKE}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}} \hookrightarrow (\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}}(\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}}) \to \mathrm{IndCoh}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}},$$

and

$$\mathrm{LKE}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}} \hookrightarrow (\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}closed}}}(\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{closed}}}) \to \mathrm{IndCoh}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}closed}}}$$

are isomorphisms.

1.4.3. Recall from Volume I, Chapter 1, Sect. 2.4 the notion of two functors obtained from each other by passing to adjoints.

Let $F : \mathbf{C}_1 \to \mathbf{C}_2$ be a functor between ∞-categories. Let $\Phi_1 : \mathbf{C}_1 \to \mathrm{DGCat}_{\mathrm{cont}}$ be a functor such that for every $\mathbf{c}_1' \to \mathbf{c}_1''$, the corresponding functor

$$\Phi_1(\mathbf{c}_1') \to \Phi_1(\mathbf{c}_1'')$$

admits a right adjoint. Let $\Psi_1 : \mathbf{C}_1^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$ be the resulting functor given by taking the right adjoints.

Let Φ_2 and Ψ_2 be the left (resp., right) Kan extension of Φ_1 (resp., Ψ_1) along F (resp., F^{op}). The following is a particular case of Volume I, Chapter 8, Proposition 2.2.7:

LEMMA 1.4.4. *Under the above circumstances, the functor Ψ_2 is obtained from Φ_2 by taking right adjoints.*

1.4.5. We apply Lemma 1.4.4 in the following situation:

$$\mathbf{C}_1 \coloneqq (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}, \ \mathbf{C}_2 \coloneqq (\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}},$$

and F to be the natural embedding. We take

$$\Phi_1 \coloneqq \mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}} \text{ and } \Psi_1 \coloneqq \mathrm{IndCoh}^!_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}}.$$

These two functors are obtained from one another by passage to adjoints, by the definiton of the functor $\mathrm{IndCoh}^!_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}}$, see Volume I, Chapter 4, Corollary 5.1.12.

COROLLARY 1.4.6. *The functor*

$$\mathrm{IndCoh}^!_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}} : ((\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

is obtained from the functor

$$\mathrm{IndCoh}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}} : (\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

by passing to right adjoints.

1.4.7.   By the above corollary and Proposition 1.4.2, for $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$ there is a canonical isomorphism

$$\mathrm{IndCoh}^!_{\mathrm{indSch}_{\mathrm{laft}}}(\mathcal{X}) \simeq \mathrm{IndCoh}_{\mathrm{indSch}_{\mathrm{laft}}}(\mathcal{X}),$$

and by definition the left hand side is $\mathrm{IndCoh}(\mathcal{X})$. Thus, given a morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$ in $\mathrm{indSch}_{\mathrm{laft}}$, the functor $\mathrm{IndCoh}_{\mathrm{indSch}_{\mathrm{laft}}}$ gives a functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}_1) \to \mathrm{IndCoh}(\mathcal{X}_2),$$

which, by definition of the functor $\mathrm{IndCoh}_{\mathrm{indSch}_{\mathrm{laft}}}$, agrees with the previously defined IndCoh direct image functor when restricted to $\mathrm{Sch}_{\mathrm{aft}}$. Furthermore, by Corollary 1.4.6, if f is ind-proper, then f_*^{IndCoh} is the left adjoint of $f^!$. In particular, for a closed embedding

$$X \xrightarrow{i} \mathcal{X}$$

of a scheme $X \in \mathrm{Sch}_{\mathrm{aft}}$ into an ind-scheme $\mathcal{X} \in \mathrm{indSch}_{\mathrm{laft}}$, the corresponding functor i_*^{IndCoh} agrees with that of Corollary 1.1.7.

1.4.8.   We can now make the following observation pertaining to the behavior of the t-structure with respect to direct images:

LEMMA 1.4.9. Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a map of ind-schemes. Then the functor f_*^{IndCoh} is left t-exact. Furthermore, if f is ind-affine, then it is t-exact.

PROOF. Let $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X}_1)^{\geq 0}$. We wish to show that $f_*^{\mathrm{IndCoh}}(\mathcal{F}) \in \mathrm{IndCoh}(\mathcal{X}_2)^{\geq 0}$. By Corollary 1.2.7, we can assume that $\mathcal{F} = (i_1)_*^{\mathrm{IndCoh}}(\mathcal{F}_1)$ for $\mathcal{F}_1 \in \mathrm{IndCoh}(X_1)^{\geq 0}$ where

$$i_1 : X_1 \to \mathcal{X}_1$$

is a closed embedding of a scheme. Now, let

$$X_1 \xrightarrow{g} X_2 \xrightarrow{i_2} \mathcal{X}_2$$

be a factorization of $f \circ i_1$, where i_2 is a closed embedding of a scheme. Thus, it suffices to show that the functor

$$f_*^{\mathrm{IndCoh}} \circ (i_1)_*^{\mathrm{IndCoh}} \simeq (i_2)_*^{\mathrm{IndCoh}} \circ g_*^{\mathrm{IndCoh}}$$

is left t-exact. However, $(i_2)_*^{\mathrm{IndCoh}}$ is t-exact by Lemma 1.2.5, while $g_*^{\mathrm{IndCoh}}(\mathcal{F}_1)$ is left t-exact, since g is a map between schemes.

Now, suppose that f is ind-affine. In this case, we wish to show that f_*^{IndCoh} is also right t-exact. Let $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X}_1)^{\leq 0}$. We can assume that $\mathcal{F} = (i_1)_*^{\mathrm{IndCoh}}(\mathcal{F}_1)$ for $\mathcal{F}_1 \in \mathrm{IndCoh}(X_1)^{\leq 0}$ where $i_1 : X_1 \to \mathcal{X}_1$ is a closed embedding. In the notation as above, it suffices to show that

$$f_*^{\mathrm{IndCoh}} \circ (i_1)_*^{\mathrm{IndCoh}} \simeq (i_2)_*^{\mathrm{IndCoh}} \circ g_*^{\mathrm{IndCoh}}$$

is t-exact.

By Lemma 1.2.5, $(i_2)_*^{\mathrm{IndCoh}}$ is t-exact. Hence, it suffices to show that g_*^{IndCoh} is t-exact. However, g is an affine map between schemes, and the assertion follows. □

2. Proper base change for ind-schemes

Base change for IndCoh is a crucial property needed for its definition as a functor out of the category of correspondences. In this section we make two (necessary) preparatory steps, establishing base change for morphisms between ind-schemes.

2.1. 1st version.

2.1.1. Recall the notion of *ind-schematic* map in PreStk, see Chapter 2, Defintion 1.6.5(a).

Let

$$\begin{array}{ccc} \mathcal{X}_1' & \xrightarrow{g_1} & \mathcal{X}_1 \\ f' \downarrow & & \downarrow f \\ \mathcal{X}_2' & \xrightarrow{g_2} & \mathcal{X}_2, \end{array}$$

be a Cartesian diagram in $\mathrm{PreStk}_{\mathrm{laft}}$ with f being ind-schematic and ind-proper. We claim:

PROPOSITION 2.1.2. *The functors $f^!$ and $(f')^!$ admit left adjoints, to be denoted f_*^{IndCoh} and $(f')_*^{\mathrm{IndCoh}}$, respectively. Moreover, the natural transformation*

$$(2.1) \qquad (f')_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\mathrm{IndCoh}},$$

arrising by adjunction from

$$g_1^! \circ f^! \simeq (f')^! \circ g_2^!,$$

is an isomorphism.

The rest of this subsection is devoted to the proof of this proposition.

2.1.3. We begin by reviewing the setting of Volume I, Chapter 1, Lemma 2.6.4:

Let $G : \mathbf{C}_2 \to \mathbf{C}_1$ be a functor between ∞-categories. Let A be a category of indices, and suppose we are given an A-family of commutative diagrams

$$\begin{array}{ccc} \mathbf{C}_1^a & \xleftarrow{i_1^a} & \mathbf{C}_1 \\ G^a \uparrow & & \uparrow G \\ \mathbf{C}_2^a & \xleftarrow{i_2^a} & \mathbf{C}_2. \end{array}$$

Assume that for each $a \in A$, the functor G^a admits a left (resp. right) adjoint F^a. Furthermore, assume that for each map $a' \to a''$ in A, the diagram

$$\begin{array}{ccc} \mathbf{C}_1^{a''} & \xleftarrow{i_1^{a',a''}} & \mathbf{C}_1^{a'} \\ F^{a''} \downarrow & & \downarrow F^{a'} \\ \mathbf{C}_2^{a''} & \xleftarrow{i_2^{a',a''}} & \mathbf{C}_2^{a'}, \end{array}$$

which a priori commutes up to a natural transformation, actually commutes.

Finally, assume that the functors

$$\mathbf{C}_1 \to \lim_{a \in A} \mathbf{C}_1^a \text{ and } \mathbf{C}_2 \to \lim_{a \in A} \mathbf{C}_2^a$$

are equivalences.

In the above situation, Volume I, Chapter 1, Lemma 2.6.4 says:

LEMMA 2.1.4. *The functor G admits a left (resp. right) adjoint F, and for every $a \in A$, the diagram*

$$
\begin{array}{ccc}
\mathbf{C}_1^a & \xleftarrow{\; i_1^a \;} & \mathbf{C}_1 \\
{\scriptstyle F^a}\downarrow & & \downarrow{\scriptstyle F} \\
\mathbf{C}_2^a & \xleftarrow{\; i_2^a \;} & \mathbf{C}_2,
\end{array}
$$

which a priori commutes up to a natural transformation, commutes.

2.1.5.   To show that (2.1) is an isomorphism, it suffices to show that it becomes an isomorphism after composing with $f^!$ for every map $f : S \to \mathcal{X}_2'$, with $S \in \mathrm{Sch}_{\mathrm{aft}}$. Therefore, we can assume that $\mathcal{X}_2' = X_2' \in \mathrm{Sch}_{\mathrm{aft}}$.

Now, we will apply Lemma 2.1.4 to the following situation. Let $\mathbf{C}_1 := \mathrm{IndCoh}(\mathcal{X}_1)$, $\mathbf{C}_2 := \mathrm{IndCoh}(\mathcal{X}_2)$ and let A be the category $(\mathrm{Sch}_{\mathrm{aft}})_{/\mathcal{X}_2}$. For each $Z \in (\mathrm{Sch}_{\mathrm{aft}})_{/\mathcal{X}_2}$, let

$$
\mathbf{C}_2^a := \mathrm{IndCoh}(Z), \quad \mathbf{C}_1^a := \mathrm{IndCoh}(Z \underset{\mathcal{X}_2}{\times} \mathcal{X}_1).
$$

Now, since X_2' is in particular an object of $(\mathrm{Sch}_{\mathrm{aft}})_{/\mathcal{X}_2}$, by Lemma 2.1.4, the assertion of Proposition 2.1.2 reduces to the case when $\mathcal{X}_2 = X_2 \in \mathrm{Sch}_{\mathrm{aft}}$ and $\mathcal{X}_2' = X_2' \in \mathrm{Sch}_{\mathrm{aft}}$. In this case $\mathcal{X}_1, \mathcal{X}_1' \in \mathrm{indSch}_{\mathrm{laft}}$ and the left adjoints exist by Corollary 1.4.6.

2.1.6.   We have

$$
\mathcal{X}_1 \simeq \operatorname*{colim}_{a \in A} X_a,
$$

where $X_a \in \mathrm{Sch}_{\mathrm{aft}}$ and $i_a : X_a \to \mathcal{X}_1$ are closed embeddings.

Set

$$
X_a' := X_2' \underset{X_2}{\times} X_a.
$$

We have:

$$
\mathcal{X}_1' \simeq \operatorname*{colim}_{a \in A} X_a',
$$

Let i_a' denote the corresponding closed embedding $X_a' \to \mathcal{X}_1'$, and let g_a denote the map $X_a' \to X_a$.

Note that the maps $f \circ i_a : X_a \to X_2$ and $f' \circ i_a' : X_a' \to X_2'$ are proper, by assumption.

2.1.7.   By Corollary 1.1.6, we have:

$$
\mathrm{Id}_{\mathrm{IndCoh}(\mathcal{X}_1)} \simeq \operatorname*{colim}_{a \in A} (i_a)_*^{\mathrm{IndCoh}} \circ (i_a)^! \quad \text{and} \quad \mathrm{Id}_{\mathrm{IndCoh}(\mathcal{X}_1')} \simeq \operatorname*{colim}_{a \in A} (i_a')_*^{\mathrm{IndCoh}} \circ (i_a')^!.
$$

Hence, we can rewrite the left-hand side in (2.1) as

$$
\operatorname*{colim}_{a \in A} (f')_*^{\mathrm{IndCoh}} \circ (i_a')_*^{\mathrm{IndCoh}} \circ (i_a')^! \circ g_1^!,
$$

and the right-hand side as

$$
\operatorname*{colim}_{a \in A} g_2^! \circ f_*^{\mathrm{IndCoh}} \circ (i_a)_*^{\mathrm{IndCoh}} \circ (i_a)^!.
$$

It follows from the construction that the map in (2.1) is given by a compatible system of maps for each $a \in A$

$$(f')_*^{\mathrm{IndCoh}} \circ (i'_a)_*^{\mathrm{IndCoh}} \circ (i'_a)^! \circ g_1^! \simeq (f' \circ i'_a)_*^{\mathrm{IndCoh}} \circ (g_1 \circ i'_a)^! \simeq$$

$$(f' \circ i'_a)_*^{\mathrm{IndCoh}} \circ (i_a \circ g_a)^! \simeq (f' \circ i'_a)_*^{\mathrm{IndCoh}} \circ g_a^! \circ i_a^! \to$$

$$\to g_2^! \circ (f \circ i_a)_*^{\mathrm{IndCoh}} \circ i_a^! \simeq g_2^! \circ f_*^{\mathrm{IndCoh}} \circ (i_a)_*^{\mathrm{IndCoh}} \circ (i_a)^!,$$

where the arrow

$$(f' \circ i'_a)_*^{\mathrm{IndCoh}} \circ g_a^! \to g_2^! \circ (f \circ i_a)_*^{\mathrm{IndCoh}}$$

is base change for the Cartesian square

$$
\begin{array}{ccc}
X'_a & \xrightarrow{\ g_a\ } & X_a \\
{\scriptstyle f' \circ i'_a}\downarrow & & \downarrow{\scriptstyle f \circ i_a} \\
X'_2 & \xrightarrow{\ g_2\ } & X_2.
\end{array}
$$

Hence, the required isomorphism follows from proper base change in the case of schemes, see Volume I, Chapter 5, Proposition 3.1.4(b).

$\square$

2.1.8. In the sequel we will need the following corollary of Proposition 2.1.2:

COROLLARY 2.1.9. *Let $\mathcal{X}' \to \mathcal{X}$ be a map of ind-schemes. Then the functor*

$$\mathrm{IndCoh}(\mathcal{X}') \to \lim_{(Z \xrightarrow{f} \mathcal{X}) \in ((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper\ over\ } \mathcal{X}})^{\mathrm{op}}} \mathrm{IndCoh}(Z \underset{\mathcal{X}}{\times} \mathcal{X}')$$

is an equivalence.

PROOF. The statement of the corollary is equivalent to the fact that for $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X}')$, the map

$$(2.2) \qquad \underset{(Z \xrightarrow{f} \mathcal{X}) \in (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper\ over\ } \mathcal{X}}}{\mathrm{colim}} f'^{\mathrm{IndCoh}}_* \circ f'^!(\mathcal{F}) \to \mathcal{F}$$

is an isomorphism, where

$$Z \underset{\mathcal{X}}{\times} \mathcal{X}' =: Z' \xrightarrow{f'} \mathcal{X}'.$$

Note that base change (i.e., Proposition 2.1.2) implies the projection formula, so for each $Z' \xrightarrow{f'} \mathcal{X}'$ as above, the natural map

$$f'^{\mathrm{IndCoh}}_* \circ f'^!(\mathcal{F}) \to f'^{\mathrm{IndCoh}}_* \circ f'^!(\omega_{\mathcal{X}}) \overset{!}{\otimes} \mathcal{F}$$

is an isomorphism.

Hence, the map in (2.2) is obtained by tensoring with $\mathcal{F}$ from the map

$$\underset{(Z \xrightarrow{f} \mathcal{X}) \in (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper\ over\ } \mathcal{X}}}{\mathrm{colim}} f'^{\mathrm{IndCoh}}_* \circ f'^!(\omega_{\mathcal{X}'}) \to \omega_{\mathcal{X}'},$$

and therefore it is sufficient to check that the latter map is an isomorphism.

However, again by base change, the latter map identifies with the pullback under $\mathcal{X}' \to \mathcal{X}$ of the map

$$\underset{(Z \xrightarrow{f} \mathcal{X}) \in (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper\ over\ } \mathcal{X}}}{\mathrm{colim}} f_*^{\mathrm{IndCoh}} \circ f^!(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}},$$

while the latter is an isomorphism by Corollary 1.3.5.

$\square$

2.2. 2nd version.

2.2.1.   Now, let

$$
\begin{array}{ccc}
\mathcal{X}_1' & \xrightarrow{\ g_1\ } & \mathcal{X}_1 \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
\mathcal{X}_2' & \xrightarrow{\ g_2\ } & \mathcal{X}_2
\end{array}
$$

be a Cartesian diagram in $\mathrm{indSch}_{\mathrm{laft}}$, such that the map g_2, and hence g_1, is ind-proper.

From the isomorphism

$$
(g_2)_*^{\mathrm{IndCoh}} \circ (f')_*^{\mathrm{IndCoh}} \simeq f_*^{\mathrm{IndCoh}} \circ (g_1)_*^{\mathrm{IndCoh}},
$$

we obtain, by adjunction, a natural transformation:

$$
(2.3) \qquad\qquad (f')_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\mathrm{IndCoh}}.
$$

We claim:

PROPOSITION 2.2.2.   *The map* (2.3) *is an isomorphism.*

The rest of this subsection is devoted to the proof of the proposition.

2.2.3.   First, suppose that f is ind-proper. In this case, the map (2.3) equals the map (2.1). Hence, it is an isomorphism by Proposition 2.1.2.

2.2.4.   We have

$$
\mathcal{X}_2' \simeq \operatorname*{colim}_{a \in A} X_{2,a}',
$$

where $X_{2,a}' \in \mathrm{Sch}_{\mathrm{aft}}$ and the maps $i_{2,a} : X_{2,a}' \to \mathcal{X}_2'$ are closed embeddings. Therefore, it suffices to show that (2.3) becomes an isomorphism after composing both sides with $i_{2,a}^!$ for every a. Thus, we can assume without loss of generality that $\mathcal{X}_2' = X_2' \in \mathrm{Sch}_{\mathrm{aft}}$.

Furthermore, by Corollary 1.1.6, we need to show that (2.3) becomes an isomorphism after precomposing both sides with the functor $(i_1)_*^{\mathrm{IndCoh}}$ for a closed embedding $i_1 : X_1 \to \mathcal{X}_1$ with $X_1 \in \mathrm{Sch}_{\mathrm{aft}}$. Consider the commutative diagram

$$
\begin{array}{ccc}
X_1' & \longrightarrow & X_1 \\
\downarrow & & \downarrow{\scriptstyle i_1} \\
\mathcal{X}_1' & \xrightarrow{\ g_1\ } & \mathcal{X}_1 \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
X_2' & \xrightarrow{\ g_2\ } & \mathcal{X}_2',
\end{array}
$$

where both squares are Cartesian. Since i_1 is ind-proper, we have that base change holds for the top square. Hence, to show that (2.3) becomes an isomorphism after precomposing with $(i_1)_*^{\mathrm{IndCoh}}$, we need to show that base change holds for the outer square. In particular, we reduce to the case when $\mathcal{X}_1 = X_1 \in \mathrm{Sch}_{\mathrm{aft}}$.

2.2.5. By Chapter 2, Corollary 1.7.5(b), we can factor the map $X_1 \to \mathcal{X}_2$ as a composition

$$X_1 \to \widetilde{X}_1 \to \mathcal{X}_2,$$

where $\widetilde{X}_1 \in \mathrm{Sch}_{\mathrm{aft}}$ and $\widetilde{X}_1 \to \mathcal{X}_2$ is a closed embedding (and in particular schematic). We have the diagram

$$
\begin{array}{ccc}
X_1' & \longrightarrow & X_1 \\
\downarrow & & \downarrow \\
\widetilde{X}_1' & \longrightarrow & \widetilde{X}_1 \\
\downarrow & & \downarrow \\
X_2' & \longrightarrow & \mathcal{X}_2
\end{array}
$$

where all the squares are Cartesian. The top square is a Cartesian square in $\mathrm{Sch}_{\mathrm{aft}}$ and hence satisfies base change by Volume I, Chapter 4, Proposition 5.2.1. In the bottom square, the map $\widetilde{X}_1 \to \mathcal{X}_2$ is ind-proper, and hence it satisfies base change by the above. Hence the outer square satisfies base change as desired. $\qquad\square$

3. IndCoh on (ind)-inf-schemes

In this section we begin the development of the theory of IndCoh on ind-inf-schemes. We will essentially bootstrap it from IndCoh on ind-schemes, using *nil base change*.

3.1. Nil base change. As just mentioned, nil base change is a crucial property of the category IndCoh. Its proof relies on the structural results on inf-schemes from Chapter 2, Sect. 4.

3.1.1. Recall the notion of an *ind-inf-schematic* map in PreStk, see Chapter 2, Definition 3.1.5.

We will show:

PROPOSITION 3.1.2. *Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a map in $\mathrm{PreStk}_{\mathrm{laft}}$, and assume that f is an inf-schematic nil-isomorphism.*

(a) *The functor $f^! : \mathrm{IndCoh}(\mathcal{X}_2) \to \mathrm{IndCoh}(\mathcal{X}_1)$ admits a left adjoint, to be denoted f_*^{IndCoh}.*

(b) *The functor $f^!$ is conservative.*

(c) *For a Cartesian daigram*

$$
\begin{array}{ccc}
\mathcal{X}_1' & \xrightarrow{\ g_1\ } & \mathcal{X}_1 \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
\mathcal{X}_2' & \xrightarrow{\ g_2\ } & \mathcal{X}_2,
\end{array}
$$

the natural transformation

$$(f')_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\mathrm{IndCoh}},$$

arrising by adjunction from

$$g_1^! \circ f^! \simeq (f')^! \circ g_2^!,$$

is an isomorphism.

3.1.3. *Proof of Proposition 3.1.2.* Using Lemma 2.1.4, we reduce the assertion to the case when $\mathcal{X}_2 = X_2 \in \mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}}$, for points (a) and (b), and further to the case when $\mathcal{X}_2' = X_2' \in \mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}}$ for point (c).

In this case $\mathcal{X}_1$ has the property that $^{\mathrm{red}}\mathcal{X}_1 = X_1 \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$. By Chapter 2, Corollary 4.3.3, we can write

$$\mathcal{X}_1 \simeq \operatorname*{colim}_{a \in A} X_{1,a},$$

where A is the category

$$(\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}})_{/\mathcal{X}_1} \underset{({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}})_{/X_1}}{\times} \{X_1\},$$

and the colimit is taken in the category $\mathrm{PreStk}_{\mathrm{laft}}$.

In particular, for every a, the resulting map $X_{1,a} \to X_2$ is a nil-isomorphism, and hence, proper. Moreover, for every morphism $a' \to a''$, the corresponding map

$$i_{a',a''} : X_{1,a'} \to X_{1,a''}$$

is also a nil-isomorphism and, in particular, is proper.

We have

$$\mathrm{IndCoh}(\mathcal{X}_1) \simeq \operatorname*{lim}_{a \in A^{\mathrm{op}}} \mathrm{IndCoh}(X_{1,a}),$$

and the fact that $f^!$ is conservative follows from the fact that each $(f \circ i_a)^!$ is conservative.

Using Volume I, Chapter 1, Proposition 2.5.7, we can therefore rewrite

(3.1) $$\mathrm{IndCoh}(\mathcal{X}_1) \simeq \operatorname*{colim}_{a \in A} \mathrm{IndCoh}(X_{1,a}),$$

where the colimit is taken with respect to the functors

$$(i_{a',a''})_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X_{1,a'}) \to \mathrm{IndCoh}(X_{1,a''}).$$

Now, the left adjoint to f is given by the compatible collection of functors

$$(f \circ i_a)_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X_{1,a}) \to \mathrm{IndCoh}(X_2).$$

Thus, it remains to establish the base change property. However, the latter follows by repeating the argument in Sects. 2.1.6-2.1.7.

$\square$

3.2. Basic properties.
We will now use nil base change to establish some basic properties of the category IndCoh on an ind-inf-scheme.

3.2.1.   First, as a corollary of Proposition 3.1.2 we obtain:

COROLLARY 3.2.2. *Let $\mathcal{X}$ be an object of* $\mathrm{indinfSch}_{\mathrm{laft}}$. *Then the category* $\mathrm{IndCoh}(\mathcal{X})$ *is compactly generated.*

PROOF. Consider the canonical map $i : {}^{\mathrm{red}}\mathcal{X} \to \mathcal{X}$. The category $\mathrm{IndCoh}({}^{\mathrm{red}}\mathcal{X})$ is compactly generated by Corollary 1.1.8. Now, Proposition 3.1.2 implies that the essential image of $i_*^{\mathrm{IndCoh}}(\mathrm{IndCoh}({}^{\mathrm{red}}\mathcal{X})^c)$ compactly generates $\mathrm{IndCoh}(\mathcal{X})$.

$\square$

3.2.3. Let

$$
\begin{array}{ccc}
\mathcal{X}_1' & \xrightarrow{\ g_1\ } & \mathcal{X}_1 \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
\mathcal{X}_2' & \xrightarrow{\ g_2\ } & \mathcal{X}_2,
\end{array}
$$

be a Cartesian diagram in $\mathrm{PreStk}_{\mathrm{laft}}$ with f being ind-inf-schematic and ind-proper. We claim:

PROPOSITION 3.2.4. *The functors $f^!$ and $(f')^!$ admit left adjoints, to be denoted f_*^{IndCoh} and $(f')_*^{\mathrm{IndCoh}}$, respectively. The natural transformation*

(3.2) $$(f')_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\mathrm{IndCoh}},$$

arrising by adjunction from

$$g_1^! \circ f^! \simeq (f')^! \circ g_2^!,$$

is an isomorphism.

PROOF. By Lemma 2.1.4, the assertion of the proposition reduces to the case when $\mathcal{X}_2$ and (resp., both $\mathcal{X}_2$ and $\mathcal{X}_2'$) belong to $\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}}$. Denote these objects by X_2 and X_2', respectively. In this case $\mathcal{X}_1$ (resp., both $\mathcal{X}_1$ and $\mathcal{X}_1'$) belong to $\mathrm{indinfSch}_{\mathrm{laft}}$. The existence of the left adjoint f_*^{IndCoh} (and therefore also $(f')_*^{\mathrm{IndCoh}}$) follows, using Chapter 2, Corollary 4.1.4 and Proposition 2.1.2, by the same argument as Proposition 3.1.2(a).

Now, let $\mathcal{X}_0$ be any object of $\mathrm{indSch}_{\mathrm{laft}}$ endowed with a nil-isomorphism to $\mathcal{X}_1$; e.g., $\mathcal{X}_0 = {}^{\mathrm{red}}\mathcal{X}_1$. Set

$$\mathcal{X}_0' := X_2' \underset{X_2}{\times} \mathcal{X}_0,$$

and consider the diagram

$$
\begin{array}{ccc}
\mathcal{X}_0' & \xrightarrow{\ g_0\ } & \mathcal{X}_0 \\
{\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\
\mathcal{X}_1' & \xrightarrow{\ g_1\ } & \mathcal{X}_1 \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
X_2' & \xrightarrow{\ g_2\ } & X_2,
\end{array}
$$

in which both squares are Cartesian.

By Proposition 3.1.2(b), it suffices to prove the assertion for the top square and for the outer square.

Now, the assertion for the outer square is given by Proposition 2.1.2, and for the top square by Proposition 3.1.2(c).

$\square$

3.3. Descent for ind-inf-schematic ind-proper maps.

3.3.1. Let $\mathcal{X}^\bullet$ be a groupoid simplicial object in $\mathrm{PreStk}_{\mathrm{laft}}$, see [**Lu1**, Definition 6.1.2.7]. Denote by

$$(3.3) \qquad\qquad\qquad p_s, p_t : \mathcal{X}^1 \rightrightarrows \mathcal{X}^0$$

the corresponding maps.

We form a co-simplicial category $\mathrm{IndCoh}(\mathcal{X}^\bullet)^!$ using the !-pullback functors, and consider its totalization $\mathrm{Tot}(\mathrm{IndCoh}(\mathcal{X}^\bullet)^!)$. Consider the functor of evaluation on 0-simplices:

$$\mathrm{ev}^0 : \mathrm{Tot}(\mathrm{IndCoh}(\mathcal{X}^\bullet)) \to \mathrm{IndCoh}(\mathcal{X}^0).$$

3.3.2. We claim:

PROPOSITION 3.3.3. *Suppose that the maps p_s, p_t in* (3.3) *are ind-inf-schematic and ind-proper. Then:*

(a) *The functor ev^0 admits a left adjoint and the adjoint pair*

$$\mathrm{IndCoh}(\mathcal{X}^0) \rightleftarrows \mathrm{Tot}(\mathrm{IndCoh}(\mathcal{X}^\bullet)^!)$$

is monadic. Furthermore, the resulting monad on $\mathrm{IndCoh}(\mathcal{X}^0)$, viewed as a plain endo-functor, is canonically isomorphic to $(p_t)_*^{\mathrm{IndCoh}} \circ (p_s)^!$.

(b) *Suppose that $\mathcal{X}^\bullet$ is the Čech nerve of a map $f : \mathcal{X}^0 \to \mathcal{Y}$, where f is ind-schematic and ind-proper. Assume also that f is surjective at the level of k-points. Then the resulting map*

$$\mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{Tot}(\mathrm{IndCoh}(\mathcal{X}^\bullet)^!)$$

is an equivalence.

PROOF. Follows by repeating the argument of Volume I, Chapter 4, Proposition 7.2.2. □

3.3.4. For $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ recall the category $\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)_{\mathrm{laft}}^{\mathrm{fake}}$, see Chapter 1, Sect. 4.3.6. From Proposition 3.3.3 and Chapter 1, Corollary 4.4.2 we obtain:

COROLLARY 3.3.5. *Let $\mathcal{X} \to \mathcal{Y}$ be a map in $\mathrm{PreStk}_{\mathrm{laft}}$, which is ind-inf-schematic, ind-proper and surjective at the level of k-points. Then the pullback functor*

$$\mathrm{Pro}(\mathrm{QCoh}(\mathcal{Y})^-)_{\mathrm{laft}}^{\mathrm{fake}} \to \mathrm{Tot}\left(\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X}^\bullet/\mathcal{Y})^-)_{\mathrm{laft}}^{\mathrm{fake}}\right)$$

is an equivalence.

3.3.6. *Descent for maps.* Let $f : \mathcal{X} \to \mathcal{Y}$ be an inf-schematic nil-isomorphism in $\mathrm{PreStk}_{\mathrm{laft}}$. We then have:

PROPOSITION 3.3.7. *For $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$, the natural map*

$$\mathrm{Tot}(\mathrm{Maps}(\mathcal{X}^\bullet, \mathcal{Z})) \to \mathrm{Maps}(\mathcal{Y}, \mathcal{Z})$$

is an isomorphism.

PROOF. The statement automatically reduces to the case when $\mathcal{Y} = Y \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$. Furthermore, by Chapter 2, Corollary 4.3.4, we can further assume that $\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$.

The assertion of the proposition is evident if Y is reduced: in this case the simplicial object $X^\bullet$ is split. Hence, by Chapter 1, Proposition 5.4.2, by induction,

it suffices to show that if the assertion holds for a given Y and we have a square-zero extension $Y \hookrightarrow Y'$ by means of $\mathcal{F} \in \mathrm{Coh}(Y)^{\leq 0}$, then the assertion holds also for $X' \in (\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ to}\ Y'}$.

Set $X := X' \underset{Y'}{\times} Y'$. By Chapter 1, Proposition 5.3.2, the map

$$X^{\bullet} \hookrightarrow X'^{\bullet}$$

has a structure of simplicial object in the category of square-zero extensions.

By the induction hypothesis, it is enough to show that for a given map $z : Y \to \mathcal{Z}$, the map

$$(3.4) \qquad \mathrm{Maps}(Y', \mathcal{Z}) \underset{\mathrm{Maps}(Y, \mathcal{Z})}{\times} \{z\} \to \mathrm{Tot}(\mathrm{Maps}(X'^{\bullet}, \mathcal{Z})) \underset{\mathrm{Tot}(\mathrm{Maps}(X^{\bullet}, \mathcal{Z}))}{\times} \{z\}$$

is an isomorphism.

Since $\mathcal{Z}$ admits deformation theory, the left-hand side in (3.4) is canonically isomorphic to the groupoid of null-homotopies of the composition

$$T_z^*(\mathcal{Z}) \to T^*(Y) \to \mathcal{F}.$$

We rewrite the right-hand side in (3.4) as the totalization of the simplicial space

$$\mathrm{Maps}(X'^{\bullet}, \mathcal{Z}) \underset{\mathrm{Maps}(X^{\bullet}, \mathcal{Z})}{\times} \{z\}.$$

The above simplicial groupoid identifies with that of null-homotopies of the composition

$$T_{z \circ f^{\bullet}}^*(\mathcal{Z}) \to T^*(X^{\bullet}) \to f^{\bullet}(\mathcal{F}),$$

where $f^{\bullet}$ denotes the map $X^{\bullet} \to Y$.

Now, the desired property follows the the descent property of $\mathrm{Pro}(\mathrm{QCoh}(-)^-)_{\mathrm{laft}}^{\mathrm{fake}}$, see Corollary 3.3.5 above.

$\square$

3.4. t-structure for ind-inf-schemes. The category IndCoh on an ind-inf-scheme also possesses a t-structure. However, it has less favorable properties than in the case of ind-schemes.

3.4.1. Let $\mathcal{X}$ be an ind-inf-scheme. We define a t-structure on the category $\mathrm{IndCoh}(\mathcal{X})$ by declaring that an object $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ belongs to $\mathrm{IndCoh}(\mathcal{X})^{\geq 0}$ if and only if $i^!(\mathcal{F})$ belongs to $\mathrm{IndCoh}(^{\mathrm{red}}\mathcal{X})^{\geq 0}$, where

$$i : {}^{\mathrm{red}}\mathcal{X} \to \mathcal{X}$$

is the canonical map.

Equivalently, we let $\mathrm{IndCoh}(\mathcal{X})^{\leq 0}$ be generated under colimits by the essential image of $\mathrm{IndCoh}(^{\mathrm{red}}\mathcal{X})^{\leq 0}$ under i_*^{IndCoh}.

It is easy to see that if f is an ind-finite map, then the functor $f^!$ is left t-exact.

3.4.2.   Suppose that $\mathcal{X}$ is actually an ind-scheme. We claim that the t-structure defined above, when we view $\mathcal{X}$ as a mere ind-inf-scheme, coincides with one for $\mathcal{X}$ considered as an ind-scheme of Sect. 1.2. This follows from the next lemma:

LEMMA 3.4.3.   *Let* $f : \mathcal{X}_1 \to \mathcal{X}_2$ *be a nil-isomorphism of ind-schemes. Then for* $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X}_2)$ *we have:*

$$\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X}_2)^{\geq 0} \Leftrightarrow f^!(\mathcal{F}) \in \mathcal{F} \in \mathrm{IndCoh}(\mathcal{X}_1)^{\geq 0}.$$

PROOF.   The $\Rightarrow$ implication is tautological. For the $\Leftarrow$ implication, by the definition of the t-structure on $\mathrm{IndCoh}(\mathcal{X}_2)$, we can assume that $\mathcal{X}_2 = X_2 \in \mathrm{Sch}_{\mathrm{aft}}$ and $\mathcal{X}_1 = X_1 \in \mathrm{Sch}_{\mathrm{aft}}$; i.e. f is a nil-isomorphism of schemes $X_1 \to X_2$.

By the definition of the t-structure on $\mathrm{IndCoh}(X_2)$ and adjunction, it suffices to show that $\mathrm{Coh}(X_2)^{\leq 0}$ is generated by the essential image of $\mathrm{Coh}(X_1)^{\leq 0}$ under f_*^{IndCoh}, which is obvious. $\square$

COROLLARY 3.4.4.   *Let* $\mathcal{X}$ *be an object of* $\mathrm{indinfSch}_{\mathrm{laft}}$.

(a) *For* $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$, *we have* $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})^{\geq 0}$ *if and only if for every nil-closed map* $f : X \to \mathcal{X}$ *with* $X \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}$ *we have*

$$f^!(\mathcal{F}) \in \mathrm{IndCoh}(X)^{\geq 0}.$$

(b) *The category* $\mathrm{IndCoh}(\mathcal{X})^{\leq 0}$ *is generated under colimits by the essential images of the categories* $\mathrm{IndCoh}(X)^{\leq 0}$ *for* $f : X \to \mathcal{X}$ *with* $X \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}$ *and* f *nil-closed.*

3.4.5.   As mentioned above, if f is an ind-finite map $\mathcal{X}_1 \to \mathcal{X}_2$ of ind-inf-schemes, then the functor $f^!$ is left t-exact. By adjunction, this implies that the functor f_*^{IndCoh} is right t-exact.

LEMMA 3.4.6.   *Let* $f : \mathcal{X}_1 \to \mathcal{X}_2$ *be an* ind-finite *and* ind-schematic *map between ind-inf-schemes. Then the functor* f_*^{IndCoh} *is t-exact.*

PROOF.   We only have to prove that f_*^{IndCoh} is left t-exact. By Proposition 3.1.2(c), the assertion reduces to the case when $\mathcal{X}_2 = \mathrm{indSch}_{\mathrm{laft}}$. In the latter case, $\mathcal{X}_1$ is an ind-scheme, and the assertion follows from the fact that the functor f_*^{IndCoh} for a map of ind-schemes is left t-exact, by Lemma 1.4.9. $\square$

REMARK 3.4.7.   It is easy to see that the assertion of the lemma is false without the assumption that f be ind-schematic, see Sect. 0.2.2

4. The direct image functor for ind-inf-schemes

In this section we construct the direct image functor on IndCoh for maps between ind-inf-schemes. The idea is that one can bootstrap it from the case of maps that are nil-closed embeddings, while for the latter the sought-for procedure is obtained as left/right Kan extension from the case of schemes.

4.1. Recovering from nil-closed embeddings.   In this subsection we show that if we take IndCoh on the category of schemes, with morphisms restricted to nil-closed maps, then its right Kan extension to ind-inf-schemes recovers the usual IndCoh.

4.1.1.   Consider the fully faithful embeddings

$$\text{Sch}_{\text{aft}}^{\text{aff}} \hookrightarrow \text{Sch}_{\text{aft}} \hookrightarrow \text{indinfSch}_{\text{laft}} \hookrightarrow \text{PreStk}_{\text{laft}} \,.$$

Denote

$$\text{IndCoh}_{\text{indinfSch}_{\text{laft}}}^{!} := \text{IndCoh}_{\text{PreStk}_{\text{laft}}}^{!}|_{(\text{indinfSch}_{\text{laft}})^{\text{op}}}\,.$$

Since

$$\text{IndCoh}_{\text{Sch}_{\text{aft}}}^{!} \to \text{RKE}_{(\text{Sch}_{\text{aft}}^{\text{aff}})^{\text{op}} \hookrightarrow (\text{Sch}_{\text{aft}})^{\text{op}}}(\text{IndCoh}_{\text{Sch}_{\text{aft}}^{\text{aff}}}^{!})$$

is an isomorphism, the map

$$(4.1) \qquad \text{IndCoh}_{\text{indinfSch}_{\text{laft}}}^{!} \to \text{RKE}_{(\text{Sch}_{\text{aft}})^{\text{op}} \hookrightarrow (\text{indinfSch}_{\text{laft}})^{\text{op}}}(\text{IndCoh}_{\text{Sch}_{\text{aft}}}^{!})$$

is an isomorphism.

4.1.2.   Let

$$(\text{indinfSch}_{\text{laft}})_{\text{nil-closed}} \subset \text{indinfSch}_{\text{laft}}$$

denote the 1-full subcategory, where we restrict 1-morphisms to be nil-closed.

Denote

$$\text{IndCoh}_{(\text{indinfSch}_{\text{laft}})_{\text{nil-closed}}}^{!} := \text{IndCoh}_{\text{indinfSch}_{\text{laft}}}^{!}|_{((\text{indinfSch}_{\text{laft}})_{\text{nil-closed}})^{\text{op}}}\,.$$

From the isomorphism (4.1), we obtain a canonically defined map

$$(4.2) \quad \text{IndCoh}_{(\text{indinfSch}_{\text{laft}})_{\text{nil-closed}}}^{!} \to$$
$$\text{RKE}_{((\text{Sch}_{\text{aft}})_{\text{nil-closed}})^{\text{op}} \hookrightarrow ((\text{indinfSch}_{\text{laft}})_{\text{nil-closed}})^{\text{op}}}(\text{IndCoh}_{(\text{Sch}_{\text{aft}})_{\text{nil-closed}}}^{!}).$$

We will prove:

PROPOSITION 4.1.3.   *The map* (4.2) *is an isomorphism.*

PROOF.   We need to show that for $\mathcal{X} \in \text{indinfSch}_{\text{laft}}$, the functor

$$\text{IndCoh}(\mathcal{X}) \to \lim_{Z \in ((\text{Sch}_{\text{aft}})_{\text{nil-closed in }\mathcal{X}})^{\text{op}}} \text{IndCoh}(Z)$$

is an equivalence.

However, this follows from Chapter 2, Corollary 4.1.4, since the functor IndCoh takes colimits in $\text{PreStk}_{\text{laft}}$ to limits.

$\square$

4.1.4.   For the sequel we will need the following observation:

COROLLARY 4.1.5.   *Let* $\mathcal{X}' \to \mathcal{X}$ *be a map of ind-inf-schemes. Then the functor*

$$\text{IndCoh}(\mathcal{X}') \to \lim_{Z \in ((\text{Sch}_{\text{aft}})_{\text{nil-closed in }\mathcal{X}})^{\text{op}}} \text{IndCoh}(Z \underset{\mathcal{X}}{\times} \mathcal{X}')$$

is an equivalence.

PROOF.   Same as that of Corollary 2.1.9. $\square$

4.2. Recovering from nil-isomorphisms. The material in this subsection is not needed for the sequel and is included for the sake of completeness.

4.2.1.   Let

$$(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}isom}} \subset \mathrm{indinfSch}_{\mathrm{laft}} \ \text{ and } \ (\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}isom}} \subset \mathrm{indSch}_{\mathrm{laft}}$$

denote the 1-full subcategories, where we restrict 1-morphisms to be nil-isomorphisms.

Denote also

$$\mathrm{IndCoh}^{!}_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}isom}}} := \mathrm{IndCoh}^{!}_{\mathrm{indinfSch}_{\mathrm{laft}}}|_{((\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}isom}})^{\mathrm{op}}}$$

and

$$\mathrm{IndCoh}^{!}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}isom}}} := \mathrm{IndCoh}^{!}_{\mathrm{indSch}_{\mathrm{laft}}}|_{((\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}isom}})^{\mathrm{op}}}$$

4.2.2.   From Proposition 4.1.3 we deduce:

COROLLARY 4.2.3.   *The natural map*

$$\mathrm{IndCoh}^{!}_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}isom}}} \to$$
$$\to \mathrm{RKE}_{((\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}isom}})^{\mathrm{op}} \to ((\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}isom}})^{\mathrm{op}}}(\mathrm{IndCoh}^{!}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}isom}}})$$

is an isomorphism.

PROOF.   By Corollary 1.3.5 and Proposition 4.1.3, the map

$$\mathrm{IndCoh}^{!}_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}closed}}} \to$$
$$\to \mathrm{RKE}_{((\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}closed}})^{\mathrm{op}} \to ((\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}closed}})^{\mathrm{op}}}(\mathrm{IndCoh}^{!}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}closed}}})$$

is an isomorphism.

Hence, it remains to show that for $\mathcal{X} \in \mathrm{indinfSch}_{\mathrm{laft}}$, the restriction map

$$\lim_{\mathcal{Y} \in ((\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}closed in }\mathcal{X}})^{\mathrm{op}}} \mathrm{IndCoh}(\mathcal{Y}) \to \lim_{\mathcal{Y} \in ((\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}isom to }\mathcal{X}})^{\mathrm{op}}} \mathrm{IndCoh}(\mathcal{Y})$$

is an isomorphism.

We claim that the map

$$(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}isom to }\mathcal{X}} \to (\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}closed in }\mathcal{X}}$$

is cofinal. Indeed, it admits a left adjoint, given by sending an object

$$(\mathcal{Y} \to \mathcal{X}) \in (\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{nil\text{-}closed in }\mathcal{X}}$$

$$\mathcal{Y} \underset{^{\mathrm{red}}\mathcal{Y}}{\sqcup} {}^{\mathrm{red}}\mathcal{X} \to \mathcal{X},$$

where the push-out is taken in the category $\mathrm{PreStk}_{\mathrm{laft}}$.

□

4.3. Constructing the direct image functor. In this subsection we finally construct the direct image functor. The crucial assertion is Theorem 4.3.3, which says that this functor is the 'right one'.

4.3.1. Consider again the functor

$$\text{IndCoh}_{\text{Sch}_{\text{aft}}} : \text{Sch}_{\text{aft}} \to \text{DGCat}_{\text{cont}},$$

where for a morphism $f : X_1 \to X_2$, the functor $\text{IndCoh}(X_1) \to \text{IndCoh}(X_2)$ is f_*^{IndCoh}.

Recall the notation:

$$\text{IndCoh}_{(\text{Sch}_{\text{aft}})_{\text{nil-closed}}} := \text{IndCoh}_{\text{Sch}_{\text{aft}}}|_{(\text{Sch}_{\text{aft}})_{\text{nil-closed}}}.$$

Denote

$$\text{IndCoh}_{\text{indinfSch}_{\text{laft}}} := \text{LKE}_{\text{Sch}_{\text{aft}} \hookrightarrow \text{indinfSch}_{\text{laft}}}(\text{IndCoh}_{\text{Sch}_{\text{aft}}}),$$

Note that by Proposition 1.4.2, the restriction of $\text{IndCoh}_{\text{indinfSch}_{\text{laft}}}$ to $\text{IndCoh}_{\text{infSch}_{\text{laft}}}$ identifies canonically with $\text{IndCoh}_{\text{indSch}_{\text{laft}}}$.

4.3.2. Denote

$$\text{IndCoh}_{(\text{indinfSch}_{\text{laft}})_{\text{nil-closed}}} := \text{IndCoh}_{\text{indinfSch}_{\text{laft}}}|_{(\text{indinfSch}_{\text{laft}})_{\text{nil-closed}}}.$$

We have a canonical map

$$(4.3) \quad \text{LKE}_{(\text{Sch}_{\text{aft}})_{\text{nil-closed}} \hookrightarrow (\text{indinfSch}_{\text{laft}})_{\text{nil-closed}}}(\text{IndCoh}_{(\text{Sch}_{\text{aft}})_{\text{nil-closed}}}) \to$$
$$\to \text{IndCoh}_{(\text{indinfSch}_{\text{laft}})_{\text{nil-closed}}}.$$

We claim:

THEOREM 4.3.3. *The map* (4.3) *is an isomorphism.*

4.3.4. Note that by combining Lemma 1.4.4 and Theorem 4.3.3, we obtain:

COROLLARY 4.3.5. *The functors*

$$\text{IndCoh}_{(\text{indinfSch}_{\text{laft}})_{\text{nil-closed}}} \text{ and } \text{IndCoh}^{!}_{(\text{indinfSch}_{\text{laft}})_{\text{nil-closed}}}$$

are obtained from one another by passing to adjoints.

REMARK 4.3.6. The concrete meaning of the combination of the above corollary and Theorem 4.3.3 is the following. Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ is a morphism between objects of $\text{indinfSch}_{\text{laft}}$. Then the claim is that we have a well-defined functor

$$f_*^{\text{IndCoh}} : \text{IndCoh}(\mathcal{X}_1) \to \text{IndCoh}(\mathcal{X}_2),$$

which tautologically agrees with the previously constructed IndCoh direct image functor when $\mathcal{X}_1, \mathcal{X}_2 \in \text{indSch}_{\text{laft}}$.

Furthermore, if f is nil-closed, then f_*^{IndCoh} is the left adjoint of $f^!$.

REMARK 4.3.7. Given an ind-proper map

$$f : \mathcal{X} \to \mathcal{Y}$$

in $\text{indinfSch}_{\text{laft}}$, we have defined two functors that we called f_*^{IndCoh}; namely, one is the functor given by $\text{IndCoh}_{\text{indinfSch}_{\text{laft}}}$ and the other is the right adjoint of $f^!$. A priori, these two functors are unrelated.

However, this abuse of notation will be justified in Corollary 5.2.3, where we will establish a canonical identification of these functors. Namely, we will show that the assertion of Theorem 4.3.3 and therefore Corollary 4.3.5 can be strengthened by replacing the class of nil-closed morphisms by that of ind-proper ones.

REMARK 4.3.8. In what follows, for $\mathcal{X}_1 = \mathcal{X}$ and $\mathcal{X}_2 = \mathrm{pt}$, we shall also use the notation

$$\Gamma(\mathcal{X}, -)^{\mathrm{IndCoh}}$$

for the functor $(p_{\mathcal{X}})_*^{\mathrm{IndCoh}}$, where $p_{\mathcal{X}} : \mathcal{X} \to \mathrm{pt}$ is the projection.

4.4. Proof of Theorem 4.3.3. Before we begin the proof, let us explain why the proof of Theorem 4.3.3 is more involved than that of Proposition 4.1.3.

The reason is that in Proposition 4.1.3, we could test the equivalence via affine schemes, and the latter we did using Chapter 2, Theorem 4.1.3. Now, for Theorem 4.3.3 affine schemes are not enough. We could have gotten away cheaply if we knew that for (a not necessarily affine) scheme $Z \in \mathrm{Sch}_{\mathrm{aft}}$ and a map $Z \to \mathcal{X}$, the category of its factorizations as

$$Z \to Z' \to \mathcal{X},$$

where $Z' \to \mathcal{X}$ is a nil-isomorphism, is contractible. However, the latter fact is simply *not true* (see Chapter 2, Remark 4.1.6).

4.4.1. *Step -1.* We need to show that for $\mathcal{X} \in \mathrm{indinfSch}_{\mathrm{laft}}$, the functor

$$\underset{Z \in (\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nil\text{-}closed\ in}\ \mathcal{X}}}{\mathrm{colim}} \mathrm{IndCoh}(Z) \to \underset{Y \in (\mathrm{Sch}_{\mathrm{aft}})_{/\mathcal{X}}}{\mathrm{colim}} \mathrm{IndCoh}(Y)$$

is an equivalence.

The convergence property of the IndCoh functor allows to replace $\mathrm{Sch}_{\mathrm{aft}}$ by $^{<\infty}\mathrm{Sch}_{\mathrm{ft}}$. Thus, we need to show that the functor

$$\underset{Z \in (^{<\infty}\mathrm{Sch}_{\mathrm{ft}})_{\mathrm{nil\text{-}closed\ in}\ \mathcal{X}}}{\mathrm{colim}} \mathrm{IndCoh}(Z) \to \underset{Y \in (^{<\infty}\mathrm{Sch}_{\mathrm{ft}})_{/\mathcal{X}}}{\mathrm{colim}} \mathrm{IndCoh}(Y)$$

is an equivalence.

4.4.2. *Step 0.* Consider the commutative diagram

(4.4)

$$
\begin{array}{ccc}
\underset{Y \in (^{<\infty}\mathrm{Sch}_{\mathrm{ft}})_{/\mathcal{X}}}{\mathrm{colim}} \mathrm{IndCoh}(Y) & \longleftarrow & \underset{S \in (^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{X}}}{\mathrm{colim}} \mathrm{IndCoh}(S) \\
\uparrow & & \uparrow \\
\underset{Z \in (^{<\infty}\mathrm{Sch}_{\mathrm{ft}})_{\mathrm{nil\text{-}closed\ in}\ \mathcal{X}}}{\mathrm{colim}} \mathrm{IndCoh}(Z) & \longleftarrow & \underset{(S \to Z \to \mathcal{X}) \in \mathbf{C}}{\mathrm{colim}} \mathrm{IndCoh}(S),
\end{array}
$$

where $\mathbf{C}$ is the category of $S \to Z \to \mathcal{X}$, with $S \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$ and

$$(Z \to \mathcal{X}) \in \left({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}} \right)_{\mathrm{nil\text{-}closed\ in}\ \mathcal{X}}.$$

We will show that the horizontal arrows and the right vertical arrow in this diagram are equivalences. This will prove that the left vertical arrow is also an equivalence.

4.4.3. *Step 1.* Consider the functor $\mathbf{C} \to (^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{X}}$, appearing in the right vertical arrow in (4.4). We claim that it is cofinal, which would prove that the right vertical arrow in (4.4) is an equivalence.

We note that the functor $\mathbf{C} \to (^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{X}}$ is a Cartesian fibration. Hence, the fact that it is cofinal is equivalent to the fact that it has contractible fibers.

The fiber over a given object $S \to \mathcal{X}$ is the category of factorizations

$$S \to Z \to \mathcal{X}, \quad (Z \to \mathcal{X}) \in (^{<\infty}\mathrm{Sch}_{\mathrm{ft}})_{\text{nil-closed in } \mathcal{X}}.$$

This category is contractible by Chapter 2, Theorem 4.1.3.

4.4.4. *Step 2.* Consider the functor $\mathbf{C} \to (^{<\infty}\mathrm{Sch}_{\mathrm{ft}})_{\text{nil-closed in } \mathcal{X}}$, appearing in the bottom horizontal arrow in (4.4). It is a co-Cartesian fibration.

Hence, in order to show that this arrow in the diagram is an equivalence, it suffices to show that for a given $Z \in (^{<\infty}\mathrm{Sch}_{\mathrm{ft}})_{\text{nil-closed in } \mathcal{X}}$, the functor

(4.5)
$$\operatorname*{colim}_{S \in (^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/Z}} \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(Z)$$

is an equivalence.

We have the following assertion, proved below:

PROPOSITION 4.4.5. *The functor* $\mathrm{IndCoh}_{\mathrm{Sch}_{\mathrm{aft}}}$, *regarded as a presheaf on* $\mathrm{Sch}_{\mathrm{aft}}$ *with values in* $(\mathrm{DGCat}_{\mathrm{cont}})^{\mathrm{op}}$ *satisfies Zariski descent.*

This proposition readily implies that (4.5) is an equivalence (for a general statement along these lines see [**Ga1**, Proposition 6.4.3]; here we apply it to $^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}} \subset {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}$.)

4.4.6. *Step 3.* To treat the top horizontal arrow in (4.4), we consider the category $\mathbf{D}$ of

$$S \to Y \to \mathcal{X}, \quad S \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}, \quad Y \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}},$$

and functor

(4.6)
$$\operatorname*{colim}_{(S \to Y \to \mathcal{X}) \in \mathbf{D}} \mathrm{IndCoh}(S) \to \operatorname*{colim}_{S \in (^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{X}}} \mathrm{IndCoh}(S).$$

We note that the functor (4.6) is an equivalence, because the corresponding forgetful functor $\mathbf{D} \to (^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{X}}$ is cofinal (it is a Cartesian fibration with contractible fibers).

Hence, it remains to show that the composition

$$\operatorname*{colim}_{(S \to Y \to \mathcal{X}) \in \mathbf{D}} \mathrm{IndCoh}(S) \to \operatorname*{colim}_{Y \in (^{<\infty}\mathrm{Sch}_{\mathrm{ft}})_{/\mathcal{X}}} \mathrm{IndCoh}(S)$$

of (4.6) with the top horizontal arrow in (4.4) is an equivalence.

The above functor corresponds to the co-Cartesian fibration $\mathbf{D} \to (^{<\infty}\mathrm{Sch}_{\mathrm{ft}})_{/\mathcal{X}}$. Hence, it suffices to show that for a fixed $Y \in (^{<\infty}\mathrm{Sch}_{\mathrm{ft}})_{/\mathcal{X}}$, the functor

$$\operatorname*{colim}_{S \in (^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/Y}} \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(Y)$$

is an equivalence.

However, this follows as in Step 2 from Proposition 4.4.5.

4.4.7. *Proof of Proposition 4.4.5.* The assertion of the proposition is equivalent to the following. Let $X \in \mathrm{Sch}_{\mathrm{aft}}$ be covered by two opens U_1 and U_2. Denote

$$U_1 \overset{j_1}{\hookrightarrow} X, \ U_2 \overset{j_2}{\hookrightarrow} X, \ U_1 \cap U_2 \overset{j_{12}}{\hookrightarrow} X,$$

$$U_1 \cap U_2 \overset{j_{12,1}}{\hookrightarrow} U_1, \ U_1 \cap U_2 \overset{j_{12,1}}{\hookrightarrow} U_2.$$

Then the claim is that the diagram

$$
\begin{array}{ccc}
\mathrm{IndCoh}(U_1 \cap U_2) & \xrightarrow{(j_{12,1})_*^{\mathrm{IndCoh}}} & \mathrm{IndCoh}(U_1) \\
{\scriptstyle (j_{12,2})_*^{\mathrm{IndCoh}}} \downarrow & & \downarrow {\scriptstyle (j_1)_*^{\mathrm{IndCoh}}} \\
\mathrm{IndCoh}(U_1) & \xrightarrow{(j_2)_*^{\mathrm{IndCoh}}} & \mathrm{IndCoh}(X)
\end{array}
$$

is a push-out square in $\mathrm{DGCat}_{\mathrm{cont}}$.

In other words, given $\mathbf{C} \in \mathrm{DGCat}$ and a triple of continuous functors

$$F_1 : \mathrm{IndCoh}(U_1) \to \mathbf{C}, \ F_2 : \mathrm{IndCoh}(U_2) \to \mathbf{C}, \ F_{12} : \mathrm{IndCoh}(U_1 \cap U_2) \to \mathbf{C}$$

endowed with isomorphisms

$$F_1 \circ (j_{12,1})_*^{\mathrm{IndCoh}} \simeq F_{12} \simeq F_2 \circ (j_{12,2})_*^{\mathrm{IndCoh}},$$

we need to show that this data comes from a uniquely defined functor

$$F : \mathrm{IndCoh}(X) \to \mathbf{C}.$$

The sought-for functor F is recovered as follows: for $\mathcal{F} \in \mathrm{IndCoh}(X)$, we have

$$F(\mathcal{F}) = F_1(j_1^!(\mathcal{F})) \underset{F_{12}(j_{12}^!(\mathcal{F}))}{\times} F_2(j_2^!(\mathcal{F})),$$

where the maps $F_i(j_i^!(\mathcal{F})) \to F_{12}(j_{12}^!(\mathcal{F}))$ are given by

$$F_i(j_i^!(\mathcal{F})) \to F_i((j_{12,i})_*^{\mathrm{IndCoh}} \circ j_{12,i}^! \circ j_i^!(\mathcal{F})) = F_i((j_{12,i})_*^{\mathrm{IndCoh}} \circ j_{12}^!(\mathcal{F}) \simeq F_{12}(j_{12}^!(\mathcal{F})).$$

$\square$

4.5. Base change. As in the case of ind-schemes, there are two types of base change isomorphism for ind-proper inf-schematic maps. The first is given by Proposition 3.2.4. Here we will prove the second.

4.5.1.   Let

$$
\begin{array}{ccc}
\mathcal{X}_1' & \xrightarrow{g_1} & \mathcal{X}_1 \\
{\scriptstyle f'} \downarrow & & \downarrow {\scriptstyle f} \\
\mathcal{X}_2' & \xrightarrow{g_2} & \mathcal{X}_2,
\end{array}
$$

be a Cartesian diagram of objects of $\mathrm{indinfSch}_{\mathrm{laft}}$ such that g_2 is an ind-closed embedding. Note that in this case, the right adjoint of $(g_2)_*^{\mathrm{IndCoh}}$ is $g_2^!$ (and similarly for g_1).

From the isomorphism

$$(g_2)_*^{\mathrm{IndCoh}} \circ (f')_*^{\mathrm{IndCoh}} \simeq f_*^{\mathrm{IndCoh}} \circ (g_1)_*^{\mathrm{IndCoh}},$$

we obtain, by adjunction, a natural transformation:

$$(4.7) \qquad\qquad (f')_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\mathrm{IndCoh}}.$$

We claim:

PROPOSITION 4.5.2. *The map (4.7) is an isomorphism.*

PROOF. Let $\mathcal{X}_0 := {}^{\mathrm{red}}\mathcal{X}_1$. Set

$$\mathcal{X}'_0 := \mathcal{X}'_1 \underset{\mathcal{X}_1}{\times} \mathcal{X}_0,$$

and consider the diagram

$$
\begin{array}{ccc}
\mathcal{X}'_0 & \xrightarrow{g_0} & \mathcal{X}_0 \\
{\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\
\mathcal{X}'_1 & \xrightarrow{g_1} & \mathcal{X}_1 \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
\mathcal{X}'_2 & \xrightarrow{g_2} & \mathcal{X}_2,
\end{array}
$$

in which both squares are Cartesian.

Since the functor $i^!$ is conservative (by Proposition 3.1.2(b)), its left adjoint i_*^{IndCoh} generates the target. Hence, it suffices to show that the outer square and the top square each satisfy base change of Proposition 4.5.2.

Note that base change for the top square is given by Proposition 3.1.2(c).

This reduces the assertion of the proposition to the case when $\mathcal{X}_1$ is a classical reduced ind-scheme. In this case the map f factors as

$$\mathcal{X}_1 \to \mathcal{X}_{3/2} \to \mathcal{X}_2,$$

where $\mathcal{X}_{3/2} = {}^{\mathrm{red}}\mathcal{X}_2$. Set $\mathcal{X}'_{3/2} := \mathcal{X}'_2 \underset{\mathcal{X}_2}{\times} \mathcal{X}_{3/2}$, and consider the diagram

$$
\begin{array}{ccc}
\mathcal{X}'_1 & \xrightarrow{g_0} & \mathcal{X}_1 \\
{\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\
\mathcal{X}'_{3/2} & \xrightarrow{g_3} & \mathcal{X}_{3/2} \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
\mathcal{X}'_2 & \xrightarrow{g_2} & \mathcal{X}_2,
\end{array}
$$

in which both squares are Cartesian.

It is sufficient to show that the two inner squares each satisfy base change of Proposition 4.5.2. For the bottom square, this is given by Proposition 3.1.2(c).

We note now that $\mathcal{X}_{3/2}$ and $\mathcal{X}'_{3/2}$ are *ind-schemes*. Hence, base change for the top square, this is given by Proposition 2.2.2.

$\square$

5. Extending the formalism of correspondences to inf-schemes

In this section we will take the formalism of IndCoh to what (in our opinion) is its ultimate domain of definition: the category of correspondences, where the objects are all prestacks (locally almost of finite type), pullbacks are taken with respect to any maps, push-forwards are taken with respect to ind-inf-schematic maps, and adjunctions hold for ind-proper maps.

5.1. Set-up for extension. As a first step, we will consider the category of correspondences, where the objects are ind-inf-schemes, pullbacks and push-forwards are taken with respect to any maps, and adjunctions are for nil-closed maps. We will construct the required functor by the Kan extension procedure from Volume I, Chapter 8, Theorem 1.1.9.

5.1.1. We consider the category $\mathrm{indinfSch}_{\mathrm{laft}}$ with the following three classes of morphisms

$$vert = \text{all}, \ horiz = \text{all}, \ adm = \text{nil-closed}.$$

Let

$$\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{all;all}}^{\mathrm{nil\text{-}closed}}$$

be the resulting $(\infty, 2)$-category of correspondences.

5.1.2. Consider also the category

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}^{\mathrm{proper}},$$

and the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}^{\mathrm{proper}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}^{\mathrm{proper}} \to \mathrm{DGCat}_{\mathrm{cont}}^{2\text{-}\mathrm{Cat}},$$

constructed in Volume I, Chapter 5, Theorem 2.1.4.

We restrict it along

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{laft}})_{\mathrm{all;all}}^{\mathrm{nil\text{-}closed}} \to \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}^{\mathrm{proper}},$$

and obtain a functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}^{\mathrm{nil\text{-}closed}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}^{\mathrm{nil\text{-}closed}} \to \mathrm{DGCat}_{\mathrm{cont}}^{2\text{-}\mathrm{Cat}}.$$

Note that by Volume I, Chapter 7, Theorem 4.1.3, this restriction does not lose any information.

We wish to extend the functor $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}^{\mathrm{nil\text{-}closed}}}$ to a functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{all;all}}^{\mathrm{nil\text{-}closed}}} : \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{all;all}}^{\mathrm{nil\text{-}closed}} \to \mathrm{DGCat}_{\mathrm{cont}}^{2\text{-}\mathrm{Cat}},$$

along the tautological functor

$$\mathrm{Sch}_{\mathrm{laft}} \to \mathrm{indinfSch}_{\mathrm{laft}}.$$

We will apply Volume I, Chapter 8, Theorem 1.1.9 to obtain this extension. In the present context, the conditions of Volume I, Chapter 8, Sect. 1.1.6 are satisfied for the following reasons:

Condition (1) is satisfied by Proposition 4.5.2;

Condition (2) is satisfied by by Theorem 4.3.3;

Condition (3) is satisfied by by Proposition 3.2.4;

Condition (4) is satisfied by Proposition 4.1.3.

Condition (*) is satisfied by Corollary 4.1.5.

5.1.3.   Applying Volume I, Chapter 8, Theorem 1.1.9, we obtain:

THEOREM 5.1.4. *There exists a uniquely defined functor*

$$\mathrm{IndCoh}_{\mathrm{Corr(indinfSch_{laft})}^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}} : \mathrm{Corr(indinfSch_{laft})}^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}},$$

whose restriction along

$$\mathrm{Corr(Sch_{aft})}^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}} \to \mathrm{Corr(indinfSch_{laft})}^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}$$

identifies with $\mathrm{IndCoh}_{\mathrm{Corr(Sch_{aft})}^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}}$.

Moreover, the restrictions of $\mathrm{IndCoh}_{\mathrm{Corr(indinfSch_{laft})}^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}}$ *to*

$$(\mathrm{indinfSch_{laft}})^{\mathrm{op}} \ and \ \mathrm{indinfSch_{laft}}$$

identify, respectively, with

$$\mathrm{IndCoh}^{!}_{\mathrm{indinfSch_{laft}}} \ and \ \mathrm{IndCoh}_{\mathrm{indinfSch_{laft}}}.$$

5.2. Adding adjunctions for ind-proper morphsisms. In this subsection we will extend the functor from the previous subsection, where we include adjunctions for ind-proper maps.

5.2.1.   Consider now the $(\infty, 2)$-category

$$\mathrm{Corr(indinfSch_{laft})}^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}},$$

where we enlarge the class of 2-morphisms to that of ind-proper maps.

Consider the 2-fully faithful functor

$$\mathrm{Corr(indinfSch_{laft})}^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}} \to \mathrm{Corr(indinfSch_{laft})}^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}.$$

We are going to prove:

THEOREM 5.2.2. *There exists a unique extension of the functor* $\mathrm{IndCoh}_{\mathrm{Corr(Sch_{aft})}^{\mathrm{proper}}_{\mathrm{all;all}}}$ *to a functor*

$$\mathrm{IndCoh}_{\mathrm{Corr(indinfSch_{laft})}^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}} : \mathrm{Corr(indinfSch_{laft})}^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}.$$

As a formal corollary, using Volume I, Chapter 7, Theorem 3.2.2, we obtain:

COROLLARY 5.2.3. *The functors*

$$\mathrm{IndCoh}_{(\mathrm{indinfSch_{laft}})_{\mathrm{ind\text{-}proper}}} := \mathrm{IndCoh}_{\mathrm{indinfSch_{laft}}}|_{(\mathrm{indinfSch_{laft}})_{\mathrm{ind\text{-}proper}}}$$

and

$$\mathrm{IndCoh}^{!}_{(\mathrm{indinfSch_{laft}})_{\mathrm{ind\text{-}proper}}} := \mathrm{IndCoh}^{!}_{\mathrm{indinfSch_{laft}}}|_{((\mathrm{indinfSch_{laft}})_{\mathrm{ind\text{-}proper}})^{\mathrm{op}}}$$

are obtained from each other by passing to adjoints.

5.2.4.   Let us explain the concrete content of Theorem 5.2.2 and Corollary 5.2.3.

First, Corollary 5.2.3 says that if $f : \mathcal{X}_1 \to \mathcal{X}_2$ is an ind-proper map between ind-inf-schemes, then the functor f_*^{IndCoh} is the left adjoint of $f^!$.

Next, let

$$
\begin{array}{ccc}
\mathcal{X}_1' & \xrightarrow{\ g_1\ } & \mathcal{X}_1 \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
\mathcal{X}_2' & \xrightarrow{\ g_2\ } & \mathcal{X}_2
\end{array}
$$

be a Cartesian diagram in indinfSch$_{\mathrm{laft}}$.

Theorem 5.1.4 says that we have a *canonical isomorphism*

(5.1) $$g_2^! \circ f_*^{\mathrm{IndCoh}} \simeq (f')_*^{\mathrm{IndCoh}} \circ g_1^!.$$

If f is ind-proper, then the morphism $\leftarrow$ in (5.1) is obtained by adjunction from the (iso)morphism

$$(f')^! \circ g_2^! \simeq g_1^! \circ f^!.$$

If g_2 is ind-proper, then the morphism $\to$ in (5.1) is obtained by adjunction from the (iso)morphism

$$f_*^{\mathrm{IndCoh}} \circ (g_1)_*^{\mathrm{IndCoh}} \simeq (g_2)_*^{\mathrm{IndCoh}} \circ (f')_*^{\mathrm{IndCoh}}.$$

In particular, a generalization of Proposition 4.5.2 holds with 'nil-closed' replaced by 'ind-proper'.

If the ind-inf-schemes in the above diagram are schemes, then the isomorphism (5.1) equals one defined a priori in this case by Volume I, Chapter 5, Corollary 3.1.4.

5.2.5.   Note that by combining Corollary 5.2.3 with Lemma 1.4.4, we obtain:

COROLLARY 5.2.6.

$$\mathrm{LKE}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}} \hookrightarrow (\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}} \left(\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}} \right) \to \mathrm{IndCoh}_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}}$$

is an isomorphism.

5.3. Proof of Theorem 5.2.2.

5.3.1. *The case of ind-schemes.* Consider the category indSch$_{\mathrm{laft}}$ with the following three classes of morphisms

$$vert = \mathrm{all}, \ horiz = \mathrm{all}, \ adm = \mathrm{ind\text{-}proper}.$$

We claim that we have the following result:

THEOREM 5.3.2. *There exists a uniquely defined functor*

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{all;all}}^{\mathrm{ind\text{-}proper}}} : (\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{all;all}}^{\mathrm{ind\text{-}proper}} \to \mathrm{DGCat}_{\mathrm{cont}}^{2\text{-Cat}},$$

whose restriction along

$$\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}^{\mathrm{proper}} \to \mathrm{Corr}(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{all;all}}^{\mathrm{ind\text{-}proper}}$$

identifies canonically with $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{all;all}}^{\mathrm{nil\text{-}closed}}}$.

PROOF. This follows from Volume I, Chapter 8, Theorem 1.1.9 applied to the functor

$$\mathrm{Sch}_{\mathrm{aft}} \to \mathrm{indSch}_{\mathrm{laft}}.$$

Here the conditions of Volume I, Chapter 8, Sect. 1.1.6 are satisfied for the following reasons:

Condition (1) holds by Proposition 2.2.2;

Condition (2) holds by Corollary 1.3.5;

Condition (3) holds by Proposition 2.1.2;

Condition (4) holds by Proposition 1.4.2.

Condition (*) holds by by Corollary 2.1.9.

$\square$

REMARK 5.3.3. The difference between the case of ind-schemes and that of inf-schemes that the fact that the map

$$\mathrm{LKE}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}} \hookrightarrow (\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}} (\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}}) \to \mathrm{IndCoh}_{(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}}$$

is an isomorphism only follows a posteriori from Theorem 5.2.2, while the corresponding fact for ind-schemes, i.e., the isomorphism

$$\mathrm{LKE}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}} \hookrightarrow (\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}} (\mathrm{IndCoh}_{(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{proper}}}) \simeq \mathrm{IndCoh}_{(\mathrm{indSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}}}$$

is given by Corollary 1.3.5.

5.3.4. We are going to deduce Theorem 5.2.2 from Theorem 5.1.4 by applying Volume I, Chapter 7, Theorem 4.1.3. Thus, we need to check that the inclusion

$$\mathrm{nil\text{-}closed} \subset \mathrm{ind\text{-}proper}$$

satisfies the condition of Volume I, Chapter 7, Sect. 4.1.2.

That is, we consider a ind-proper morphism

$$f : \mathcal{X}_1 \to \mathcal{X}_2$$

of objects of $\mathrm{indinfSch}_{\mathrm{laft}}$, and the Cartesian square:

$$\begin{array}{ccc} \mathcal{X}_1 \underset{\mathcal{X}_2}{\times} \mathcal{X}_1 & \xrightarrow{\ p_2\ } & \mathcal{X}_1 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{X}_1 & \xrightarrow{\ f\ } & \mathcal{X}_2. \end{array}$$

The diagonal map

$$\Delta_{\mathcal{X}_1/\mathcal{X}_2} : \mathcal{X}_1 \to \mathcal{X}_1 \underset{\mathcal{X}_2}{\times} \mathcal{X}_1$$

is nil-closed. Hence, from the $((\Delta_{\mathcal{X}_1/\mathcal{X}_2})^{\mathrm{IndCoh}}_*, (\Delta_{\mathcal{X}_1/\mathcal{X}_2})^!)$-adjunction, we obtain a natural transformation

$$(\Delta_{\mathcal{X}_1/\mathcal{X}_2})^{\mathrm{IndCoh}}_* \circ (\Delta_{\mathcal{X}_1/\mathcal{X}_2})^! \to \mathrm{Id}_{\mathrm{IndCoh}(\mathcal{X}_1 \underset{\mathcal{X}_2}{\times} \mathcal{X}_1)}.$$

By composing, the latter natural transformation gives rise to

$$(5.2) \quad \mathrm{Id}_{\mathrm{IndCoh}(\mathcal{X}_1)} \simeq (\mathrm{id}_{\mathcal{X}_1})^{\mathrm{IndCoh}}_* \circ (\mathrm{id}_{\mathcal{X}_1})^! \simeq$$

$$\simeq (p_1)^{\mathrm{IndCoh}}_* \circ (\Delta_{\mathcal{X}_1/\mathcal{X}_2})^{\mathrm{IndCoh}}_* \circ (\Delta_{\mathcal{X}_1/\mathcal{X}_2})^! \circ p_2^! \to (p_1)^{\mathrm{IndCoh}}_* \circ p_2^! \simeq f^! \circ f^{\mathrm{IndCoh}}_*,$$

where the last isomorphism is due to the existence of the functor $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})_{\mathrm{all;all}}^{\mathrm{nil\text{-}closed}}}$, see Sect. 5.2.4.

We need to show that the natural transformation (5.2) is the unit of an adjunction. I.e., that for $\mathcal{F}_1 \in \mathrm{IndCoh}(\mathcal{X}_1)$ and $\mathcal{F}_2 \in \mathrm{IndCoh}(\mathcal{X}_2)$, the map

$$(5.3) \quad \mathrm{Maps}(f_*^{\mathrm{IndCoh}}(\mathcal{F}_1), \mathcal{F}_2) \to \mathrm{Maps}(f^! \circ f_*^{\mathrm{IndCoh}}(\mathcal{F}_1), f^!(\mathcal{F}_2)) \simeq$$
$$\simeq \mathrm{Maps}((p_1)_*^{\mathrm{IndCoh}} \circ p_2^!(\mathcal{F}_1), f^!(\mathcal{F}_2)) \to \mathrm{Maps}(\mathcal{F}_1, f^!(\mathcal{F}_2))$$

is an isomorphism.

We note that by Theorem 5.1.4 the map (5.3) is the unit for the $(f_*^{\mathrm{IndCoh}}, f^!)$ adjunction, when f is nil-closed.

5.3.5.   Note that the natural transformation (5.2) is defined for any map f which is *nil-separated*, i.e., one for which $\Delta_{\mathcal{X}_1/\mathcal{X}_2}$ is nil-closed.

Let $g : \mathcal{X}_0 \to \mathcal{X}_1$ be another nil-separated map between objects of $\mathrm{indinfSch}_{\mathrm{laft}}$. Diagram chase implies:

LEMMA 5.3.6.   *For $\mathcal{F}_0 \in \mathrm{IndCoh}(\mathcal{X}_0)$ and $\mathcal{F}_2 \in \mathrm{IndCoh}(\mathcal{X}_2)$, the diagram*

$$
\begin{array}{ccc}
\mathrm{Maps}(g_*^{\mathrm{IndCoh}}(\mathcal{F}_0), f^!(\mathcal{F}_2)) & \xrightarrow{(5.3)} & \mathrm{Maps}(\mathcal{F}_0, g^! \circ f^!(\mathcal{F}_2)) \\
{\scriptstyle (5.3)} \uparrow & & \uparrow {\scriptstyle \mathrm{id}} \\
\mathrm{Maps}(f_*^{\mathrm{IndCoh}} \circ g_*^{\mathrm{IndCoh}}(\mathcal{F}_0), \mathcal{F}_2) & \xrightarrow{(5.3)} & \mathrm{Maps}(\mathcal{F}_0, g^! \circ f^!(\mathcal{F}_2))
\end{array}
$$

commutes.

5.3.7.   Let us take $\mathcal{X}_0 := {}^{\mathrm{red}}\mathcal{X}_1$ and g to be the canonical embedding. By Proposition 3.1.2, it is sufficient to show that (5.3) is an isomorphism for $\mathcal{F}_1$ of the form $g_*^{\mathrm{IndCoh}}(\mathcal{F}_0)$ for $\mathcal{F}_0 \in \mathrm{IndCoh}(\mathcal{X}_0)$.

Using Lemma 5.3.6, and the fact that the map

$$\mathrm{Maps}(g_*^{\mathrm{IndCoh}}(\mathcal{F}_0), f^!(\mathcal{F}_2)) \to \mathrm{Maps}(\mathcal{F}_0, g^! \circ f^!(\mathcal{F}_2))$$

is an isomorphism in this case, since g is nil-closed, we obtain that it is sufficient to show that (5.3) is an isomorphism, when the initial map f is replaced by $f \circ g$.

Thus, in proving that (5.3) is an isomorphism, we can assume that $\mathcal{X}_1$ is a reduced ind-scheme.

5.3.8.   Let us now factor f as

$$\mathcal{X}_1 \to \mathcal{X}_{3/2} \to \mathcal{X}_2,$$

where $\mathcal{X}_{3/2} := {}^{\mathrm{red}}\mathcal{X}_2$. Applying Lemma 5.3.6 again, we obtain that it is enough to show that (5.3) is an isomorphism for f replaced by $\mathcal{X}_1 \to \mathcal{X}_{3/2}$ and $\mathcal{X}_{3/2} \to \mathcal{X}_2$ separately.

For the map $\mathcal{X}_{3/2} \to \mathcal{X}_2$, the assertion follows from the fact that the map in question is nil-closed.

5.3.9. Hence, we are further reduced to the case when f is a ind-proper map between ind-schemes. However, in this case, the required isomorphism follows from Theorem 5.3.2: it follows by Volume I, Chapter 7, Theorem 4.1.3 from the existence of the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indSch}_{\mathrm{laft}})^{\mathrm{nil\text{-}proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{indSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{\text{2-Cat}}_{\mathrm{cont}},$$

whose restriction to $\mathrm{Corr}(\mathrm{indSch}_{\mathrm{laft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}$ is isomorphic to

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{infSch}_{\mathrm{laft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}}\big|_{\mathrm{Corr}(\mathrm{indSch}_{\mathrm{laft}})^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}}.$$

$\square$

5.4. Extending to prestacks. In this subsection, we will finally extend the formalism to the category of correspondences that has all laft prestacks as objects.

5.4.1. Consider the category $\mathrm{PreStk}_{\mathrm{laft}}$, and the three classes of morphisms

$$\mathrm{indinfsch}, \ \mathrm{all}, \ \mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper},$$

where 'indinfsch' stands for the class of ind-inf-schematic morphisms, and 'ind-proper' for the class of morphisms that are ind-proper.

Consider the tautological embedding

$$\mathrm{indinfSch}_{\mathrm{laft}} \hookrightarrow \mathrm{PreStk}_{\mathrm{laft}}.$$

It satisfies the conditions of Volume I, Chapter 8, Theorem 6.1.5, with respect to the classes

$$(\mathrm{all}, \mathrm{all}, \mathrm{ind\text{-}proper}) \to (\mathrm{indinfsch}, \mathrm{all}, \mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}).$$

Now, consider the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}} : \mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{\text{2-Cat}}_{\mathrm{cont}},$$

and the corresponding functor

$$\mathrm{IndCoh}^{!}_{\mathrm{indinfSch}_{\mathrm{laft}}} : (\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

Clearly, the map

$$\mathrm{IndCoh}^{!}_{\mathrm{PreStk}_{\mathrm{laft}}} \to \mathrm{RKE}_{(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{op}}\hookrightarrow(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}}(\mathrm{IndCoh}^{!}_{\mathrm{indinfSch}_{\mathrm{laft}}})$$

is an isomorphism.

5.4.2. Hence, by Volume I, Chapter 8, Theorem 6.1.5, from Theorem 5.1.4, we obtain the following theorem, which is *for us* the ultimate version of the formalism of ind-coherent sheaves:

THEOREM 5.4.3. *There exists a uniquely defined functor*

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}} \to \mathrm{DGCat}^{\text{2-Cat}}_{\mathrm{cont}},$$

equipped with isomorphisms

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}}\big|_{(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}}} \simeq \mathrm{IndCoh}^{!}_{\mathrm{PreStk}_{\mathrm{laft}}}$$

and

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indinfsch;all}}}\big|_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}} \simeq \mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{indinfSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}},$$

where the latter two isomorphisms are compatible in a natural sense.

5.4.4.   The concrete meaning of Theorem 5.4.3 is analogous to that of Theorem 5.1.4, with the difference that we can now consider the direct image functor f_*^{IndCoh} when f is an ind-inf-schematic map

$$f : \mathcal{X}_1 \to \mathcal{X}_2,$$

with $\mathcal{X}_1, \mathcal{X}_2$ being objects of $\mathrm{PreStk}_{\mathrm{laft}}$, and not necessarily ind-inf-schemes. The functor f_*^{IndCoh} satisfies base change for Cartesian squares

$$
\begin{array}{ccc}
\mathcal{X}_1' & \xrightarrow{\;g_1\;} & \mathcal{X}_1 \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
\mathcal{X}_2' & \xrightarrow{\;g_2\;} & \mathcal{X}_2,
\end{array}
$$

with vertical maps being ind-inf-schematic:

$$(f')_*^{\mathrm{IndCoh}} \circ g_1^! \to g_2^! \circ f_*^{\mathrm{IndCoh}}.$$

Moreover, for f ind-inf-schematic and ind-proper, the functor f_*^{IndCoh} is the left adjoint of $f^!$. In this case the base change isomorphism comes by adjunction from

$$(f')^! \circ g_2^! \simeq g_1^! \circ f^!.$$

If g_2 is ind-inf-schematic and ind-proper, the base change isomorphism comes by adjunction from

$$f_*^{\mathrm{IndCoh}} \circ (g_1)_*^{\mathrm{IndCoh}} \simeq (f')_*^{\mathrm{IndCoh}} \circ (g_2)_*^{\mathrm{IndCoh}}.$$

5.5. Open embeddings. The formalism of Theorem 5.4.3 contains the $\left(f_*^{\mathrm{IndCoh}}, f^!\right)$ adjunction for f proper.

However, it does not explicitly contain the $(f^!, f_*^{\mathrm{IndCoh}})$-adjunction for f which is an open embedding. In this subsection we will show that the latter follows automatically.

5.5.1.   Let $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch;all}}}$ denote the restriction of the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch;all}}^{\mathrm{indinfsch \,\&\, ind\text{-}proper}}}$$

to

$$\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch;all}} \subset \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch;all}}^{\mathrm{indinfsch \,\&\, ind\text{-}proper}}.$$

We regard it as a functor of $(\infty, 1)$-categories

$$\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch;all}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

Consider the $(\infty, 2)$-category $\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch;all}}^{\mathrm{open}}$.

5.5.2. We claim:

PROPOSITION 5.5.3. *There exists a unique extension of* $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch;all}}}$ *to a functor*

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{open}}_{\mathrm{indinfsch;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{open}}_{\mathrm{indinfsch;all}} \to \left(\mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}\right)^{2\text{-}\mathrm{op}}.$$

PROOF. We start with the three classes of 1-morphisms in $\mathrm{PreStk}_{\mathrm{laft}}$

$$\mathrm{indinfsch,\ all,\ isom,}$$

and enlarge it to

$$\mathrm{indinfsch,\ all,\ open}.$$

This enlargement satisfies the assumptions of Volume I, Chapter 8, Sect. 6.1.1. Hence, if the functor $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{open}}_{\mathrm{indinfsch;all}}}$ exists, then it is unique.

Furthermore, to prove the existence, it is sufficient to do so for the pair of categories

$$\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{open;all}} \subset \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{open}}_{\mathrm{open;all}},$$

and the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{open;all}}} := \mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch;all}}}\big|_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})\mathrm{open;all}}.$$

To construct the sought-for functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{open}}_{\mathrm{open;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{open}}_{\mathrm{open;all}} \to \left(\mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}\right)^{2\text{-}\mathrm{op}}$$

we proceed as follows.

We start with the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{open;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{open;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}},$$

and we recall that by construction (see Volume I, Chapter 5, Sect. 2.1.2), it extends to a functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{open}}_{\mathrm{open;all}}} : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{open;all}} \to \left(\mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}\right)^{2\text{-}\mathrm{op}}.$$

Now, the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{open}}_{\mathrm{open;all}}}$$

is obtained from $\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{open}}_{\mathrm{open;all}}}$ by Volume I, Chapter 8, Theorem 6.1.5 for the functor

$$\mathrm{Sch}_{\mathrm{aft}} \hookrightarrow \mathrm{PreStk}_{\mathrm{laft}}.$$

$\square$

6. Self-duality and multiplicative structure of IndCoh on ind-inf-schemes

In this section we will show how the formalism of IndCoh as a functor out of the category of correspondences of ind-inf-schemes defines Serre duality on ind-inf-schemes. This is parallel to Volume I, Chapter 5, Sect. 4.

6.1. The multiplicative structure. In this subsection we discuss a canonical symmetric monoidal structure on IndCoh.

6.1.1.   Recall that the functor

$$\mathrm{IndCoh}_{\mathrm{Corr(Sch_{aft})}^{\mathrm{proper}}_{\mathrm{all;all}}} : \mathrm{Corr(Sch_{aft})}^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{\mathrm{2\text{-}Cat}}_{\mathrm{cont}}$$

is endowed with a symmetric monoidal structure, see Volume I, Chapter 5, Theorem 4.1.2. Hence, the same is true for its restriction

$$\mathrm{IndCoh}_{\mathrm{Corr(Sch_{aft})}^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}} : \mathrm{Corr(Sch_{aft})}^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{\mathrm{2\text{-}Cat}}_{\mathrm{cont}} .$$

Applying Volume I, Chapter 9, Proposition 3.3.3, we obtain:

COROLLARY 6.1.2. *The functor*

$$\mathrm{IndCoh}_{\mathrm{Corr(indinfSch_{laft})}^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}} : \mathrm{Corr(indinfSch_{laft})}^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{\mathrm{2\text{-}Cat}}_{\mathrm{cont}}$$

carries a unique symmetric monoidal structure extending one on $\mathrm{IndCoh}_{\mathrm{Corr(Sch_{aft})}^{\mathrm{nil\text{-}closed}}_{\mathrm{all;all}}}$.

Applying Volume I, Chapter 9, Proposition 3.1.2, from Corollary 6.1.2, we obtain:

COROLLARY 6.1.3. *The functor*

$$\mathrm{IndCoh}_{\mathrm{(indinfSch_{laft})}^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}} : \mathrm{Corr(indinfSch_{laft})}^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}} \to \mathrm{DGCat}^{\mathrm{2\text{-}Cat}}_{\mathrm{cont}}$$

carries a unique symmetric monoidal structure extending one on $\mathrm{IndCoh}_{\mathrm{Corr(Sch_{aft})}^{\mathrm{proper}}_{\mathrm{all;all}}}$.

6.2. Duality. In this subsection we show that the symmetric monoidal structure on IndCoh gives rise to Serre duality. The idea is that an ind-inf-scheme $\mathcal{X}$ is canonically self-dual as an object of the category of correspondences equipped with its natural monoidal structure.

6.2.1.   By restricting the functor $\mathrm{IndCoh}_{\mathrm{(indinfSch_{laft})}^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}}$ to

$$\mathrm{Corr(indinfSch_{laft})}_{\mathrm{all;all}} \subset \mathrm{Corr(indinfSch_{laft})}^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}},$$

we obtain a symmetric monoidal structure on the functor

$$\mathrm{IndCoh}_{\mathrm{Corr(indinfSch_{laft})_{all;all}}} : \mathrm{Corr(indinfSch_{laft})}_{\mathrm{all;all}} \to \mathrm{DGCat}_{\mathrm{cont}} .$$

As in Volume I, Chapter 5, Theorem 4.2.5, we deduce:

THEOREM 6.2.2. *We have a commutative diagram of functors*

$$
\begin{array}{ccc}
(\mathrm{Corr(indinfSch_{laft})_{all;all}})^{\mathrm{op}} & \xrightarrow{\;\left(\mathrm{IndCoh}_{\mathrm{(indinfSch_{laft})_{all;all}}}\right)^{\mathrm{op}}\;} & \left(\mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}}\right)^{\mathrm{op}} \\
{\scriptstyle \varpi}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{dualization}} \\
\mathrm{Corr(indinfSch_{laft})_{all;all}} & \xrightarrow{\;\mathrm{IndCoh}_{\mathrm{(indinfSch_{laft})_{all;all}}}\;} & \mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}} .
\end{array}
$$

As in Volume I, Chapter 5, Sect. 4.2.2, the functor ϖ is the natural anti-equivalence on the category $\mathrm{Corr(indinfSch_{laft})_{all;all}}$ corresponding to interchanging the roles of vertical and horizontal arrows. The right vertical arrow is the functor of passage to the dual category.

6.2.3. Let us explain the concrete meaning of Theorem 6.2.2. This is parallel to Volume I, Chapter 5, Sect. 4.2.6.

For an individual object $\mathcal{X} \in \text{indinfSch}_{\text{laft}}$ it says that there is a natural self-duality data on the category $\text{IndCoh}(\mathcal{X})$, i.e.,

$$(6.1) \qquad \qquad \mathbf{D}_{\mathcal{X}}^{\text{Serre}} : \text{IndCoh}(\mathcal{X})^{\vee} \simeq \text{IndCoh}(\mathcal{X}).$$

Furthermore, for a map $f : \mathcal{X}_1 \to \mathcal{X}_2$, there is a canonical identification

$$(6.2) \qquad \qquad f^! \simeq (f_*^{\text{IndCoh}})^{\vee}.$$

6.2.4. Below we shall write down explicitly the unit and counit functors

$$\epsilon_{\mathcal{X}} : \text{IndCoh}(\mathcal{X}) \otimes \text{IndCoh}(\mathcal{X}) \to \text{Vect} \ \text{ and } \ \mu_{\mathcal{X}} : \text{Vect} \to \text{IndCoh}(\mathcal{X}) \otimes \text{IndCoh}(\mathcal{X})$$

that define the identification (6.1).

REMARK 6.2.5. We observe that the fact that the functors $\epsilon_{\mathcal{X}}$ and $\mu_{\mathcal{X}}$ do indeed define an isomorphism (6.1) is easy to check directly. I.e., this does not require the full statement of Theorem 6.2.2.

6.2.6. The pairing

$$\epsilon_{\mathcal{X}} : \text{IndCoh}(\mathcal{X}) \otimes \text{IndCoh}(\mathcal{X}) \to \text{Vect}$$

is the composition

$$\text{IndCoh}(\mathcal{X}) \otimes \text{IndCoh}(\mathcal{X}) \simeq \text{IndCoh}(\mathcal{X} \times \mathcal{X}) \xrightarrow{\Delta^!} \text{IndCoh}(\mathcal{X}) \xrightarrow{(p_{\mathcal{X}})_*^{\text{IndCoh}}} \text{Vect}\,.$$

Here $p_{\mathcal{X}}$ is the map $\mathcal{X} \to \text{pt}$, so $(p_{\mathcal{X}})_*^{\text{IndCoh}} \simeq \Gamma^{\text{IndCoh}}(\mathcal{X}, -)$. The first map is an isomorphism due to the fact that $\text{IndCoh}(\mathcal{X})$ is dualizable as a DG category.

The unit functor

$$\mu_{\mathcal{X}} : \text{Vect} \to \text{IndCoh}(\mathcal{X}) \otimes \text{IndCoh}(\mathcal{X})$$

is the composition

$$\text{Vect} \xrightarrow{p_{\mathcal{X}}^!} \text{IndCoh}(\mathcal{X}) \xrightarrow{\Delta_*^{\text{IndCoh}}} \text{IndCoh}(\mathcal{X} \times \mathcal{X}) \simeq \text{IndCoh}(\mathcal{X}) \otimes \text{IndCoh}(\mathcal{X}).$$

One can explicitly verify that $(\epsilon_{\mathcal{X}}, \mu_{\mathcal{X}})$ specified above define an identification

$$\text{IndCoh}(\mathcal{X})^{\vee} \simeq \text{IndCoh}(\mathcal{X})$$

by calculating the composition

$$\text{IndCoh}(\mathcal{X}) \xrightarrow{\text{Id} \otimes \mu_{\mathcal{X}}} \text{IndCoh}(\mathcal{X}) \otimes \text{IndCoh}(\mathcal{X}) \otimes \text{IndCoh}(\mathcal{X}) \xrightarrow{\epsilon_{\mathcal{X}} \otimes \text{Id}} \text{IndCoh}(\mathcal{X}).$$

Indeed, it can be calculated via the commutative diagram

$$
\begin{array}{ccccc}
\text{IndCoh}(\mathcal{X}) & \xleftarrow{\ \Delta^!\ } & \text{IndCoh}(\mathcal{X} \times \mathcal{X}) & \xleftarrow{\ p_1^!\ } & \text{IndCoh}(\mathcal{X}) \\
{\scriptstyle \Delta_*^{\text{IndCoh}}} \downarrow & & \downarrow {\scriptstyle (\text{id} \times \Delta)_*^{\text{IndCoh}}} & & \\
\text{IndCoh}(\mathcal{X} \times \mathcal{X}) & \xleftarrow[(\Delta \times \text{id})^!]{} & \text{IndCoh}(\mathcal{X} \times \mathcal{X} \times \mathcal{X}) & & \\
{\scriptstyle (p_2)_*^{\text{IndCoh}}} \downarrow & & & & \\
\text{IndCoh}(\mathcal{X}), & & & &
\end{array}
$$

and the base chase isomorphism (5.1) isomorphs it with the identity functor. The other composition is calculated in the same way by symmetry.

6.2.7.   For the sake of completeness, let us explicitly perform the calculation that defines an identification (6.2).

We can think of both functors as given by objects of

$$\mathrm{IndCoh}(\mathcal{X}_1) \otimes \mathrm{IndCoh}(\mathcal{X}_2) \simeq \mathrm{IndCoh}(\mathcal{X}_1 \times \mathcal{X}_2)$$

and diagram chase shows that both are given by the object

$$(\Gamma_f)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}_1}),$$

where $\Gamma_f : \mathcal{X}_1 \to \mathcal{X}_1 \times \mathcal{X}_2$ is the graph of f, and

$$\omega_{\mathcal{X}_1} := p_{\mathcal{X}_1}^!(k).$$

6.2.8.   The datum of self-duality

$$\mathbf{D}_{\mathcal{X}}^{\mathrm{Serre}} : \mathrm{IndCoh}(\mathcal{X})^\vee \simeq \mathrm{IndCoh}(\mathcal{X})$$

is equivalent to that of an equivalence

$$(\mathrm{IndCoh}(\mathcal{X})^c)^{\mathrm{op}} \to \mathrm{IndCoh}(\mathcal{X})^c.$$

We shall refer to the above functor as 'Serre duality' on $\mathcal{X}$, and denote it by $\mathbb{D}_{\mathcal{X}}^{\mathrm{Serre}}$.

From Theorem 5.1.4, isomorphism (6.2) and Volume I, Chapter 1, Proposition 7.3.5, we obtain:

COROLLARY 6.2.9. *For an ind-proper map* $f : \mathcal{X}_1 \to \mathcal{X}_2$ *of ind-inf-schemes, we have a commutative diagram*

$$
\begin{array}{ccc}
(\mathrm{IndCoh}(\mathcal{X}_1)^c)^{\mathrm{op}} & \xrightarrow{\mathbb{D}_{\mathcal{X}_1}^{\mathrm{Serre}}} & \mathrm{IndCoh}(\mathcal{X}_1)^c \\
{\scriptstyle (f_*^{\mathrm{IndCoh}})^{\mathrm{op}}} \downarrow & & \downarrow {\scriptstyle f_*^{\mathrm{IndCoh}}} \\
(\mathrm{IndCoh}(\mathcal{X}_2)^c)^{\mathrm{op}} & \xrightarrow{\mathbb{D}_{\mathcal{X}_2}^{\mathrm{Serre}}} & \mathrm{IndCoh}(\mathcal{X}_2)^c.
\end{array}
$$

6.3. Convolution categories and algebras.

6.3.1.   As in Volume I, Chapter 5, Sect. 4.1.5, from Corollary 6.1.3 we obtain that the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch;all}}^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch;all}}^{\mathrm{indinfsch}\,\&\,\mathrm{ind\text{-}proper}} \to \mathrm{DGCat}_{\mathrm{cont}}^{2\text{-}\mathrm{Cat}}$$

also carries a canonical right-lax symmetric monoidal structure.

6.3.2.   This allows to extend the formalism in Volume I, Chapter 5, Sect. 5 by replacing

–the class of schematic quasi-compact maps by the class of ind-inf-schematic maps;

-the class of schematic and proper maps by the class of maps that are ind-inf-schematic and ind-proper.

CHAPTER 4

An appliction: crystals

Introduction

In this Chapter we will establish one of the goals indicated in the Introduction to Part I: we will show that inf-schemes give a common framework for ind-coherent sheaves and D-modules. In particular, we will show that the induction and forgetful functors

$$(0.1) \qquad \mathbf{ind}_X : \mathrm{IndCoh}(X) \rightleftarrows \mathrm{Dmod}(X) : \mathbf{oblv}_X$$

interact with the direct and inverse image functors in the expected way.

0.1. Let's do D-modules! The usual definition of the category of D-modules on a smooth affine scheme X is as the category

$$\mathrm{Diff}_X\text{-mod},$$

where Diff_X is the (classical) ring of Grothendieck operations.

This approach to D-modules is very explicit, and is indispensable for concrete applications (e.g. to define regular D-modules and study the notion of holonomicity). However, this approach is not particularly convenient for setting up the theory from the point of view of higher category theory.

Here are some typical issues that become painful in this approach.

0.1.1. One often encounters the question of how to define the category of D-modules on a singular scheme X? The usual answer is that we first assume that X is affine, and choose an embedding $X \hookrightarrow Y$, where Y is smooth. Now, define $\mathrm{Dmod}(X)$ to be $\mathrm{Dmod}(Y)_X$, i.e., the full subcategory of $\mathrm{Dmod}(Y)$ consisting of objects with set-theoretic support on X.

Then, using Kashiwara's lemma, one shows that this construction is canonically independent of the choice of Y. For general X, one considers an affine Zariski cover and glues the corresponding categories.

Note, however, that the words 'choose an embedding $X \hookrightarrow Y$' mean that in the very definition, we appeal to resolutions. From the homotopical point of view, this exacts a substantial price and is too cumbersome to be convenient.

0.1.2. Another example is the definition of the direct image functor. For a morphism $f : X \to Y$ between smooth affine schemes, one introduces an explicit object

$$\mathrm{Diff}_{X,Y} : (\mathrm{Diff}_Y \otimes \mathrm{Diff}_X^{\mathrm{op}})\text{-mod},$$

which defines the desired functor

$$\mathrm{Diff}_X\text{-mod} \to \mathrm{Diff}_Y\text{-mod}.$$

When X and Y are not necessarily smooth, one again embeds this situation into one where X and Y are smooth. When X and Y are non-affine, this is performed locally on X and Y.

All of this can be made to work for an individual morphism: we can prove the proper adjunction between pullbacks and pushforwards, and the base change isomorphism. However, it is not clear how to establish the full functoriality of the category D-mod in this way; namely, as a functor out of the category of correspondences.

0.1.3. Another layer of complexity (=homotopical nuisance) arises when one wants to construct D-modules together with the adjoint pair (0.1).

0.2. D-modules via crystals. In this book, we take a different approach to the theory of D-modules. We *define* the category of D-modules as crystals, establish all the needed functorialities, and then in the case of smooth schemes and morphisms between them identify the resulting categories and functors with the classical ones from the theory of D-modules.

0.2.1. By definition, for a laft prestack $\mathcal{Z}$, the category of crystals on $\mathcal{Z}$ is

$$\mathrm{Crys}(\mathcal{Z}) := \mathrm{IndCoh}(\mathcal{Z}_{\mathrm{dR}}),$$

where $\mathcal{Z}_{\mathrm{dR}}$ is the de Rham prestack of $\mathcal{Z}$.

Let $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ be a map of laft prestacks. Then !-pullback on IndCoh defines a functor

$$f_{\mathrm{dR}}^! : \mathrm{Crys}(\mathcal{Z}_2) \to \mathrm{Crys}(\mathcal{Z}_1).$$

This is the pullback functor for crystals.

0.2.2. Assume now that f is ind-nil-schematic, which means that the corresponding morphism $^{\mathrm{red}}\mathcal{Z}_1 \to {}^{\mathrm{red}}\mathcal{Z}_2$ is ind-schematics. Then one (easily) sees that the resulting morphism

$$(f_{\mathrm{dR}}) : (\mathcal{Z}_1)_{\mathrm{dR}} \to (\mathcal{Z}_2)_{\mathrm{dR}}$$

is ind-inf-schematic. Now, using Chapter 3, Sect. 4, we define the functor

$$f_{\mathrm{dR},*} : \mathrm{Crys}(\mathcal{Z}_1) \to \mathrm{Crys}(\mathcal{Z}_2)$$

to be the functor $(f_{\mathrm{dR}})_*$. This is the de Rham direct image functor.

0.2.3. Taking $\mathcal{Z}_1 = \mathcal{Z}$ (so that $^{\mathrm{red}}\mathcal{Z}$ is an ind-scheme) and $\mathcal{Z}_2 = \mathrm{pt}$, we obtain the functor of de Rham sections

$$\Gamma_{\mathrm{dR}}(\mathcal{Z}, -) : \mathrm{Crys}(\mathcal{Z}) \to \mathrm{Vect}.$$

Moreover, the above constructions automatically extend to the data of a functor out of a suitable $(\infty, 2)$-category of correspondences. Namely, we consider the category $\mathrm{PreStk}_{\mathrm{laft}}$ equipped with the following classes of functors:

–'horizontal' maps are all maps in $\mathrm{PreStk}_{\mathrm{laft}}$;

–'vertical' maps are those maps f that $^{\mathrm{red}}f$ is ind-schematic (we call them *ind-nil-schematic*);

–'admissible' maps are those vertical maps that are also ind-proper.

One shows that the assignment $\mathcal{Z} \mapsto \mathcal{Z}_{\mathrm{dR}}$ defines a functor

$$\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indnilsch;all}}^{\mathrm{indnilsch\,\&\,ind\text{-}proper}} \to \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch;all}}^{\mathrm{indinfsch\,\&\,ind\text{-}proper}}.$$

Composing with the functor

$$\mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch\,\&\,ind\text{-}proper}}_{\mathrm{indinfsch;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indinfsch\,\&\,ind\text{-}proper}}_{\mathrm{indinfsch;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}},$$

we obtain a functor

$$\mathrm{Crys}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indnilsch\,\&\,ind\text{-}proper}}_{\mathrm{indnilsch;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indnilsch\,\&\,ind\text{-}proper}}_{\mathrm{indnilsch;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}.$$

The above functor $\mathrm{Crys}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indnilsch\,\&\,ind\text{-}proper}}_{\mathrm{indnilsch;all}}}$ is the desired expression of functoriality of the assignment

$$\mathcal{Z} \mapsto \mathrm{Crys}(\mathcal{Z}).$$

0.2.4.   Now suppose that $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft}}$ admits deformation theory. One shows that in the case the tautological map

$$p_{\mathrm{dR},\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}_{\mathrm{dR}}$$

is an inf-schematic nil-isomorphism. Hence, by Chapter 3, Prop. 3.1.2, the functor

$$p^{!}_{\mathrm{dR},\mathcal{Z}} : \mathrm{Crys}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{Z})$$

admits the left adjoint.

Thus, we obtain the desired adjoint pair:

$$\mathbf{ind}_{\mathrm{dR},\mathcal{Z}} : \mathrm{IndCoh}(\mathcal{Z}) \rightleftarrows \mathrm{Crys}(\mathcal{Z}) : \mathbf{oblv}_{\mathrm{dR},\mathcal{Z}}.$$

0.2.5. *But what does this have to do with D-modules?* The basic observation, essentially due to Grothendieck[1], is that for a smooth scheme X, the category $\mathrm{Crys}(X)$, together with the forgetful functor

$$\Psi_X \circ \mathbf{oblv}_{\mathrm{dR},X} : \mathrm{Crys}(X) \to \mathrm{QCoh}(X),$$

is canonically equivalent to the category of right D-modules, together with its tautological forgetful functor to $\mathrm{QCoh}(X)$.

We describe this identification in Sect. 4 of this Chapter. We also show that the functors on the category of crystals (direct and inverse image for a map $f : X \to Y$) described above map to the corresponding functors for D-modules under this identification.

This is thus our ansatz to the construction of the theory of D-modules: instead of developing the theory of D-modules directly, we develop the theory of crystals, and then identity it with D-modules when D-modules are conveniently defined; namely, in the case of smooth schemes.

0.3. What else is done in this chapter?

---

[1] We learned it from A. Beilinson.

0.3.1.   In Sect. 1 we introduce the category of crystals $\mathrm{Crys}(\mathcal{Z})$, where $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft}}$.

The key observation here is the following: let $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ be a map between prestacks such that the induced map

$$^{\mathrm{red}}\mathcal{Z}_1 \to {}^{\mathrm{red}}\mathcal{Z}_2$$

is (ind)-schematic. Then we show that the resulting map

$$(\mathcal{Z}_1)_{\mathrm{dR}} \to (\mathcal{Z}_2)_{\mathrm{dR}}$$

is (ind)-inf-schematic.

This observation, along with the fact that pushforward is defined on IndCoh for (ind)-nil-schematic morphisms, is what makes the theory work. I.e., this is the framework that allows to treat the de Rham pushfoward (in particular, de Rham (co)homology) on the same footing as the $\mathcal{O}$-module pushforward (in its IndCoh variant).

We then establish some properties, expected from the theory of D-modules:

(i) For a closed embedding $i : \mathcal{Y} \hookrightarrow \mathcal{Z}$, the functor $i_{\mathrm{dR},*} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Z})$ is fully faithful;

(ii) If $\mathcal{Z}$ is an (ind)-nil-scheme, the category $\mathrm{Crys}(\mathcal{Z})$ is compactly generated and has a reasonably behaved t-structure.

0.3.2.   In Sect. 2 we apply the results of Chapter 3, Sect. 5 and 6 and construct Crys as a functor out of the category of correspondences.

We show that when evaluated on ind-nil-schemes, this gives rise to the operation of Verdier duality.

0.3.3.   In Sect. 3 we study the functor of forgetting the crystal structure:

$$\mathbf{oblv}_{\mathrm{dR},\mathcal{Z}} : \mathrm{Crys}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{Z}),$$

which, in our framework, is just the pullback functor for the morphism

$$p_{\mathrm{dR},\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}_{\mathrm{dR}}.$$

The key observation is that if $\mathcal{Z}$ admits deformation-theory, then the map $p_{\mathrm{dR},\mathcal{Z}}$ is inf-schematic. Hence, in this case the functor $\mathbf{oblv}_{\mathrm{dR},\mathcal{Z}}$ admits a left adjoint, given by $(p_{\mathrm{dR},\mathcal{Z}})_*^{\mathrm{IndCoh}}$. This left adjoint, denoted $\mathbf{ind}_{\mathrm{dR},\mathcal{Z}}$, is the functor of induction from ind-coherent sheaves to crystals.

When $\mathcal{Z} = X$ is a smooth affine scheme, under the identification

$$\mathrm{Crys}(X) \simeq (\mathrm{Diff}_X^{\mathrm{op}})\text{-mod},$$

the functor $\mathbf{ind}_{\mathrm{dR},\mathcal{Z}}$ corresponds to

$$\mathcal{F} \mapsto \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathrm{Diff}_X .$$

We show that if $\mathcal{Z}$ is an ind-scheme, then the morphism $p_{\mathrm{dR},\mathcal{Z}}$ is *ind-schematic*. We use this fact to deduce that the functor $\mathbf{ind}_{\mathrm{dR},\mathcal{Z}}$ is t-exact.

0.3.4. In Sect. 3.3, we develop the theory of crystals *relative* to a given prestack $\mathcal{Y}$. Namely, for $\mathcal{Z}$ over $\mathcal{Y}$, we set

$$\mathcal{Z}_{/\mathcal{Y}_{\mathrm{dR}}} \coloneqq \mathcal{Z}_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}$$

and we set

$$^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z}) \coloneqq \mathrm{IndCoh}(\mathcal{Z}_{/\mathcal{Y}_{\mathrm{dR}}}).$$

When $\mathcal{Z} = X$ and $\mathcal{Y} = Y$ are smooth affine schemes, and the map $X \to Y$ is smooth, category $^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z})$ identifies with

$$(\mathrm{Diff}_{X/Y})^{\mathrm{op}}\text{-mod},$$

where $\mathrm{Diff}_{X/Y}$ is the (classical) ring of *vertical* differential operators (i.e., the sub-ring of Diff_X consisting of elements that commute with functions on Y).

If $\mathcal{Z}$ *admits deformation theory relative to* $\mathcal{Y}$, then the morphism

$$p_{/\mathcal{Y}_{\mathrm{dR}},\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}_{/\mathcal{Y}_{\mathrm{dR}}}$$

is again inf-schematic, and hence the forgetful functor

$$(p_{/\mathcal{Y}_{\mathrm{dR}},\mathcal{Z}})^! : {}^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{Z})$$

admits a left adjoint, given by $(p_{/\mathcal{Y}_{\mathrm{dR}},\mathcal{Z}})^{\mathrm{IndCoh}}_*$.

0.3.5. In Sect. 4 we show how to identify the theory of crystals with D-modules in the case of smooth schemes. Our exposition here is not self contained: we make frequent references to [**GaRo2**].

We first consider the case of *left* D-modules, and we show that the category $\mathrm{Crys}^l(X)$ of *left* crystals on a smooth affine scheme X, defined as $\mathrm{QCoh}(X_{\mathrm{dR}})$, identifies with Diff_X-mod.

We then show that the category of right crystals (i.e., the usual category of crystals)

$$\mathrm{Crys}^r(X) \coloneqq \mathrm{Crys}(X) \coloneqq \mathrm{IndCoh}(X_{\mathrm{dR}})$$

identifies with $(\mathrm{Diff}_X)^{\mathrm{op}}$-mod.

Next, we show that the functor

$$\Upsilon_{X_{\mathrm{dR}}} : \mathrm{QCoh}(X_{\mathrm{dR}}) \to \mathrm{IndCoh}(X_{\mathrm{dR}})$$

that identifies $\mathrm{Crys}^l(X)$ with $\mathrm{Crys}^r(X)$ corresponds under the above equivalences

(0.2) $\qquad \mathrm{Crys}^l(X) \simeq \mathrm{Diff}_X$-mod and $\mathrm{Crys}^r(X) \simeq (\mathrm{Diff}_X)^{\mathrm{op}}$-mod

with the functor

$$\mathrm{Crys}^l(X) \to \mathrm{Crys}^r(X), \quad \mathcal{M} \mapsto M \otimes \det(T^*(X))[\dim(X)].$$

Finally, we show that for a map between $f : X \to Y$ between smooth schemes, under the identifications (0.2), the functor

$$f^{\blacktriangle,l} : \mathrm{Diff}_Y\text{-mod} \to \mathrm{Diff}_X\text{-mod}$$

from the theory of D-modules corresponds to pullback

$$f^*_{\mathrm{dR}} : \mathrm{QCoh}(Y_{\mathrm{dR}}) \to \mathrm{QCoh}(X_{\mathrm{dR}}),$$

and the functor

$$f_{\mathrm{Dmod},*} : \mathrm{Diff}_X\text{-mod} \to \mathrm{Diff}_Y\text{-mod}$$

from the theory of D-modules corresponds to push-forward

$$f_{\mathrm{dR},*} : \mathrm{QCoh}(X_{\mathrm{dR}}) \to \mathrm{QCoh}(Y_{\mathrm{dR}}).$$

1. Crystals on prestacks and inf-schemes

In this section we will reap the fruits of the work done in Chapter 3. Namely, we will show how the theory of IndCoh gives rise to the theory of *crystals*.

1.1. The de Rham functor and crystals: recollections. The category $\mathrm{Crys}(\mathcal{X})$ of crystals on a prestack $\mathcal{X}$ is defined to be IndCoh on the corresponding prestack $\mathcal{X}_{\mathrm{dR}}$. In this subsection we recall the functor $\mathcal{X} \mapsto \mathcal{X}_{\mathrm{dR}}$ and study its basic properties.

1.1.1. For $\mathcal{Z} \in \mathrm{PreStk}$, we denote by $\mathcal{Z}_{\mathrm{dR}}$ the corresponding de Rham prestack, defined as

$$\mathrm{Maps}(S, \mathcal{Z}_{\mathrm{dR}}) := \mathrm{Maps}(^{\mathrm{red}}S, \mathcal{Z}),$$

for $S \in \mathrm{Sch}^{\mathrm{aff}}$.

For a morphism $f : \mathcal{Z}^1 \to \mathcal{Z}^2$, let $f_{\mathrm{dR}} : \mathcal{Z}^1_{\mathrm{dR}} \to \mathcal{Z}^2_{\mathrm{dR}}$ denote the corresponding morphism between deRham prestacks.

1.1.2. Note that the functor dR commutes both with limits and colimits.

Also, note that

$$\mathcal{Z}_{\mathrm{dR}} \simeq (^{\mathrm{red}}\mathcal{Z})_{\mathrm{dR}}.$$

So, if a morphism $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ is a nil-isomorphism (i.e., $^{\mathrm{red}}\mathcal{Z}_1 \to {}^{\mathrm{red}}\mathcal{Z}_1$ is an isomorphism), then $(\mathcal{Z}_1)_{\mathrm{dR}} \to (\mathcal{Z}_2)_{\mathrm{dR}}$ is an isomorhism.

1.1.3. We claim:

PROPOSITION 1.1.4. *The functor* dR *takes* $\mathrm{PreStk}_{\mathrm{laft}}$ *to* $\mathrm{PreStk}_{\mathrm{laft}}$.

PROOF. Let $\mathcal{Z}$ be an object of $\mathrm{PreStk}_{\mathrm{laft}}$. We need to show that $\mathcal{Z}_{\mathrm{dR}}$ satisfies:
- It is convergent;
- For every n, the truncation $^{\leq n}\mathcal{Z}$ belongs to $^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$.

The convergence of $\mathcal{Z}_{\mathrm{dR}}$ is obvious. To show that $^{\leq n}\mathcal{Z} \in {}^{\leq n}\mathrm{PreStk}_{\mathrm{lft}}$, it suffices to show that $\mathcal{Z}_{\mathrm{dR}}$ takes filtered limits in $\mathrm{Sch}^{\mathrm{aff}}$ to colimits in Spc. However, this follows from the fact that the functor

$$S \mapsto {}^{\mathrm{red}}S, \quad \mathrm{Sch}^{\mathrm{aff}} \to {}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}$$

preserves filtered limits, and the fact that $^{\mathrm{red}}\mathcal{Z} \in {}^{\mathrm{red}}\mathrm{PreStk}_{\mathrm{lft}}$.

$\square$

1.2. Crystals. In this subsection we introduce the category of crystals.

1.2.1. Composing the functor $\mathrm{dR} : \mathrm{PreStk}_{\mathrm{laft}} \to \mathrm{PreStk}_{\mathrm{laft}}$ with

$$\mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

we obtain a functor denoted by

$$\mathrm{Crys}^!_{\mathrm{PreStk}_{\mathrm{laft}}} : (\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

This is the functor which is denoted $\mathrm{Crys}^r_{\mathrm{PreStk}_{\mathrm{laft}}}$ in [**GaRo2**, Sect 2.3.2].

1.2.2. For $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft}}$ we shall denote the value of $\mathrm{Crys}^!_{\mathrm{PreStk}_{\mathrm{laft}}}$ on $\mathcal{Z}$ by $\mathrm{Crys}(\mathcal{Z})$. For a morphism $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ in $\mathrm{PreStk}_{\mathrm{laft}}$, we shall denote by $f^!_{\mathrm{dR}}$ the resulting functor

$$\mathrm{Crys}(\mathcal{Z}_2) \to \mathrm{Crys}(\mathcal{Z}_1).$$

Note that if a morphism $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ is a nil-isomorphism, then

$$f^!_{\mathrm{dR}} : \mathrm{Crys}(\mathcal{Z}_2) \to \mathrm{Crys}(\mathcal{Z}_1)$$

is an equivalence.

1.2.3. For $\mathcal{Z} \in \mathrm{PreStk}$, we let $p_{\mathrm{dR},\mathcal{Z}}$ denote the tautological projection:

$$\mathcal{Z} \to \mathcal{Z}_{\mathrm{dR}}.$$

The map $p_{\mathrm{dR},\mathcal{Z}}$ gives rise to a natural transformation of functors

$$\mathbf{oblv}_{\mathrm{dR}} : \mathrm{Crys}^!_{\mathrm{PreStk}_{\mathrm{laft}}} \to \mathrm{IndCoh}^!_{\mathrm{PreStk}_{\mathrm{laft}}}.$$

For a map $f : \mathcal{Z}_1 \to \mathcal{Z}_2$, we have a commutative square of functors:

$$
\begin{array}{ccc}
\mathrm{Crys}(\mathcal{Z}_1) & \xrightarrow{\ \mathbf{oblv}_{\mathrm{dR},\mathcal{Z}_1}\ } & \mathrm{IndCoh}(\mathcal{Z}_1) \\
{\scriptstyle f^!_{\mathrm{dR}}} \big\uparrow & & \big\uparrow {\scriptstyle f^!} \\
\mathrm{Crys}(\mathcal{Z}_2) & \xrightarrow{\ \mathbf{oblv}_{\mathrm{dR},\mathcal{Z}_2}\ } & \mathrm{IndCoh}(\mathcal{Z}_2).
\end{array}
$$

1.2.4. Finally, we make the following observation:

PROPOSITION 1.2.5. *For $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft}}$, the functor*

$$\mathrm{Crys}(\mathcal{Z}) \to \lim_{Z \in (\mathbf{C}_{/\mathcal{Z}})^{\mathrm{op}}} \mathrm{Crys}(Z)$$

is an equivalence, where $\mathbf{C}$ is any of the following categories:

$${}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}, \ {}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}, \ {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}, \ \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}, \ {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}, \ {}^{\mathrm{cl}}\mathrm{Sch}_{\mathrm{ft}}, \ {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}, \ \mathrm{Sch}_{\mathrm{aft}} \ .$$

PROOF. It is enough to show that the functor

$$\mathrm{dR} : \mathrm{PreStk}_{\mathrm{laft}} \to \mathrm{PreStk}_{\mathrm{laft}}$$

is isomorphic to the left Kan extension of its restriction to $\mathbf{C} \subset \mathrm{PreStk}_{\mathrm{laft}}$ for $\mathbf{C}$ as above. It is sufficient to consider the case of $\mathbf{C} = {}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

First, we note that the functor dR commutes with colimits. This implies that dR is isomorphic to the left Kan extension of its restriction to ${}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$. Hence, it suffices to show that the functor

$$\mathrm{dR} : \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}} \to \mathrm{PreStk}_{\mathrm{laft}}$$

is isomorphic to the left Kan extension of its restriction to ${}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

In other words, we have to show that given $Z \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$ and a map

$${}^{\mathrm{red}}S \to Z,$$

the category of its factorizations as

$${}^{\mathrm{red}}S \to Z' \to Z$$

with $Z' \in {}^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}$, is contractible.

However, the latter is obvious as the above category has a final object, namely, $Z' := {}^{\mathrm{red}}Z$.

$\square$

1.3. Crystals and (ind)-nil-schemes. In this subsection we introduce the class of prestacks that we call *(ind)-nil-schemes*, and study the category of crystals on such prestacks. (Ind)-nil-schemes play the same role vis-à-vis Crys as (ind)-inf-schemes do for IndCoh.

1.3.1.   Consider the full subcategories

$$\mathrm{indnilSch}_{\mathrm{laft}} := \mathrm{PreStk}_{\mathrm{laft}} \underset{{}^{\mathrm{red}}\mathrm{PreStk}_{\mathrm{lft}}}{\times} {}^{\mathrm{red}}\mathrm{indSch} \subset \mathrm{PreStk}_{\mathrm{laft}}$$

and

$$\mathrm{nilSch}_{\mathrm{laft}} := \mathrm{PreStk}_{\mathrm{laft}} \underset{{}^{\mathrm{red}}\mathrm{PreStk}_{\mathrm{lft}}}{\times} {}^{\mathrm{red}}\mathrm{Sch} \subset \mathrm{PreStk}_{\mathrm{laft}},$$

where $\mathrm{PreStk}_{\mathrm{laft}} \to {}^{\mathrm{red}}\mathrm{PreStk}_{\mathrm{lft}}$ is the functor $\mathcal{Z} \mapsto {}^{\mathrm{red}}\mathcal{Z}$.

In other words, $\mathcal{Z}$ belongs to $\mathrm{indnilSch}_{\mathrm{laft}}$ (resp., $\mathrm{nilSch}_{\mathrm{laft}}$) if and only if ${}^{\mathrm{red}}\mathcal{Z}$ is a reduced ind-scheme (resp., scheme).

For example, we have

$$\mathrm{infSch}_{\mathrm{laft}} \subset \mathrm{nilSch}_{\mathrm{laft}} \ \text{ and } \ \mathrm{indinfSch}_{\mathrm{laft}} \subset \mathrm{indnilSch}_{\mathrm{laft}} .$$

We shall refer to objects of $\mathrm{indnilSch}_{\mathrm{laft}}$ (resp., $\mathrm{nilSch}_{\mathrm{laft}}$) as *ind-nil-schemes* (resp., *nil-schemes*).

1.3.2.   We claim:

LEMMA 1.3.3.   *The functor* dR *takes objects of* $\mathrm{indnilSch}_{\mathrm{laft}}$ *(resp.,* $\mathrm{nilSch}_{\mathrm{laft}}$*) to* $\mathrm{indinfSch}_{\mathrm{laft}}$ *(resp.,* $\mathrm{infSch}_{\mathrm{laft}}$*).*

PROOF.   We have

$${}^{\mathrm{red}}(\mathcal{Z}_{\mathrm{dR}}) = {}^{\mathrm{red}}\mathcal{Z}.$$

Now, we claim that for *any* $\mathcal{Z} \in \mathrm{PreStk}$, the corresponding $\mathcal{Z}_{\mathrm{dR}}$ admits deformation theory. In fact, it admits an ∞-connective deformation theory: all of its cotangent spaces are zero.

$\square$

1.3.4.   Recall from Chapter 2, Definitions 1.6.5(a), 1.6.7(c) and 1.6.11(c), the notions of *(ind)-schematic* and *(ind)-proper* maps of prestacks, as well as *(ind)-closed embeddings* of prestacks.

DEFINITION 1.3.5.

(a) *We shall say that a map of prestacks is* (ind)-nil-schematic *if the map of the corresponding reduced prestacks is (ind)-schematic.*

(b) *We shall say that a map of prestacks is an* nil-closed-embedding(ind)-nil-closed embedding *if the map of the corresponding reduced prestacks is an ind-closed embedding.*

Recall the notion of an (ind)-inf-schematic map of prestacks, see Chapter 2, Definitions 3.1.5. We have:

COROLLARY 1.3.6.   *The functor* dR *takes (ind)-nil-schematic maps in* $\mathrm{PreStk}_{\mathrm{laft}}$ *to (ind)-inf-schematic maps.*

PROOF. For a map of prestacks $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ and $S \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/(\mathcal{Z}_2)_{\mathrm{dR}}}$, the Cartesian product

$$S \underset{(\mathcal{Z}_2)_{\mathrm{dR}}}{\times} (\mathcal{Z}_1)_{\mathrm{dR}}$$

identifies with

$$S \underset{S_{\mathrm{dR}}}{\times} ({}^{\mathrm{red}}S \underset{\mathcal{Z}_2}{\times} \mathcal{Z}_1)_{\mathrm{dR}}.$$

Now, we use Lemma 1.3.3 and the fact that the subactegory $\mathrm{indinfSch}_{\mathrm{laft}}$ is preserved by finite limits.

$\square$

1.3.7. We claim:

LEMMA 1.3.8. *Let $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ be an ind-nil-proper map in $\mathrm{PreStk}_{\mathrm{laft}}$. Then:*
(a) *The functor $f_{\mathrm{dR},*} : \mathrm{Crys}(\mathcal{Z}_1) \to \mathrm{Crys}(\mathcal{Z}_2)$, left adjoint to $f^!_{\mathrm{dR}}$, is well-defined, and satisfies base change with respect to !-pullbacks.*
(b) *If f is an ind-nil-closed embedding, then $f_{\mathrm{dR},*}$ is fully faithful.*

PROOF. Point (a) follows from Corollary 1.3.6 and Chapter 3, Proposition 3.2.4.

To prove point (b), we need to show that the unit of the adjunction

$$\mathrm{Id}_{\mathrm{Crys}(\mathcal{Z}_1)} \to f^!_{\mathrm{dR}} \circ f_{\mathrm{dR},*}$$

is an isomorphism.

Consider the Cartesian square:

$$
\begin{CD}
\mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} \mathcal{Z}_1 @>{p_1}>> \mathcal{Z}_1 \\
@V{p_2}VV @VVV \\
\mathcal{Z}_1 @>>> \mathcal{Z}_2.
\end{CD}
$$

The above unit of the adjunction equals the composite map

$$\mathrm{Id}_{\mathrm{Crys}(\mathcal{Z}_1)} \simeq (p_2)_{\mathrm{dR},*} \circ (\Delta_{\mathcal{Z}_1})_{\mathrm{dR},*} \circ (\Delta_{\mathcal{Z}_1})^!_{\mathrm{dR}} \circ (p_1)^!_{\mathrm{dR}} \to (p_2)_{\mathrm{dR},*} \circ (p_1)^!_{\mathrm{dR}} \to f^!_{\mathrm{dR}} \circ f_{\mathrm{dR},*},$$

where $\Delta_{\mathcal{Z}_1}$ is the diagonal map

$$\mathcal{Z}_1 \to \mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} \mathcal{Z}_1,$$

and second arrow is the co-unit of the $((\Delta_{\mathcal{Z}_1})_{\mathrm{dR},*}, (\Delta_{\mathcal{Z}_1})^!_{\mathrm{dR}})$-adjunction.

Now, by base change,

$$(p_2)_{\mathrm{dR},*} \circ (p_1)^!_{\mathrm{dR}} \to f^!_{\mathrm{dR}} \circ f_{\mathrm{dR},*}$$

is an isomorphism. Hence, it is enough to show that

$$(p_2)_{\mathrm{dR},*} \circ (\Delta_{\mathcal{Z}_1})_{\mathrm{dR},*} \circ (\Delta_{\mathcal{Z}_1})^!_{\mathrm{dR}} \circ (p_1)^!_{\mathrm{dR}} \to (p_2)_{\mathrm{dR},*} \circ (p_1)^!_{\mathrm{dR}}$$

is an isomorphism as well. However, the map

$$(\Delta_{\mathcal{Z}_1})_{\mathrm{dR},*} \circ (\Delta_{\mathcal{Z}_1})^!_{\mathrm{dR}} \to \mathrm{Id}_{\mathrm{Crys}(\mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} \mathcal{Z}_1)}$$

is an isomorphism, since $(\Delta_{\mathcal{Z}_1})^!_{\mathrm{dR}}$ is an equivalence (because the map $\Delta_{\mathcal{Z}_1}$ is a nil-isomorphism).

$\square$

1.4. The functor of de Rham direct image. In this subsection we develop the functor of de Rham direct image (a.k.a., pushforward) for crystals.

1.4.1.   Recall the functor

$$\mathrm{IndCoh}_{\mathrm{indinfSch}_{\mathrm{laft}}} : \mathrm{indinfSch}_{\mathrm{laft}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

that sends a morphism f to the functor f_*^{IndCoh}, see Chapter 3, Sect. 4.3.

Precomposing it with the functor

$$\mathrm{dR} : \mathrm{indnilSch}_{\mathrm{laft}} \to \mathrm{indinfSch}_{\mathrm{laft}}$$

we obtain a functor

(1.1) $$\mathrm{Crys}_{\mathrm{indnilSch}_{\mathrm{laft}}} : \mathrm{indnilSch}_{\mathrm{laft}} \to \mathrm{DGCat}_{\mathrm{cont}} .$$

1.4.2.   For a morphism $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ in $\mathrm{indnilSch}_{\mathrm{laft}}$ we shall denote the resulting functor

$$\mathrm{Crys}(\mathcal{Z}_1) \to \mathrm{Crys}(\mathcal{Z}_2)$$

by $f_{\mathrm{dR},*}$.

In other words,

$$f_{\mathrm{dR},*} = (f_{\mathrm{dR}})_*^{\mathrm{IndCoh}}.$$

1.4.3.   From Chapter 3, Corollary 5.2.3, we obtain:

COROLLARY 1.4.4. *The restriction of the functor* $\mathrm{Crys}_{\mathrm{indnilSch}_{\mathrm{laft}}}$ *to the* 1-*full subcategory*

$$(\mathrm{indnilSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}} \subset \mathrm{indnilSch}_{\mathrm{laft}}$$

is obtained by passing to left adjoints from the restriction functor $\mathrm{Crys}^!_{\mathrm{indnilSch}_{\mathrm{laft}}}$ *to*

$$((\mathrm{indnilSch}_{\mathrm{laft}})_{\mathrm{ind\text{-}proper}})^{\mathrm{op}} \subset (\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{op}}.$$

REMARK 1.4.5. Note that we have used the notation $f_{\mathrm{dR},*}$ when f is ind-proper earlier (in Lemma 1.3.8), to denote the left adjoint of $f^!_{\mathrm{dR}}$. The above corollary implies that the notations are consistent.

1.5. Crystals on ind-nil-schemes as extended from schemes. The material of this subsection will not be used in the sequel and is included for the sake of completeness. We show that the theory of Crys on ind-nil-schemes can be obtained by extending the same theory on schemes.

1.5.1.   Consider the category $^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}$, and consider the functors

$$\mathrm{Crys}^!_{^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}} : (^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

and

$$\mathrm{Crys}_{^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}} : {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}} \to \mathrm{DGCat}_{\mathrm{cont}} .$$

From Proposition 1.2.5 we obtain:

COROLLARY 1.5.2. *The natural map*

$$\mathrm{Crys}^!_{\mathrm{indnilSch}_{\mathrm{laft}}} \to \mathrm{RKE}_{(^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}})^{\mathrm{op}} \to (\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{op}}} (\mathrm{Crys}^!_{^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}})$$

is an isomorphism.

We are going to prove the following:

PROPOSITION 1.5.3. *The natural map*

$$\mathrm{LKE}_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}\hookrightarrow\mathrm{indnilSch}_{\mathrm{laft}}}(\mathrm{Crys}_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}}) \to \mathrm{Crys}_{\mathrm{indnilSch}_{\mathrm{laft}}}$$

is an isomorphism.

The rest of this subsection is devoted to the proof of this proposition.

1.5.4.   Consider the 1-full subcategory of $\mathrm{indnilSch}_{\mathrm{laft}}$ equal to

$$(\mathrm{indnilSch}_{\mathrm{laft}})_{\text{nil-closed}} = \mathrm{PreStk}_{\mathrm{laft}} \underset{{}^{\mathrm{red}}\mathrm{PreStk}_{\mathrm{lft}}}{\times} ({}^{\mathrm{red}}\mathrm{indSch}_{\mathrm{laft}})_{\text{closed}}.$$

I.e., we restrict 1-morphisms to be nil-closed maps.

It is enough to show that the map in Proposition 1.5.3 becomes an isomorphism when restricted to the above subcategory. This follows by Chapter 3, Corollary 4.1.4 from Proposition 1.4.4 and the following statement:

PROPOSITION 1.5.5.

(a) *The map*

$$\left(\left(\mathrm{RKE}_{({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}})^{\mathrm{op}}\to(\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{op}}\right)(\mathrm{Crys}^{!}_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}})\right)\big|_{((\mathrm{indnilSch}_{\mathrm{laft}})_{\text{nil-closed}})^{\mathrm{op}}} \to$$
$$\to \mathrm{RKE}_{(({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}})_{\text{closed}})^{\mathrm{op}}\to((\mathrm{indnilSch}_{\mathrm{laft}})_{\text{nil-closed}})^{\mathrm{op}}}\left(\mathrm{Crys}^{!}_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}}\big|_{(({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}})_{\text{closed}})^{\mathrm{op}}}\right)$$

is an isomorphism.

(b) *The map*

$$\mathrm{LKE}_{({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}})_{\text{closed}}\hookrightarrow(\mathrm{indnilSch}_{\mathrm{laft}})_{\text{nil-closed}}}\left(\mathrm{Crys}_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}}\big|_{({}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}})_{\text{closed}}}\right) \to$$
$$\to \left(\mathrm{LKE}_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}\hookrightarrow\mathrm{indnilSch}_{\mathrm{laft}}}(\mathrm{Crys}_{{}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}})\right)\big|_{(\mathrm{indnilSch}_{\mathrm{laft}})_{\text{nil-closed}}}$$

is an isomorphism.

PROOF. Follows from the fact that for

$$\mathcal{Z} \in \mathrm{indnilSch}_{\mathrm{laft}},$$

the category

$$\{f : Z \to \mathcal{Z}, \quad Z \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}, \quad f \text{ is nil-closed}\}$$

is cofinal in

$$\{f : Z \to \mathcal{Z}, \quad Z \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}\},$$

by Chapter 2, Corollary 1.7.5(b)

$\square$

1.6. Properties of the category of crystals on (ind)-nil-schemes. In this subsection we study properties of the category $\mathrm{Crys}(\mathcal{Z})$ on a given object $\mathcal{Z} \in \mathrm{indnilSch}_{\mathrm{laft}}$.

1.6.1.   We claim:

PROPOSITION 1.6.2. *The functor*

$$\mathrm{Crys}(\mathcal{Z}) \to \lim_{f:Z\to\mathcal{Z}} \mathrm{Crys}(Z)$$

is an equivalence, where the limit is taken over the index ∞-category

$$\{f : Z \to \mathcal{Z}, \ Z \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}, \ f \text{ is nil-closed}\}.$$

For every $f : Z \to \mathcal{Z}$ as above, the corresponding functor

$$f_{\mathrm{dR},*} : \mathrm{Crys}(Z) \to \mathrm{Crys}(\mathcal{Z})$$

is fully faithful.

PROOF. The first assertion follows from Proposition 1.2.5 for $\mathbf{C} = {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$ and Chapter 2, Corollary 1.7.5(b).

The second assertion follows from Lemma 1.3.8(b).

$\square$

1.6.3. *Compact generation.* From Chapter 3, Corollary 3.2.2 and , we obtain:

COROLLARY 1.6.4. *The category* $\mathrm{Crys}(\mathcal{Z})$ *is compactly generated.*

From Proposition 1.6.2, combined with [**DrGa2**, Corollary 1.9.4 and Lemma 1.9.5], we have the following more explicit description of the subcategory

$$\mathrm{Crys}(\mathcal{Z})^c \subset \mathrm{Crys}(\mathcal{Z}).$$

COROLLARY 1.6.5. *Compact objects of* $\mathrm{Crys}(\mathcal{Z})$ *are those that can be obtained as*

$$f_{\mathrm{dR},*}(\mathcal{M}), \quad \mathcal{M} \in \mathrm{Crys}(Z)^c, \ Z \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}} \ and \ f \ is \ a \ nil\text{-}closed \ map \ Z \to \mathcal{Z}.$$

1.6.6. *t-structure.* According to Chapter 3, Sect. 3.4, the category $\mathrm{Crys}(\mathcal{Z})$ carries a canonical t-structure. It is characterized by the following property:

$$\mathcal{M} \in \mathrm{Crys}(\mathcal{Z})^{\geq 0} \Leftrightarrow \mathbf{oblv}_{\mathrm{dR},\mathcal{Z}}(\mathcal{M}) \in \mathrm{IndCoh}(\mathcal{Z})^{\geq 0}.$$

In addition, from Chapter 3, Corollary 3.4.4, we obtain:

COROLLARY 1.6.7.

(a) *An object $\mathcal{M} \in \mathrm{Crys}(\mathcal{Z})$ lies in $\mathrm{Crys}(\mathcal{Z})^{\geq 0}$ if and only if for every nil-closed map $f : Z \to \mathcal{Z}$ with $Z \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}$ we have*

$$f_{\mathrm{dR}}^!(\mathcal{M}) \in \mathrm{Crys}(Z)^{\geq 0}.$$

(b) *The category $\mathrm{Crys}(\mathcal{Z})^{\leq 0}$ is generated under colimits by the essential images of $\mathrm{Crys}(Z)^{\leq 0}$ for $f : Z \to \mathcal{Z}$ with $Z \in {}^{\mathrm{red}}\mathrm{Sch}_{\mathrm{ft}}$ and f nil-closed.*

2. Crystals as a functor out of the category of correspondences

In this section we extend the formalism of crystals to a functor out of the category of correspondences.

2.1. Correspondences and the de Rham functor. In this subsection we show that the de Rham functor turns (ind)-nil-schematic morphisms into (ind)-inf-schematic ones.

2.1.1. Recall that the functor dR commutes with Cartesian products. Combining this observation with Lemma 1.3.6, we obtain that dR gives rise to a functor of $(\infty, 2)$-categories:

$$\text{Corr}(\text{dR})^{\text{ind-proper}}_{\text{indnilsch;all}} : \text{Corr}(\text{PreStk}_{\text{laft}})^{\text{indnilsch \& ind-proper}}_{\text{indnilsch;all}} \to \text{Corr}(\text{PreStk}_{\text{laft}})^{\text{indinfsch \& ind-proper}}_{\text{indinfsch;all}}.$$

Hence, from Chapter 3, Theorem 5.4.3 and Proposition 5.5.3, we obtain:

THEOREM 2.1.2. *There exists a canonically defined functor*

$$\text{Crys}_{\text{Corr}(\text{PreStk}_{\text{laft}})^{\text{indnilsch \& ind-proper}}_{\text{indnilsch;all}}} : \text{Corr}(\text{PreStk}_{\text{laft}})^{\text{indnilsch \& ind-proper}}_{\text{indnilsch;all}} \to \text{DGCat}^{\text{2-Cat}}_{\text{cont}},$$

equipped with an isomorphism

$$\text{Crys}_{\text{Corr}(\text{PreStk}_{\text{laft}})^{\text{indnilsch \& ind-proper}}_{\text{indnilsch;all}}} \big|_{(\text{PreStk}_{\text{laft}})^{\text{op}}} \simeq \text{Crys}^{!}_{\text{PreStk}_{\text{laft}}}.$$

Furthermore, the restriction

$$\text{Crys}_{\text{Corr}(\text{PreStk}_{\text{laft}})_{\text{indnilsch;all}}} := \text{Crys}_{\text{Corr}(\text{PreStk}_{\text{laft}})^{\text{indnilsch \& ind-proper}}_{\text{indnilsch;all}}} \big|_{\text{Corr}(\text{PreStk}_{\text{laft}})_{\text{indnilsch;all}}}$$

uniquely extends to a functor

$$\text{Crys}_{\text{Corr}(\text{PreStk}_{\text{laft}})^{\text{nil-open}}_{\text{indnilsch;all}}} : \text{Corr}(\text{PreStk}_{\text{laft}})^{\text{nil-open}}_{\text{indnilsch;all}} \to \left(\text{DGCat}^{\text{2-Cat}}_{\text{cont}}\right)^{\text{2-op}}.$$

2.1.3. As in the case of Chapter 3, Theorem 5.5.3, the content of Theorem 2.1.2 is the existence of the functor

$$f_{\text{dR},*} : \text{Crys}(\mathcal{Z}_1) \to \text{Crys}(\mathcal{Z}_2)$$

for ind-nil-schematic morphisms of prestacks $f : \mathcal{Z}_1 \to \mathcal{Z}_2$, and of the base change isomorphisms compatible with proper and nil-open adjunctions. Namely, for a Cartesian diagram of prestacks

$$
\begin{array}{ccc}
\mathcal{Z}'_1 & \xrightarrow{g_1} & \mathcal{Z}_1 \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
\mathcal{Z}'_2 & \xrightarrow{g_2} & \mathcal{Z}_2,
\end{array}
$$

with f ind-nil-schematic, we have *a canonical isomorphism*

(2.1) $$f'_{\text{dR},*} \circ g^{!}_{1,\text{dR}} \xrightarrow{\sim} g^{!}_{2,\text{dR}} \circ f_{\text{dR},*}.$$

Moreover, if f is ind-proper, then $f_{\text{dR},*}$ is the left adjoint of $f^{!}_{\text{dR}}$. Furthermore, the isomorphism (2.1) is the one arising by adjunction if either f_X or g_2 is ind-proper.

If f is a nil-open embedding (i.e., the map of the corresponding reduced prestacks is an open embedding), then $f_{\text{dR},*}$ is the right adjoint of $f^{!}_{\text{dR}}$. Furthermore, the isomorphism (2.1) is the one arising by adjunction if either f_X or g_2 is a nil-open embedding.

2.1.4.   Now, let us restrict the functor $\mathrm{Crys}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indnilsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}}}$ to

$$\mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}} \subset \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indnilsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}}.$$

We denote the resulting functor by $\mathrm{Crys}_{\mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}}$. From Chapter 3, Theorems 5.2.2 and 5.4.3 we obtain:

COROLLARY 2.1.5.   *The restriction of* $\mathrm{Crys}_{\mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}}$ *to*

$$\mathrm{indnilSch}_{\mathrm{laft}} \subset \mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}}$$

identifies canonically with the functor $\mathrm{Crys}_{\mathrm{indnilSch}_{\mathrm{laft}}}$ *of* (1.1).

2.1.6.   Further restricting along

$$\mathrm{Corr}(\mathrm{nilSch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}} \to \mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all;all}},$$

we obtain a functor

$$\mathrm{Crys}_{(\mathrm{nilSch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}} \to (\mathrm{DGCat}_{\mathrm{cont}})^{2\text{-}\mathrm{Cat}}$$

denoted by $\mathrm{Crys}_{(\mathrm{nilSch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}$.

In particular, we obtain a functor

$$\mathrm{Crys}_{\mathrm{nilSch}_{\mathrm{aft}}} := \mathrm{Crys}_{(\mathrm{Sch}_{\mathrm{aft}})^{\mathrm{proper}}_{\mathrm{all;all}}}\big|_{\mathrm{nilSch}_{\mathrm{aft}}},$$

which is also isomorphic to

$$\mathrm{Crys}_{\mathrm{indnilSch}_{\mathrm{laft}}}\big|_{\mathrm{nilSch}_{\mathrm{aft}}}.$$

2.2. The multiplicative structure of the functor of crystals. In this subsection we show how the formalism of crystals as a functor out of the category of correspondences gives rise to Verdier duality.

2.2.1.   *Duality.* From Chapter 3, Theorem 6.2.2, we obtain:

THEOREM 2.2.2.   *We have a commutative diagram of functors*

$$
\begin{array}{ccc}
(\mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})_{\mathrm{all;all}})^{\mathrm{op}} & \xrightarrow{\left(\mathrm{Crys}_{\mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})_{\mathrm{all;all}}}\right)^{\mathrm{op}}} & \left(\mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}}\right)^{\mathrm{op}} \\
{\scriptstyle\varpi}\downarrow & & \downarrow{\scriptstyle\mathrm{dualization}} \\
\mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})_{\mathrm{all;all}} & \xrightarrow[\mathrm{Crys}_{\mathrm{Corr}(\mathrm{indnilSch}_{\mathrm{laft}})_{\mathrm{all;all}}}]{} & \mathrm{DGCat}^{\mathrm{dualizable}}_{\mathrm{cont}}.
\end{array}
$$

2.2.3.   Concretely, this theorem says that for $\mathcal{Z} \in \mathrm{indnilSch}_{\mathrm{laft}}$ there is a canonical involutive equivalence

(2.2) $$\mathbf{D}^{\mathrm{Verdier}}_{\mathcal{Z}} : \mathrm{Crys}(\mathcal{Z})^{\vee} \simeq \mathrm{Crys}(\mathcal{Z}),$$

and for a map $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ in $\mathrm{indnilSch}_{\mathrm{laft}}$ there is a canonical identification

$$f^{!}_{\mathrm{dR}} \simeq (f_{\mathrm{dR},*})^{\vee}.$$

2.2.4.   As in Chapter 3, Sect. 6.2.6, we can write the unit and counit maps

$$\mu_{\mathcal{Z}_{\mathrm{dR}}} : \mathrm{Vect} \to \mathrm{Crys}(\mathcal{Z}) \otimes \mathrm{Crys}(\mathcal{Z}) \text{ and } \epsilon_{\mathcal{Z}_{\mathrm{dR}}} : \mathrm{Crys}(\mathcal{Z}) \otimes \mathrm{Crys}(\mathcal{Z}) \to \mathrm{Vect}$$

explicitly.

Namely, $\epsilon_{\mathcal{Z}_{\mathrm{dR}}}$ is the composition

$$\mathrm{Crys}(\mathcal{Z}) \otimes \mathrm{Crys}(\mathcal{Z}) \simeq \mathrm{Crys}(\mathcal{Z} \times \mathcal{Z}) \overset{\Delta^!_{\mathcal{Z},\mathrm{dR}}}{\longrightarrow} \mathrm{Crys}(\mathcal{Z}) \overset{\Gamma_{\mathrm{dR}}(\mathcal{Z},-)}{\longrightarrow} \mathrm{Vect},$$

where

$$\Gamma_{\mathrm{dR}}(\mathcal{Z},-) \coloneqq (p_{\mathcal{Z}})_{\mathrm{dR},*},$$

and $\mu_{\mathcal{Z}_{\mathrm{dR}}}$ is the composition

$$\mathrm{Vect} \overset{\omega_{\mathcal{Z}_{\mathrm{dR}}}}{\longrightarrow} \mathrm{Crys}(\mathcal{Z}) \overset{(\Delta_{\mathcal{Z}})_{\mathrm{dR},*}}{\longrightarrow} \mathrm{Crys}(\mathcal{Z} \times \mathcal{Z}) \simeq \mathrm{Crys}(\mathcal{Z}) \otimes \mathrm{Crys}(\mathcal{Z}).$$

2.2.5. *Verdier duality.* For $\mathcal{Z} \in \mathrm{indnilSch}_{\mathrm{laft}}$, let $\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Z}}$ denote the canonical equivalence

$$(\mathrm{Crys}(\mathcal{Z})^c)^{\mathrm{op}} \to \mathrm{Crys}(\mathcal{Z})^c,$$

corresponding to the isomorphism (2.2).

In other words,

$$\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Z}} = \mathbb{D}^{\mathrm{Serre}}_{\mathcal{Z}_{\mathrm{dR}}}.$$

2.2.6.   As a particular case of Chapter 3, Corollary 6.2.9, we obtain:

COROLLARY 2.2.7. *Let $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ be an ind-proper map in* $\mathrm{indnilSch}_{\mathrm{laft}}$. *Then we have a commutative diagram:*

$$
\begin{array}{ccc}
(\mathrm{Crys}(\mathcal{Z}_1)^c)^{\mathrm{op}} & \overset{\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Z}_1}}{\longrightarrow} & \mathrm{Crys}(\mathcal{Z}_1)^c \\
{\scriptstyle (f_{\mathrm{dR},*})^{\mathrm{op}}} \downarrow & & \downarrow {\scriptstyle f_{\mathrm{dR},*}} \\
(\mathrm{Crys}(\mathcal{Z}_2)^c)^{\mathrm{op}} & \overset{\mathbb{D}^{\mathrm{Verdier}}_{\mathcal{Z}_2}}{\longrightarrow} & \mathrm{Crys}(\mathcal{Z}_2)^c.
\end{array}
$$

In view of Corollary 1.6.5, the above corollary gives an expression of the Verdier duality functor on $\mathcal{Z} \in \mathrm{indnilSch}_{\mathrm{laft}}$ in terms of that on schemes.

2.2.8. *Convolution for crystals.*

Returning to the entire $(\infty, 2)$-category $\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indnilsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}}$ and the corresponding functor

$$\mathrm{IndCoh}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indnilsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}}},$$

we obtain, from Chapter 3, Sect. 6.3, that the functor

$$\mathrm{Crys}_{\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indnilsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})^{\mathrm{indnilsch}\,\&\,\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch;all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}$$

carries a canonical right-lax symmetric monoidal structure.

As in Chapter 3, Sect. 6.3.2, we have:

(i) Given a Segal object $\mathcal{R}^\bullet$ of $\mathrm{PreStk}_{\mathrm{laft}}$, with the target and composition maps ind-nil-schematic, the category $\mathrm{Crys}(\mathcal{R})$ acquires a monoidal structure given by convolution, and as such it acts on $\mathrm{Crys}(\mathcal{X})$ (here, as in Volume I, Chapter 5, Sect. 5.1.1, $\mathcal{X} = \mathcal{R}^0$ and $\mathcal{R} = \mathcal{R}^1$).

(ii) If the composition map is ind-proper, then $\omega_{\mathcal{R}} \in \mathrm{Crys}(\mathcal{R})$ acquires the structure of an algebra in $\mathrm{Crys}(\mathcal{R})$. The action of this algebra on $\mathrm{IndCoh}(\mathcal{X})$, viewed as a plain endo-functor, is given by

$$(p_t)_{\mathrm{dR},*} \circ (p_s)^!_{\mathrm{dR}}.$$

3. Inducing crystals

In this section we study the interaction between the functors IndCoh and Crys.

3.1. The functor of induction. In this subsection we show that the forgetful functor

$$\mathrm{Crys}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{Z})$$

admits a left adjoint, provided that $\mathcal{Z}$ is a prestack that admits deformation theory.

3.1.1.   For an object $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft}}$ consider the canonical map

$$p_{\mathrm{dR},\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}_{\mathrm{dR}}.$$

We claim:

PROPOSITION 3.1.2. *Suppose that $\mathcal{Z}$ admits deformation theory. Then the map $p_{\mathrm{dR},\mathcal{Z}}$ is an inf-schematic nil-isomorphism.*

PROOF.   We need to show that for $S \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/\mathcal{Z}_{\mathrm{dR}}}$, the Cartesian product

$$(3.1) \qquad\qquad\qquad\qquad S \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{Z}$$

is an inf-scheme.

Clearly, the above Cartesian product belongs to $\mathrm{PreStk}_{\mathrm{laft}}$, and its underlying reduced prestack identifies with $^{\mathrm{red}}S$. Hence, it remains to show that (3.1) admits deformation theory. This holds because the category $\mathrm{PreStk}_{\mathrm{def-laft}}$ is closed under finite limits.

$$\square$$

3.1.3.   From Proposition 3.1.2 and Chapter 3, Proposition 3.1.2(a) we obtain:

COROLLARY 3.1.4. *Let $\mathcal{Z}$ be an object of $\mathrm{PreStk}_{\mathrm{def-laft}}$. Then the functor*

$$\mathbf{oblv}_{\mathrm{dR},\mathcal{Z}} : \mathrm{Crys}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{Z})$$

admits a left adjoint.

We denote the left adjoint to $\mathbf{oblv}_{\mathrm{dR},\mathcal{Z}}$, whose existence is given by the above corollary, by $\mathbf{ind}_{\mathrm{dR},\mathcal{Z}}$.

3.1.5.   Thus, for $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{def-laft}}$, we obtain an adjoint pair

$$(3.2) \qquad\qquad \mathbf{ind}_{\mathrm{dR},\mathcal{Z}} : \mathrm{IndCoh}(\mathcal{Z}) \rightleftarrows \mathrm{Crys}(\mathcal{Z}) : \mathbf{oblv}_{\mathrm{dR},\mathcal{Z}}.$$

We claim:

LEMMA 3.1.6. *The pair (3.2) is monadic.*

PROOF.   Since $\mathbf{oblv}_{\mathrm{dR},\mathcal{Z}}$ is continuous, we only need to check that it is conservative. However, this follows from Chapter 3, Proposition 3.1.2(b). $\square$

3.1.7.   The next corollary of Proposition 3.1.2 expresses the functoriality of the operation of induction:

COROLLARY 3.1.8.   *There is a canonically defined natural transformation*

$$\mathbf{ind}_{dR} : \mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch}}} |_{(\mathrm{PreStk}_{\mathrm{def\text{-}laft}})_{\mathrm{indinfsch}}} \Rightarrow \mathrm{Crys}_{(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch}}} |_{(\mathrm{PreStk}_{\mathrm{def\text{-}laft}})_{\mathrm{indinfsch}}},$$

as functors

$$(\mathrm{PreStk}_{\mathrm{def\text{-}laft}})_{\mathrm{indinf\text{-}sch}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

In particular, the above corollary says that for an ind-inf-schematic morphism $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ of objects of $\mathrm{PreStk}_{\mathrm{def\text{-}laft}}$, the following diagram of functors commutes:

$$
\begin{array}{ccc}
\mathrm{IndCoh}(\mathcal{Z}_1) & \xrightarrow{\ \mathbf{ind}_{dR,\mathcal{Z}_1}\ } & \mathrm{Crys}(\mathcal{Z}_1) \\
{\scriptstyle f_*^{\mathrm{IndCoh}}}\downarrow & & \downarrow{\scriptstyle f_{dR,*}} \\
\mathrm{IndCoh}(\mathcal{Z}_2) & \xrightarrow{\ \mathbf{ind}_{dR,\mathcal{Z}_2}\ } & \mathrm{Crys}(\mathcal{Z}_2).
\end{array}
$$

3.2. Induction on ind-inf-schemes. In this subsection, let $\mathcal{Z}$ be an object of $\mathrm{indinfSch}_{\mathrm{laft}}$. We study the interaction of the induction functor with that of Serre and Verdier dualities.

3.2.1.   We have:

LEMMA 3.2.2.   *The functor* $\mathbf{ind}_{dR,\mathcal{Z}}$ *sends* $\mathrm{IndCoh}(\mathcal{Z})^c$ *to* $\mathrm{Crys}(\mathcal{Z})^c$.

PROOF.   Follows from the fact that the functor $\mathbf{oblv}_{dR,\mathcal{Z}}$ is continuous and conservative.

$\square$

3.2.3.   *Induction and duality.* Let us apply isomorphism Chapter 3, Equation (6.2) to the map

$$p_{dR,\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}_{dR}.$$

We obtain:

COROLLARY 3.2.4.   *Under the isomorphisms*

$$\mathbf{D}_{\mathcal{Z}}^{\mathrm{Serre}} : \mathrm{IndCoh}(\mathcal{Z})^{\vee} \simeq \mathrm{IndCoh}(\mathcal{Z}) \ \textit{and} \ \mathbf{D}_{\mathcal{Z}}^{\mathrm{Verdier}} : \mathrm{Crys}(\mathcal{Z})^{\vee} \simeq \mathrm{Crys}(\mathcal{Z}),$$

we have a canonical identification

$$(\mathbf{oblv}_{dR,\mathcal{Z}})^{\vee} \simeq \mathbf{ind}_{dR,\mathcal{Z}}.$$

In addition, by Chapter 3, Corollary 6.2.9

COROLLARY 3.2.5.   *The following diagram of functors commutes:*

$$
\begin{array}{ccc}
(\mathrm{IndCoh}(\mathcal{Z})^c)^{\mathrm{op}} & \xrightarrow{\ \mathbb{D}_{\mathcal{Z}}^{\mathrm{Serre}}\ } & \mathrm{IndCoh}(\mathcal{Z})^c \\
{\scriptstyle (\mathbf{ind}_{dR,\mathcal{Z}})^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle \mathbf{ind}_{dR,\mathcal{Z}}} \\
(\mathrm{Crys}(\mathcal{Z})^c)^{\mathrm{op}} & \xrightarrow{\ \mathbb{D}_{\mathcal{Z}}^{\mathrm{Verdier}}\ } & \mathrm{Crys}(\mathcal{Z})^c.
\end{array}
$$

3.2.6. *Induction and t-structure.* Recall that by the definition of the t-structure on $\mathrm{Crys}(\mathcal{Z})$, the functor $\mathbf{oblv}_{\mathrm{dR},\mathcal{Z}}$ is left t-exact. We claim:

COROLLARY 3.2.7. *Assume that $\mathcal{Z}$ is an ind-scheme. Then the functor $\mathbf{ind}_{\mathrm{dR},\mathcal{Z}}$ is t-exact.*

PROOF. The fact that $\mathbf{ind}_{\mathrm{dR},\mathcal{Z}}$ is right t-exact follows by adjunction. To show that it is left t-exact we use Chapter 3, Lemma 3.4.6: we have to show that the $p_{\mathrm{dR},\mathcal{Z}}$ is ind-schematic.

Indeed, for $S \in \mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}}$ and $S \to \mathcal{Z}_{\mathrm{dR}}$, the Cartesian product

$$S \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{Z}$$

identifies with the formal completion of $S \times \mathcal{Z}$ along the graph of the map $^{\mathrm{red}}S \to \mathcal{Z}$. $\qquad\square$

3.3. Relative crystals. In this subsection we describe how the discussion of crystals generalizes to the relative situation.

3.3.1. Let $\mathcal{Y}$ be a fixed object of $\mathrm{PreStk}_{\mathrm{laft}}$. Consider the ∞-category

$$(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}$$

and the corresponding $(\infty, 2)$-category

$$\mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indinfsch};\mathrm{all}}^{\mathrm{indinfsch\,\&\,ind\text{-}proper}}.$$

Restricting the functor $\mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indinfsch};\mathrm{all}}^{\mathrm{indinfsch\,\&\,ind\text{-}proper}}}$ along the forgetful functor

$$\mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indinfsch};\mathrm{all}}^{\mathrm{indinfsch\,\&\,ind\text{-}proper}} \to \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{inf\text{-}sch};\mathrm{all}}^{\mathrm{inf\text{-}proper}},$$

we obtain the functor

$$\mathrm{IndCoh}_{\mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indinfsch};\mathrm{all}}^{\mathrm{indinfsch\,\&\,ind\text{-}proper}}} : \mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indinfsch};\mathrm{all}}^{\mathrm{indinfsch\,\&\,ind\text{-}proper}} \to \mathrm{DGCat}_{\mathrm{cont}}^{2\text{-}\mathrm{Cat}},$$

with properties specified by Chapter 3, Theorem 5.4.3.

In particular, let

$$\mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}}^{!} : ((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

be the resulting functor.

3.3.2. The category $(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}$ has an endo-functor, denoted by $^{/\mathcal{Y}}\mathrm{dR}$:

$$\mathcal{Z} \mapsto \mathcal{Z}_{/\mathcal{Y}\mathrm{dR}} := \mathcal{Z}_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}.$$

Corollary 1.3.6 implies that the functor $^{/\mathcal{Y}}\mathrm{dR}$ gives rise to a functor

$$((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indnilSch}} \to ((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indinfsch}}.$$

Hence, $^{/\mathcal{Y}}\mathrm{dR}$ induces a functor

$$(3.3) \quad \mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indnilsch};\mathrm{all}}^{\mathrm{ind\text{-}proper}} \to \mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indinfsch};\mathrm{all}}^{\mathrm{indinfsch\,\&\,ind\text{-}proper}}.$$

Thus, precomposing $\mathrm{IndCoh}_{\mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indinfsch};\mathrm{all}}^{\mathrm{indinfsch\,\&\,ind\text{-}proper}}}$ with $^{/\mathcal{Y}}\mathrm{dR}$, we obtain the functor

$$^{/\mathcal{Y}}\mathrm{Crys}_{\mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indnilsch};\mathrm{all}}^{\mathrm{ind\text{-}proper}}} : \mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indnilsch};\mathrm{all}}^{\mathrm{ind\text{-}proper}} \to \mathrm{DGCat}_{\mathrm{cont}}^{2\text{-}\mathrm{Cat}}.$$

3.3.3.   Let

$$^{/\mathcal{Y}}\mathrm{Crys}^!_{(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}} : ((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

and

$$^{/\mathcal{Y}}\mathrm{Crys}_{((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indnilsch}}} : ((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})_{\mathrm{indnilsch}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

denote the corresponding functors obtained by restriction.

For a map $\mathcal{Z}_1 \to \mathcal{Z}_2$ in $(\mathrm{PreStk}_{\mathrm{laft}})_{\mathrm{indnilsch}}$, we shall denote by $f^!_{\mathcal{Y}_{\mathrm{dR}}}$ and $f_{\mathcal{Y}_{\mathrm{dR}},*}$ the corresponding functors

$$^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z}_1) \leftrightarrows {}^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z}_2).$$

These functors are adjoint if f is ind-proper/nil-open.

3.3.4.   Let $^{/\mathcal{Y}}\mathrm{indnilSch}_{\mathrm{laft}}$ denote the full subcategory of $(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}$ given by the preimage of $(\mathrm{indinfSch}_{\mathrm{laft}})_{/\mathcal{Y}}$ under the functor $^{/\mathcal{Y}}\mathrm{dR}$.

Restricting the functor $^{/\mathcal{Y}}\mathrm{Crys}_{\mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch};\mathrm{all}}}$ to

$$\mathrm{Corr}(^{/\mathcal{Y}}\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all};\mathrm{all}} \subset \mathrm{Corr}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}})^{\mathrm{ind\text{-}proper}}_{\mathrm{indnilsch};\mathrm{all}},$$

we obtain the functor

$$^{/\mathcal{Y}}\mathrm{Crys}_{\mathrm{Corr}(^{/\mathcal{Y}}\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all};\mathrm{all}}} : \mathrm{Corr}(^{/\mathcal{Y}}\mathrm{indnilSch}_{\mathrm{laft}})^{\mathrm{ind\text{-}proper}}_{\mathrm{all};\mathrm{all}} \to \mathrm{DGCat}^{2\text{-}\mathrm{Cat}}_{\mathrm{cont}}.$$

Furthermore, for an object $\mathcal{Z}$ of $^{/\mathcal{Y}}\mathrm{indnilSch}_{\mathrm{laft}}$, the category $^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z})$ satisfies the following properties:

(1) The category $^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z})$ is compactly generated, and is self-dual in the sense of Theorem 2.2.2.

(2) The category $^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z})$ carries a t-structure in which an object $\mathcal{F}$ is coconnective if and only if its image under the forgetful functor

$$\mathbf{oblv}_{/\mathcal{Y}_{\mathrm{dR}},\mathcal{Z}} : {}^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{Z})$$

is coconnective, for $\mathbf{oblv}_{/\mathcal{Y}_{\mathrm{dR}},\mathcal{Z}} := (p_{/\mathcal{Y}_{\mathrm{dR}},\mathcal{Z}})^!$, where $p_{/\mathcal{Y}_{\mathrm{dR}},\mathcal{Z}}$ denotes the canonical morphism

$$\mathcal{Z} \to \mathcal{Z}_{/\mathcal{Y}_{\mathrm{dR}}}.$$

(3) If $\mathcal{Z}$ admits deformation theory *over* $\mathcal{Y}$ (see Chapter 1, Sect. 7.1.6 for what this means), then the morphism $p_{/\mathcal{Y}_{\mathrm{dR}},\mathcal{Z}}$ is an inf-schematic nil-isomorphism, and hence the functor $\mathbf{oblv}_{/\mathcal{Y}_{\mathrm{dR}},\mathcal{Z}}$ admits a left adjoint, denoted $\mathbf{ind}_{/\mathcal{Y}_{\mathrm{dR}},\mathcal{Z}}$.

REMARK 3.3.5. The essential difference between Crys and $^{/\mathcal{Y}}\mathrm{Crys}$ is that for $\mathcal{Z} \in (\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}}$, the category $^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z})$ depends *not just* on the underlying reduced prestack. E.g., for $\mathcal{Z} = \mathcal{Y}$,

$$^{/\mathcal{Y}}\mathrm{Crys}(\mathcal{Z}) = \mathrm{IndCoh}(\mathcal{Y}).$$

3.3.6.   Assume for a moment that $\mathcal{Y} = Y$ is a smooth classical scheme, and $\mathcal{Z} = Z$ is also a classical scheme smooth over Y. Then, as in [**GaRo2**, Sect. 4.7], one shows that the category

$$^{/Y}\mathrm{Crys}(Z)$$

identifies with the DG category associated with the abelian category of quasi-coherent sheaves of modules on Y with respect to the algebra of 'vertical' differential operators, i.e., the subalgebra of $\mathrm{Diff}(Z)$ that consists of differential operators that commute with $\mathcal{O}_Y$.

4. Comparison with the classical theory of D-modules

In this section we will identify the theory of crystals as developed in the previous sections with the theory of D-modules.

This section can be regarded as a companion to [**GaRo2**, Sects. 6 and 7], and we shall assume the reader's familiarity with the contents of *loc.cit.*

4.1. Left D-modules and left crystals. In this subsection we will recollect (and rephrase) the contents of [**GaRo2**, Sect. 5]. Specifically, we will discuss the equivalence between the category of *left D-modules* on a smooth scheme X and the category of *left crystals* on X.

4.1.1.   Let X be a classical scheme of finite type. Consider the category $\mathrm{QCoh}(X \times X)^\heartsuit$ and its full subcategory $(\mathrm{QCoh}(X \times X)_{\Delta_X})^\heartsuit$, consisting of objects that are set-theoretically supported on the diagonal. Let

$$(\mathrm{QCoh}(X \times X)_{\Delta_X})^\heartsuit_{\mathrm{rel.flat}} \subset \mathrm{QCoh}(X \times X)_{\Delta_X})^\heartsuit$$

be the full subcategory, consisting of objects that are X-flat with respect to both projections

$$p_s, p_t : X \times X \to X.$$

The category $(\mathrm{QCoh}(X \times X)_{\Delta_X})^\heartsuit_{\mathrm{rel.flat}}$ has a naturally defined monoidal structure, given by convolution.

Moreover, we have a canonically defined fully faithful *monoidal* (!) functor

$$(4.1) \qquad (\mathrm{QCoh}(X \times X)_{\Delta_X})^\heartsuit_{\mathrm{rel.flat}} \to \mathrm{QCoh}(X \times X),$$

where $\mathrm{QCoh}(X \times X)$ is a monoidal category as in Volume I, Chapter 5, Sects. 5.2.3 and 5.3.3.

4.1.2.   Now suppose that X is smooth. In this case, we have a canonically defined object

$$\mathrm{Diff}_X \in \mathrm{AssocAlg}\big((\mathrm{QCoh}(X \times X)_{\Delta_X})^\heartsuit_{\mathrm{rel.flat}}\big),$$

namely, the Grothendieck algebra of differential operators.

Composing (4.1) with the monoidal equivalence

$$\mathrm{QCoh}(X \times X) \to \mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(X), \mathrm{QCoh}(X)),$$

we obtain that Diff_X gives rise to a monad on $\mathrm{QCoh}(X)$.

We consider the category $\mathrm{Diff}_X\text{-mod}(\mathrm{QCoh}(X))$. It is equipped with a t-structure, characterized by the property that the tautological forgetful functor $\mathrm{Diff}_X\text{-mod}(\mathrm{QCoh}(X)) \to \mathrm{QCoh}(X)$ is t-exact.

The category $\mathrm{Dmod}^l(X)$ of *left* D-modules on X is defined as the canonical DG model of the derived category of the abelian category $(\mathrm{Diff}_X\text{-mod}(\mathrm{QCoh}(X)))^\heartsuit$.

As in [**GaRo2**, Proposition 4.7.3] one shows that the canonical functor

$$(4.2) \qquad\qquad \mathrm{Dmod}^l(X) \to \mathrm{Diff}_X\text{-mod}(\mathrm{QCoh}(X))$$

is an equivalence.

4.1.3.   Let X be any scheme almost of finite type. Recall the category

$$\mathrm{Crys}^l(X) := \mathrm{QCoh}(X_{\mathrm{dR}}),$$

see [**GaRo2**, Sect. 2.1]. It is equipped with a forgetful functor

$$\mathbf{oblv}^l_{\mathrm{dR},X} : \mathrm{Crys}^l(X) \to \mathrm{QCoh}(X).$$

Assume now that X is eventually coconnective. According to [**GaRo2**, Proposition 3.4.11], in this case the functor $\mathbf{oblv}^l_{\mathrm{dR},X}$ admits a left adjoint, denoted $\mathbf{ind}^l_{\mathrm{dR},X}$, and the resulting adjoint pair of functors

$$\mathbf{ind}^l_{\mathrm{dR},X} : \mathrm{QCoh}(X) \rightleftarrows \mathrm{Crys}^l(X) : \mathbf{oblv}^l_{\mathrm{dR},X},$$

is monadic.

The corresponding monad is given by an object

$$\mathcal{D}^l_X \in \mathrm{AssocAlg}(\mathrm{QCoh}(X \times X)_{\Delta_X}).$$

I.e., we have an equivalence

$$\mathrm{Crys}^l(X) \simeq \mathcal{D}^l_X\text{-mod}(\mathrm{QCoh}(X)).$$

4.1.4.   Again, assume that X is smooth. In this case one easily shows (see, e.g., [**GaRo2**, Proposition 5.3.6]) that

$$\mathcal{D}^l_X \in (\mathrm{QCoh}(X \times X)_{\Delta_X})^\heartsuit_{\mathrm{rel.flat}}.$$

Moreover, it is a classical fact (reproved for completeness in [**GaRo2**, Lemma 5.4.3]) that there is a canonical isomorphism in $\mathrm{AssocAlg}\left((\mathrm{QCoh}(X \times X)_{\Delta_X})^\heartsuit_{\mathrm{rel.flat}}\right)$:

$$(4.3) \qquad\qquad\qquad \mathcal{D}^l_X \simeq \mathrm{Diff}_X.$$

In particular, we obtain a canonical equivalence of categories

$$\mathrm{Dmod}^l(X) \simeq \mathrm{Diff}_X\text{-mod}(\mathrm{QCoh}(X)) \simeq \mathcal{D}^l_X\text{-mod}(\mathrm{QCoh}(X)) \simeq \mathrm{Crys}^l(X),$$

compatible with the forgetful functors to $\mathrm{QCoh}(X)$.

We denote the resulting equivalence $\mathrm{Dmod}^l(X) \to \mathrm{Crys}^l(X)$ by F^l_X.

4.1.5. Let $f : X \to Y$ be a morphism between smooth classical schemes. In the classical theory of D-modules, one defines a functor

$$f^{\blacktriangle,l} : \mathrm{Dmod}^l(Y) \to \mathrm{Dmod}^l(X)$$

that makes the diagram

(4.4)
$$
\begin{array}{ccc}
\mathrm{Dmod}^l(Y) & \xrightarrow{\ f^{\blacktriangle,l}\ } & \mathrm{Dmod}^l(X) \\
\downarrow & & \downarrow \mathbf{oblv}^l_{\mathrm{dR},X} \\
\mathrm{QCoh}(Y) & \xrightarrow{\ f^*\ } & \mathrm{QCoh}(X)
\end{array}
$$

commute.

Also recall (see [**GaRo2**, Sect. 2.1.2]) that for a map $f : X \to Y$ between arbitrary schemes almost of finite type, we have a functor

$$f^{\dagger,l} : \mathrm{Crys}^l(Y) \to \mathrm{Crys}^l(X),$$

that makes the diagram

(4.5)
$$
\begin{array}{ccc}
\mathrm{Crys}^l(Y) & \xrightarrow{\ f^{\dagger,l}\ } & \mathrm{Crys}^l(X) \\
\downarrow & & \downarrow \\
\mathrm{QCoh}(Y) & \xrightarrow{\ f^*\ } & \mathrm{QCoh}(X)
\end{array}
$$

commute.

The following can be established by a direct calculation:

LEMMA 4.1.6. *The following diagram of functors naturally commutes*

$$
\begin{array}{ccc}
\mathrm{Dmod}^l(Y) & \xrightarrow{\ f^{\blacktriangle,l}\ } & \mathrm{Dmod}^l(X) \\
F^l_Y \downarrow & & \downarrow F^l_X \\
\mathrm{Crys}^l(X) & \xrightarrow{\ f^{\dagger,l}\ } & \mathrm{Crys}^l(X),
\end{array}
$$

in a way compatible with the forgeftul functors to $\mathrm{QCoh}(-)$.

4.2. Right D-modules and right crystals. In this subsection we will discuss the equivalence between the category of right D-modules on a smooth scheme X, and the category $\mathrm{Crys}(X)$, considered in the earlier sections of this Chapter.

4.2.1. Note that for any scheme X, the monoidal category $\mathrm{QCoh}(X \times X)$ carries a canonical anti-involution, denoted σ, corresponding to the transposition acting on $X \times X$.

In terms of the identification

$$\mathrm{QCoh}(X \times X) \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(X), \mathrm{QCoh}(X)),$$

we have

$$\sigma(F) \simeq F^\vee, \quad F \in \mathrm{Funct}_{\mathrm{cont}}(\mathrm{QCoh}(X), \mathrm{QCoh}(X)),$$

where we use the canonical identification

$$\mathbf{D}^{\mathrm{naive}}_X : \mathrm{QCoh}(X)^\vee \simeq \mathrm{QCoh}(X),$$

of Volume I, Chapter 6, Equation (4.2).

In particular for an algebra object A in $\mathrm{QCoh}(X \times X)$, we can regard $\sigma(A^{\mathrm{op}})$ again as an algebra object in $\mathrm{QCoh}(X \times X)$, and we have

$$(4.6) \qquad\qquad (M_A)^\vee \simeq M_{\sigma(A^{\mathrm{op}})},$$

where M_B denotes the monad on $\mathrm{QCoh}(X)$, corresponding to an algebra object $B \in \mathrm{QCoh}(X)$.

4.2.2.   Let X be smooth. Consider the object

$$\sigma(\mathrm{Diff}_X^{\mathrm{op}}) \in \mathrm{AssocAlg}\left((\mathrm{QCoh}(X \times X)_{\Delta_X})_{\mathrm{rel.flat}}^\heartsuit\right),$$

and the corresponding category

$$\sigma(\mathrm{Diff}_X^{\mathrm{op}})\text{-mod}(\mathrm{QCoh}(X)).$$

The category $\mathrm{Dmod}^r(X)$ of *right* D-modules on X is defined as the canonical DG model of the derived category of the abelian category $\left(\sigma(\mathrm{Diff}_X^{\mathrm{op}})\text{-mod}(\mathrm{QCoh}(X))\right)^\heartsuit$.

As in [**GaRo2**, Proposition 4.7.3] one shows that the canonical functor

$$(4.7) \qquad\qquad \mathrm{Dmod}^r(X) \to \sigma(\mathrm{Diff}_X^{\mathrm{op}})\text{-mod}(\mathrm{QCoh}(X))$$

is an equivalence.

4.2.3.   Let X be a scheme almost of finite type. For the duration of this section, we will denote by

$$\mathrm{Crys}^r(X) \coloneqq \mathrm{Crys}(X) \coloneqq \mathrm{IndCoh}(X_{\mathrm{dR}}),$$

where the latter is defined as in Sect. 1.2.2, and by

$$\mathbf{ind}_{\mathrm{dR},X}^r : \mathrm{IndCoh}(X) \rightleftarrows \mathrm{Crys}^r(X) : \mathbf{oblv}_{\mathrm{dR},X}^r$$

the corresponding pair of adjoint functors from (3.2). I.e., we are adding the superscript 'r' (for 'right') to the notation from Sect. 1.2.2 to emphasize the comparison with right D-modules.

Let $\mathcal{D}_X^r$ be the object of $\mathrm{AssocAlg}(\mathrm{IndCoh}(X \times X)_{\Delta_X})$, corresponding to the monad

$$\mathbf{oblv}_{\mathrm{dR},X}^r \circ \mathbf{ind}_{\mathrm{dR},X}^r$$

via the equivalence of monoidal categories

$$\mathrm{IndCoh}(X \times X) \to \mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(X), \mathrm{IndCoh}(X)).$$

By Lemma 3.1.6, we have:

$$\mathcal{D}_X^r\text{-mod}(\mathrm{IndCoh}(X)) \simeq \mathrm{Crys}^r(X).$$

4.2.4.   Suppose that X is smooth. Recall that in this case the adjoint pairs

$$\Xi_X : \mathrm{QCoh}(X) \rightleftarrows \mathrm{IndCoh}(X) : \Psi_X$$

and

$$\Xi_X^\vee : \mathrm{QCoh}(X) \rightleftarrows \mathrm{IndCoh}(X) : \Psi_X^\vee = \Upsilon_X$$

are both equivalences.

In this case one shows as in [**GaRo2**, Sect. 5.5] that there is a canonical equivalence of categories

(4.8) $$F_X^r : \mathrm{Dmod}^r(X) \simeq \mathrm{Crys}^r(X),$$

that makes the diagram

$$
\begin{array}{ccc}
\mathrm{Dmod}^r(X) & \xrightarrow{\ F_X^r\ } & \mathrm{Crys}^r(X) \\
\Big\downarrow & & \Big\downarrow {\scriptstyle \mathbf{oblv}^r_{\mathrm{dR},X}} \\
\mathrm{QCoh}(X) & \xrightarrow{\ \Xi_X\ } & \mathrm{IndCoh}(X)
\end{array}
$$

commute.

REMARK 4.2.5. One can obtain the equivalence of (4.8) formally from the corresponding computation for left D-modules.

Namely, taking into account the equivalences

$$(\mathbf{oblv}^l_{\mathrm{dR},X} \circ \mathbf{ind}^l_{\mathrm{dR},X})^\vee\text{-mod}(\mathrm{QCoh}(X)) \simeq \sigma(\mathrm{Diff}_X^{\mathrm{op}})\text{-mod}(\mathrm{QCoh}(X)) \simeq \mathrm{Dmod}^r(X)$$

(where the first equivalence comes from (4.3) and (4.6)), and

$$(\mathbf{oblv}^r_{\mathrm{dR},X} \circ \mathbf{ind}^r_{\mathrm{dR},X})\text{-mod}(\mathrm{IndCoh}(X)) \simeq \mathrm{Crys}^r(X),$$

it suffices to construct an isomorphism of the monads

(4.9) $$\Psi_X \circ (\mathbf{oblv}^r_{\mathrm{dR},X} \circ \mathbf{ind}^r_{\mathrm{dR},X}) \circ \Xi_X \text{ and } (\mathbf{oblv}^l_{\mathrm{dR},X} \circ \mathbf{ind}^l_{\mathrm{dR},X})^\vee,$$

acting on $\mathrm{QCoh}(X)$.

The latter isomorphism holds for any eventually coconnective X, and follows from the isomorphism of monads

$$\mathbf{oblv}^l_{\mathrm{dR},X} \circ \mathbf{ind}^l_{\mathrm{dR},X} \simeq \Xi_X^\vee \circ (\mathbf{oblv}^r_{\mathrm{dR},X} \circ \mathbf{ind}^r_{\mathrm{dR},X}) \circ \Psi_X^\vee,$$

see [**GaRo2**, Lemma 3.4.9].

REMARK 4.2.6. The monads in (4.9) correspond to the pair of adjoint functors

$$'\mathbf{ind}^r_{\mathrm{dR},X} : \mathrm{QCoh}(X) \rightleftarrows \mathrm{Crys}(X) : {}'\mathbf{oblv}^r_{\mathrm{dR},X}$$

of [**GaRo2**, Sect. 4.6]. Furthermore, in terms of the

$$\mathbf{D}_X^{\mathrm{naive}} : \mathrm{QCoh}(X)^\vee \simeq \mathrm{QCoh}(X) \text{ and } \mathbf{D}_X^{\mathrm{Verdier}} : \mathrm{Crys}(X)^\vee \simeq \mathrm{Crys}(X),$$

we have the isomorphisms

$$('\mathbf{ind}^r_{\mathrm{dR},X})^\vee \simeq \mathbf{oblv}^l_{\mathrm{dR},X} \text{ and } ('\mathbf{oblv}^r_{\mathrm{dR},X})^\vee \simeq \mathbf{ind}^l_{\mathrm{dR},X}.$$

4.3. Passage between left and right D-modules/crystals. In this subsection we will compare the abstractly defined functor

$$\Upsilon_{X_{\mathrm{dR}}} : \mathrm{Crys}^l(X) \to \mathrm{Crys}^r(X)$$

from Volume I, Chapter 6 and the 'hands-on' functor

$$\mathrm{Dmod}^l(X) \to \mathrm{Dmod}^r(X),$$

given by tensoring a given left D-module with the right D-module

$$\det(T^*(X))[\dim(X)].$$

4.3.1.   According to [**GaRo2**, Proposition 2.2.4], for any scheme X almost of finite type we have a canonically defined equivalence

$$\Upsilon_{X_{\mathrm{dR}}} : \mathrm{Crys}^l(X) \to \mathrm{Crys}^r(X),$$

that makes the diagram

$$
\begin{array}{ccc}
\mathrm{Crys}^l(X) & \xrightarrow{\;\Upsilon_{X_{\mathrm{dR}}}\;} & \mathrm{Crys}^r(X) \\
{\scriptstyle \mathbf{oblv}^l_{\mathrm{dR},X}} \downarrow & & \downarrow {\scriptstyle \mathbf{oblv}^r_{\mathrm{dR},X}} \\
\mathrm{QCoh}(X) & \xrightarrow{\;\Upsilon_X\;} & \mathrm{IndCoh}(X)
\end{array}
$$

commute.

Concretely, the functor $\Upsilon_{X_{\mathrm{dR}}}$ is the functor from Volume I, Chapter 6, Sect. 3.3.4 applied to X_{dR}, and it is given by

$$\mathcal{M} \mapsto \mathcal{M} \otimes \omega_{X_{\mathrm{dR}}},$$

where $\otimes$ is the action of $\mathrm{QCoh}(X_{\mathrm{dR}})$ ($= \mathrm{Crys}^l(X)$) on $\mathrm{IndCoh}(X_{\mathrm{dR}})$ ($= \mathrm{Crys}^r(X)$).

4.3.2.   Now, suppose that X is a smooth classical scheme. Recall that in this case there is a canonical equivalence

$$(4.10) \qquad\qquad \mathrm{Dmod}^l(X) \to \mathrm{Dmod}^r(X),$$

given by tensoring a given left D-module with the right D-module

$$\omega_{\mathrm{Dmod},X} := \det(T^*(X))[\dim(X)].$$

We denote the above functor by $\Upsilon_{\mathrm{Dmod},X}$.

4.3.3.   We will prove:

THEOREM 4.3.4.   *The following diagram of functors canonically commutes:*

$$(4.11) \qquad
\begin{array}{ccc}
\mathrm{Dmod}^l(X) & \xrightarrow{\;F^l_X\;} & \mathrm{Crys}^l(X) \\
{\scriptstyle \Upsilon_{\mathrm{Dmod},X}} \downarrow & & \downarrow {\scriptstyle \Upsilon_{X_{\mathrm{dR}}}} \\
\mathrm{Dmod}^r(X) & \xrightarrow{\;F^r_X\;} & \mathrm{Crys}^r(X)
\end{array}
$$

By applying Theorem 4.3.4 to $\mathcal{O}_X \in \mathrm{Dmod}^l(X)$, we obtain:

COROLLARY 4.3.5.   *There exists a canonical isomorphism in* $\mathrm{Crys}^r(X)$*:*

$$(4.12) \qquad\qquad F^r_X(\omega_{\mathrm{Dmod},X}) \simeq \omega_{X_{\mathrm{dR}}}.$$

4.3.6.   Applying the forgetful functor

$$\mathbf{oblv}^r_{\mathrm{dR},X} : \mathrm{Crys}^r(X) \to \mathrm{IndCoh}(X),$$

to the isomorphism of (4.12), we obtain:

COROLLARY 4.3.7. *There exists a canonical isomorphism in* $\mathrm{IndCoh}(X)$:

(4.13) $\Xi_X(\det(T^*(X))[\dim(X)] \simeq \omega_X.$

Note that latter corollary is the well-known identification of the abstractly defined dualizing sheaf with the shifted line bundle of top forms.

REMARK 4.3.8. The isomorphism (4.13) can be proved without involving D-modules or crystals, by an argument along the same lines as that proving the isomorphism (4.12) in Sect. 4.4 below. This argument is given in a more general context in Chapter 8, Proposition 7.3.4.

4.4. Proof of Theorem 4.3.4.

4.4.1.   First, we make the following observation, which follows from the constructions:

LEMMA 4.4.2. *For* $\mathcal{M} \in \mathrm{Dmod}^l(X)$ *and* $\mathcal{N} \in \mathrm{Dmod}^r(X)$ *we have a canonical isomorphism*

$$F^r_X(\mathcal{M} \otimes \mathcal{N}) \simeq F^l_X(\mathcal{M}) \otimes F^r_X(\mathcal{N}).$$

This lemma reduces the assertion of Theorem 4.3.4 to establishing the isomorphism (4.12).

4.4.3.   Recall that for a map $f : X \to Y$ between smooth schemes, one *defines* the functor

$$f^{\blacktriangle,r} : \mathrm{Dmod}^r(Y) \to \mathrm{Dmod}^r(X)$$

by requiring that the diagram

(4.14)

$$
\begin{CD}
\mathrm{Dmod}^l(Y) @>{f^{\blacktriangle,l}}>> \mathrm{Dmod}^l(X) \\
@V{\Upsilon_{\mathrm{Dmod},Y}}VV @VV{\Upsilon_{\mathrm{Dmod},X}}V \\
\mathrm{Dmod}^r(Y) @>{f^{\blacktriangle,r}}>> \mathrm{Dmod}^r(X)
\end{CD}
$$

commute.

Assume for the moment that f is a closed embedding of smooth schemes. Let

$$\mathrm{Dmod}^r(Y)_X \subset \mathrm{Dmod}^r(Y)$$

be the full subcategory consisting of objects with set-theoretic support on X.

Recall that in this case we have Kashiwara's lemma which says that the functor $f^{\blacktriangle,r}$ induces an equivalence $\mathrm{Dmod}^r(Y)_X \to \mathrm{Dmod}^r(X)$.

4.4.4.   For a morphism $f : X \to Y$ between schemes almost of finite type, let

$$f^{\dagger,r} : \mathrm{Crys}^r(Y) \to \mathrm{Crys}^r(X)$$

be the corresponding pullback functor, see [**GaRo2**, Sect. 2.3.4], i.e., $f^{\dagger,r} = f^!_{\mathrm{dR}}$.

Assume for the moment that f is a closed embedding. Let $\mathrm{Crys}^r(Y)_X \subset \mathrm{Crys}^r(Y)$ be the full subcategory consisting of objects with set-theoretic support on X.

Recall (see [**GaRo2**, Proposition 2.5.6]) that in this case the functor $f^{\dagger,r}$ induces an equivalence $\mathrm{Crys}^r(Y)_X \to \mathrm{Crys}^r(X)$.

4.4.5.   Let $f : X \to Y$ be a closed embedding of smooth classical schemes. The next assertion also follows from the constructions:

LEMMA 4.4.6.   *Under the equivalences*

$$f^{\blacktriangle,r} : \mathrm{Dmod}^r(Y)_X \to \mathrm{Dmod}^r(X) \ \text{and} \ f^{\dagger,r} : \mathrm{Crys}^r(Y)_X \to \mathrm{Crys}(X),$$

the diagram

$$
\begin{array}{ccc}
\mathrm{Dmod}^r(Y)_X & \xrightarrow{F^r_Y} & \mathrm{Crys}^r(Y)_X \\
{\scriptstyle f^{\blacktriangle,r}}\Big\downarrow & & \Big\downarrow{\scriptstyle f^{\dagger,r}} \\
\mathrm{Dmod}^r(X) & \xrightarrow{F^r_X} & \mathrm{Crys}^r(X)
\end{array}
$$

commutes.

As a corollary, we obtain:

COROLLARY 4.4.7.   *For a closed embedding of smooth schemes $f : X \to Y$, the diagram*

$$
\begin{array}{ccc}
\mathrm{Dmod}^r(Y) & \xrightarrow{F^r_Y} & \mathrm{Crys}^r(Y) \\
{\scriptstyle f^{\blacktriangle,r}}\Big\downarrow & & \Big\downarrow{\scriptstyle f^{\dagger,r}} \\
\mathrm{Dmod}^r(X) & \xrightarrow{F^r_X} & \mathrm{Crys}^r(X)
\end{array}
$$

commutes.

PROOF.   Follows from the fact that the functor $f^{\blacktriangle,r}$ (resp., $f^{\dagger,r}$) factors through the co-localization $\mathrm{Dmod}^r(Y) \to \mathrm{Dmod}^r(Y)_X$ (resp., $\mathrm{Crys}^r(Y) \to \mathrm{Crys}^r(Y)_X$).   □

4.4.8.   We are finally ready to construct the isomorphism (4.12) and thereby prove Theorem 4.3.4.

Consider the object

$$\omega_{\mathrm{Dmod},X} \boxtimes \omega_{\mathrm{Dmod},X} = \omega_{\mathrm{Dmod},X\times X} \in \mathrm{Dmod}^r(X \times X).$$

Consider the isomorphism

$$(4.15) \qquad F^r_X \circ \Delta^{\blacktriangle,r}_X (\omega_{\mathrm{Dmod},X\times X}) \simeq \Delta^{\dagger,r} \circ F^r_{X\times X}(\omega_{\mathrm{Dmod},X} \boxtimes \omega_{\mathrm{Dmod},X})$$

of Corollary 4.4.7.

On the one hand,

$$F^r_X \circ \Delta^{\blacktriangle,r}_X (\omega_{\mathrm{Dmod},X\times X}) \simeq F^r_X \circ \Delta^{\blacktriangle,r}_X \circ \Upsilon_{\mathrm{Dmod},X\times X}(\mathcal{O}_{X\times X}) \simeq$$

$$\simeq F^r_X \circ \Upsilon_{\mathrm{Dmod},X} \circ \Delta^{\blacktriangle,l}_X(\mathcal{O}_{X\times X}) \simeq F^r_X \circ \Upsilon_{\mathrm{Dmod},X}(\mathcal{O}_X) \simeq F^r_X(\omega_{\mathrm{Dmod},X}).$$

On the other hand,

$$\Delta^{\dagger,r} \circ F^r_{X \times X} \left(\omega_{\mathrm{Dmod},X} \boxtimes \omega_{\mathrm{Dmod},X} \right) \simeq \Delta^{\dagger,r} \left(F^r_X(\omega_{\mathrm{Dmod},X}) \boxtimes F^r_X(\omega_{\mathrm{Dmod},X}) \right) \simeq$$

$$\simeq F^r_X(\omega_{\mathrm{Dmod},X}) \overset{!}{\otimes} F^r_X(\omega_{\mathrm{Dmod},X}),$$

where $\overset{!}{\otimes}$ denotes the symmetric monoidal operation on $\mathrm{Crys}^r(X)$, i.e., the $\overset{!}{\otimes}$ tensor product on $\mathrm{IndCoh}(X_{\mathrm{dR}})$.

Thus, from (4.15) we obtain an isomorphism

$$F^r_X(\omega_{\mathrm{Dmod},X}) \simeq F^r_X(\omega_{\mathrm{Dmod},X}) \overset{!}{\otimes} F^r_X(\omega_{\mathrm{Dmod},X})$$

in $\mathrm{Crys}^r(X)$.

Now, it is easy to see that $F^r_X(\omega_{\mathrm{Dmod},X})$ is *invertible* as an object of the symmetric monoidal category $\mathrm{Crys}^r(X)$.

This implies that $F^r_X(\omega_{\mathrm{Dmod},X})$ is canonically isomorphic to the unit object, i.e., $\omega_{X_{\mathrm{dR}}}$, as required.

4.5. Identification of functors. In this subsection we will show that the pullback and push-forward functors on crystals correspond to the pullback and push-forward functors defined classically for D-modules.

4.5.1.   We now show:

PROPOSITION 4.5.2. *Let $f : X \to Y$ be a morphism between smooth schemes. Then the diagram of functors*

$$
\begin{array}{ccc}
\mathrm{Dmod}^r(Y) & \xrightarrow{\ f^{\blacktriangle,r}\ } & \mathrm{Dmod}^r(X) \\
{\scriptstyle F^r_Y}\downarrow & & \downarrow{\scriptstyle F^r_X} \\
\mathrm{Crys}^r(Y) & \xrightarrow{\ f^{\dagger,r}\ } & \mathrm{Crys}^r(X)
\end{array}
$$

canonically commutes.

REMARK 4.5.3. It follows from the construction given below that when f is a closed embedding, the isomorphism of functors of Proposition 4.5.2 identifies canonically with one in Corollary 4.4.7.

PROOF. Follows by combining the following five commutative diagrams:

$$
\begin{array}{ccc}
\mathrm{Dmod}^l(Y) & \xrightarrow{\ f^{\blacktriangle,l}\ } & \mathrm{Dmod}^l(X) \\
{\scriptstyle F^l_Y}\downarrow & & \downarrow{\scriptstyle F^l_X} \\
\mathrm{Crys}^l(Y) & \xrightarrow{\ f^{\dagger,l}\ } & \mathrm{Crys}^l(X)
\end{array}
$$

(of Lemma 4.1.6);

$$
\begin{array}{ccc}
\mathrm{Dmod}^l(Y) & \xrightarrow{\ f^{\blacktriangle,l}\ } & \mathrm{Dmod}^l(X) \\
{\scriptstyle \Upsilon_{\mathrm{Dmod},Y}}\downarrow & & \downarrow{\scriptstyle \Upsilon_{\mathrm{Dmod},X}} \\
\mathrm{Dmod}^r(Y) & \xrightarrow{\ f^{\blacktriangle,r}\ } & \mathrm{Dmod}^r(X)
\end{array}
$$

(of diagram (4.14));

$$\text{Crys}^l(Y) \xrightarrow{f^{\dagger,l}} \text{Crys}^l(X)$$

$$\Upsilon_{Y_{\text{dR}}} \downarrow \qquad\qquad \downarrow \Upsilon_{X_{\text{dR}}}$$

$$\text{Crys}^r(Y) \xrightarrow{f^{\dagger,r}} \text{Crys}^r(X),$$

(of Volume I, Chapter 6, Sect. 3.3) and finally the diagrams (4.11) for X and Y, respectively.

$\square$

4.5.4. Recall that for a map $f : X \to Y$ between smooth schemes, we have a canonically defined functor

$$f_{\text{Dmod},*} : \text{Dmod}^r(X) \to \text{Dmod}^r(Y).$$

For a smooth scheme X we let $\Gamma_{\text{Dmod}}(X,-)$ denote the functor

$$\text{Dmod}^r(X) \to \text{Vect}$$

equal to $(p_X)_{\text{Dmod},*}$.

Note that Verdier duality defines an equivalence

$$\mathbf{D}_X^{\text{Verdier}} : \text{Dmod}^r(X)^\vee \to \text{Dmod}^r(X),$$

characterized by the fact that its unit and counit maps are

$$\mu_{\text{Dmod},X} : \text{Vect} \xrightarrow{\omega_{\text{Dmod},X}} \text{Dmod}^r(X) \xrightarrow{(\Delta_X)_{\text{Dmod},*}} \text{Dmod}^r(X{\times}X) \simeq \text{Dmod}^r(X){\otimes}\text{Dmod}^r(X),$$

and

$$\epsilon_{\text{Dmod},X} : \text{Dmod}^r(X){\otimes}\text{Dmod}^r(X) \simeq \text{Dmod}^r(X{\times}X) \xrightarrow{(\Delta_X)^{\blacktriangle,r}} \text{Dmod}^r(X) \xrightarrow{\Gamma_{\text{Dmod}}(X,-)} \text{Vect},$$

respectively.

4.5.5. We claim:

PROPOSITION 4.5.6. *The diagram of functors*

$$\begin{array}{ccc} \text{Dmod}^r(X)^\vee & \xrightarrow{\mathbf{D}_X^{\text{Verdier}}} & \text{Dmod}^r(X) \\ {\scriptstyle (F_X^r)^\vee} \uparrow & & \downarrow {\scriptstyle F_X^r} \\ \text{Crys}^r(X)^\vee & \xrightarrow{\mathbf{D}_X^{\text{Verdier}}} & \text{Crys}^r(X) \end{array}$$

canonically commutes.

PROOF. It is enough to establish the commutation of the following diagram:

$$\begin{array}{ccc} \text{Vect} & \xrightarrow{\mu_{\text{Dmod},X}} & \text{Dmod}^r(X) \otimes \text{Dmod}^r(X) \\ {\scriptstyle \text{Id}} \downarrow & & \downarrow {\scriptstyle F_X^r \otimes F_X^r} \\ \text{Vect} & \xrightarrow{\mu_{X_{\text{dR}}}} & \text{Crys}^r(X) \otimes \text{Crys}^r(X). \end{array}$$

Recall the description of the functor $\epsilon_{X_{\mathrm{dR}}}$ is Sect. 2.2.4. Thus, taking into account the isomorphism (4.12), it suffices to show that the diagram

$$
\begin{array}{ccc}
\mathrm{Dmod}^r(X) & \xrightarrow{(\Delta_X)_{\mathrm{Dmod},*}} & \mathrm{Dmod}^r(X \times X) \\
{\scriptstyle F_X^r}\Big\downarrow & & \Big\downarrow{\scriptstyle F_{X \times X}^r} \\
\mathrm{Crys}^r(X) & \xrightarrow{(\Delta_X)_{\mathrm{dR},*}} & \mathrm{Crys}^r(X \times X)
\end{array}
$$

commutes.

However, this follows by adjunction from the commutation of the diagram

$$
\begin{array}{ccc}
\mathrm{Dmod}^r(X) & \xleftarrow{(\Delta_X)^{\blacktriangle,r}} & \mathrm{Dmod}^r(X \times X) \\
{\scriptstyle F_X^r}\Big\downarrow & & \Big\downarrow{\scriptstyle F_{X \times X}^r} \\
\mathrm{Crys}^r(X) & \xleftarrow{(\Delta_X)^{\dagger,r}} & \mathrm{Crys}^r(X \times X),
\end{array}
$$

while the latter commutes by Proposition 4.5.2.

$\square$

As a consequence of Proposition 4.5.6, we obtain:

COROLLARY 4.5.7. *For a smooth scheme X, the following diagram of functors canonically commutes*

$$
\begin{array}{ccc}
\mathrm{Dmod}^r(X) & \xrightarrow{F_X^r} & \mathrm{Crys}^r(X) \\
{\scriptstyle \Gamma_{\mathrm{Dmod}}(X,-)}\Big\downarrow & & \Big\downarrow{\scriptstyle \Gamma_{\mathrm{dR}}(X,-)} \\
\mathrm{Vect} & \xrightarrow{\mathrm{Id}} & \mathrm{Vect}.
\end{array}
$$

PROOF. Obtained by passing to the dual functors in the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Dmod}^r(X) & \xrightarrow{F_X^r} & \mathrm{Crys}^r(X) \\
{\scriptstyle \omega_{\mathrm{Dmod},X}}\Big\uparrow & & \Big\uparrow{\scriptstyle \omega_{X_{\mathrm{dR}}}} \\
\mathrm{Vect} & \xrightarrow{\mathrm{Id}} & \mathrm{Vect}.
\end{array}
$$

$\square$

4.5.8. Finally, we claim:

PROPOSITION 4.5.9. *For a map $f : X \to Y$ between smooth schemes, the following diagram of functors canonically commutes:*

$$
\begin{array}{ccc}
\mathrm{Dmod}^r(X) & \xrightarrow{f_{\mathrm{Dmod},*}} & \mathrm{Dmod}^r(Y) \\
{\scriptstyle F_X^r}\Big\downarrow & & \Big\downarrow{\scriptstyle F_Y^r} \\
\mathrm{Crys}^r(X) & \xrightarrow{f_{\mathrm{dR},*}} & \mathrm{Crys}^r(Y)
\end{array}
$$

PROOF. We factor the map f as

$$
X \xrightarrow{f_1} X \times Y \xrightarrow{f_2} Y,
$$

where f_1 is the graph of f, and f_2 is the projection to the second factor.

Hence, it is enough to establish the commutativity of the diagrams

(4.16)
$$
\begin{CD}
\mathrm{Dmod}^r(X) @>{(f_1)_{\mathrm{Dmod},*}}>> \mathrm{Dmod}^r(X \times Y) \\
@V{F_X^r}VV @VV{F_{X\times Y}^r}V \\
\mathrm{Crys}^r(X) @>{(f_1)_{\mathrm{dR},*}}>> \mathrm{Crys}^r(X \times Y)
\end{CD}
$$

and

(4.17)
$$
\begin{CD}
\mathrm{Dmod}^r(X \times Y) @>{(f_2)_{\mathrm{Dmod},*}}>> \mathrm{Dmod}^r(Y) \\
@V{F_{X\times Y}^r}VV @VV{F_Y^r}V \\
\mathrm{Crys}^r(X \times Y) @>{(f_2)_{\mathrm{dR},*}}>> \mathrm{Crys}^r(Y),
\end{CD}
$$

respectively.

Now, the commutation of (4.16) follows by adjunction from the commutation of

$$
\begin{CD}
\mathrm{Dmod}^r(X) @<{(f_1)^{\blacktriangle,r}}<< \mathrm{Dmod}^r(X \times Y) \\
@V{F_X^r}VV @VV{F_{X\times Y}^r}V \\
\mathrm{Crys}^r(X) @<{(f_1)^{\dagger,r}}<< \mathrm{Crys}^r(X \times Y),
\end{CD}
$$

given by Proposition 4.5.2.

To establish the commutatition of (4.17) we rewrite it as

$$
\begin{CD}
\mathrm{Dmod}^r(X) \otimes \mathrm{Dmod}^r(Y) @>{\Gamma_{\mathrm{Dmod}}(X,-)\otimes\mathrm{Id}}>> \mathrm{Dmod}^r(Y) \\
@V{F_X^r \otimes F_Y^r}VV @VV{F_Y^r}V \\
\mathrm{Crys}^r(X) \otimes \mathrm{Dmod}^r(Y) @>{\Gamma_{\mathrm{dR}}(X,-)\otimes\mathrm{Id}}>> \mathrm{Crys}^r(Y),
\end{CD}
$$

and the result follows from Corollary 4.5.7.

$\square$

Part II

Formal geometry

Introduction

1. What is formal geometry?

By 'formal geometry' we mean the study of the category, whose objects are $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$, and whose morphisms are nil-isomorphisms of prestacks.

In the course of this part, we will see that this category provides a convenient and flexible framework for many geometric operations:

- Taking quotients with respect to a groupoid;
- Correspondence between group-objects (over a given base $\mathcal{X}$) and Lie algebras in the symmetric monoidal category $\mathrm{IndCoh}(\mathcal{X})$;
- Considering differential-geometric constructions such as Lie algebroids, their universal enveloping algebras, Hodge filtration, etc.

One of the features of the theory presented in this part is that it is really very general. E.g., when establishing the correspondence between formal groups over $\mathcal{X}$ and Lie algebras in $\mathrm{IndCoh}(\mathcal{X})$, there are no additional conditions: we really take *all* group-objects and *all* Lie algebras (no finiteness conditions).

1.1. We begin this part with the short Chapter IV.1 that discusses formal moduli problems. The main theorem of this chapter says the following:

For an object $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$, consider the following two categories: one is the category
$$\mathrm{FormMod}_{\mathcal{X}/} := (\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{\mathrm{nil\text{-}isom\ from}\ \mathcal{X}}.$$
I.e., it consists of prestacks locally almost of finite type that admit deformation theory and receive a nil-isomorphism from $\mathcal{X}$.

Another is the category $\mathrm{FormGrpoid}(\mathcal{X})$ of groupoid objects in $(\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{\mathrm{nil\text{-}isom}}$ acting on $\mathcal{X}$ (i.e., the groupoids whose 0-th space is $\mathcal{X}$ itself).

The Čech nerve construction defines a functor

(1.1) $$\mathrm{FormMod}_{\mathcal{X}/} \to \mathrm{FormGrpoid}(\mathcal{X}).$$

Now, the main result of this chapter, Chapter 5, Theorem 2.3.2, says that the functor (1.1) is an equivalence.

1.2. We denote by $B_{\mathcal{X}}$ the functor inverse to (1.1). This is the functor of taking the quotient with respect to a groupoid.

A feature of our proof of the equivalence (1.1) is that it is constructive. I.e., given a groupoid $\mathcal{R}^\bullet$ over $\mathcal{X}$ (i.e., $\mathcal{R}^0 = \mathcal{X}$), we explicitly describe the prestack $B_{\mathcal{X}}(\mathcal{R}^\bullet)$.

We note, however, that the natural map
$$|\mathcal{R}^\bullet| \to B_{\mathcal{X}}(\mathcal{R}^\bullet),$$

is *not* an isomorphism, where $|\mathcal{R}^\bullet|$ is understood as a geometric realization in the category $\mathrm{PreStk}_{\mathrm{laft}}$. The problem is that $|\mathcal{R}^\bullet|$ understood in the above way will *not* in general admit deformation theory.

1.3.   By a formal moduli problem *over* a given object $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ we mean an object

$$\mathcal{Y} \in (\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}},$$

such that the morphism $\mathcal{Y} \to \mathcal{X}$ is an *inf-schematic nil-isomorphism*. I.e., for any $S \to \mathcal{X}$ with $S \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$, the prestack $S \underset{\mathcal{X}}{\times} \mathcal{Y}$ should admit deformation theory and the map

$$\mathrm{red}(S \underset{\mathcal{X}}{\times} \mathcal{Y}) \to \mathrm{red}S$$

should be an isomorphism.

Let $\mathrm{FormMod}_{/\mathcal{X}}$ denote the category of formal moduli problems over $\mathcal{X}$.

By a *formal group* over $\mathcal{X}$ we mean an object of the category $\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$. It follows formally from the equivalence (1.1) that the loop functor defines an equivalence

$$\Omega_X : \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}).$$

We denote by $B_\mathcal{X}$ the inverse equivalence. Thus, we obtain that any $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$ admits a classifying space

$$B_\mathcal{X}(\mathcal{H}) \in \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}).$$

1.4.   Let us add a comment here that explains the link between our theory and that developed in [**Lu6**].

Suppose that $\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$. We have the forgetful functors

$$\mathrm{FormMod}_{X/} \to (\mathrm{PreStk}_{\mathrm{laft}})_{X/} \text{ and } \mathrm{FormMod}_{/X} \to (\mathrm{PreStk}_{\mathrm{laft}})_{/X}$$

I.e., objects of the category $\mathrm{FormMod}_{X/}$ (resp., $\mathrm{FormMod}_{/X}$) are prestacks (locally almost of finite type) under X (resp., over X) satisfying a certain condition. I.e., at the end of the day, they are functors

$$({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}.$$

We show, however, that the information of an object of $\mathrm{FormMod}_{X/}$ (resp., $\mathrm{FormMod}_{/X}$) is completely determined by the restriction of the corresponding functor to a much smaller category. Namely, the category in question is

(1.2) $(({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ from}\ X})^{\mathrm{op}}$

in the case of $\mathrm{FormMod}_{\mathcal{X}/}$ and

(1.3) $(({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ to}\ X})^{\mathrm{op}}$

in the case of $\mathrm{FormMod}_{/X}$.

I.e., in order to 'know' a formal moduli problem under X, it suffices to know how it behaves on schemes infinitesimally close to X.

For example, if $X = \mathrm{pt}$, the categories (1.2) and (1.3) both identify with the category of connective k-algebras A with finite-dimensional total cohomologies, and $H^0(A)$ being local. So, functors out of this category (satisfying the appropriate

deformation theory condition) are indeed what is traditionally called a 'formal moduli problem'.

2. Lie algebras

In Chapter IV.2 we make a digression to discuss the general theory of Lie algebras (in a symmetric monoidal DG category $\mathbf{O}$).

The material from this chapter will be extensively used in Chapter IV.3, where we study the relation between formal groups and Lie algebras.

2.1. The main actors in this chapter are the mutually adjoint functors

$$(2.1) \qquad \mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathbf{O}) \rightleftarrows \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) : \mathrm{coChev}^{\mathrm{enh}}$$

that relate the category $\mathrm{LieAlg}(\mathbf{O})$ of Lie algebras in $\mathbf{O}$ to the category $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$ of augmented co-commutative co-algebras in $\mathbf{O}$.

The main point is that the functors in (2.1) are *not* mutually inverse equivalences. But they are close to be such.

2.2. We remind that the composition of $\mathrm{Chev}^{\mathrm{enh}}$ with the forgetful functor

$$\mathbf{oblv}_{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}$$

is the the functor, denoted Chev, which is by definition the left adjoint to

$$\mathbf{triv}_{\mathrm{Lie}} \circ [-1] : \mathbf{O} \to \mathrm{LieAlg}(\mathbf{O}),$$

where $\mathbf{triv}_{\mathrm{Lie}}$ is the functor of the 'trivial Lie algebra'.

The composition of $\mathrm{coChev}^{\mathrm{enh}}$ with the forgetful functor

$$\mathbf{oblv}_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathbf{O}$$

is the functor, denoted coChev, which is by definition the right adjoint to the functor

$$\mathbf{triv}_{\mathrm{Cocom}} \circ [1] : \mathbf{O} \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}),$$

where $\mathbf{triv}_{\mathrm{Cocom}}$ is the functor of the 'trivial co-commutative co-algebra'.

In other words, the functor

$$[1] \circ \mathrm{coChev} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}$$

is the functor Prim of primitive elements.

2.3. Let us now describe the two main results of this chapter, Theorems 4.4.6 and 6.1.2.

Consider the composition

$$(2.2) \qquad \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomHopf}(\mathbf{O}).$$

Theorem 4.4.6 says that the functor (2.2) is fully faithful. I.e., although the functor $\mathrm{Chev}^{\mathrm{enh}}$ fails to be fully faithful, if we compose it with loop functor and retain the group structure, it becomes fully faithful.

Theorem 6.1.2 says that the functor (2.2) identifies canonically with the functor U^{Hopf} of universal enveloping algebra (viewed as a Hopf algebra).

2.4.   Note also that the right adjoint of the functor (2.2) is a functor

(2.3) $\mathrm{CocomHopf}(\mathbf{O}) \to \mathrm{LieAlg}(\mathbf{O})$

that makes the following diagram commutative:

$$
\begin{array}{ccc}
\mathrm{CocomHopf}(\mathbf{O}) & \xrightarrow{\;\mathbf{oblv}_{\mathrm{Assoc}}\;} & \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) \\
\downarrow & & \downarrow{\scriptstyle\mathrm{Prim}} \\
\mathrm{LieAlg}(\mathbf{O}) & \longrightarrow & \mathbf{O},
\end{array}
$$

where $\mathbf{oblv}_{\mathrm{Assoc}}$ is the natural forgetful functor, and Prim is the functor of primitive elements.

The above commutative diagram may be viewed as an ultimate answer to the question of why the tangent space to a Lie group has a structure of Lie algebra: because the tangent fiber of a co-commutative Hopf algebra, viewed as a mere augmented co-commutative co-algebra, has a structure of Lie algebra.

The latter observation will be extensively used in the next chapter, i.e., Chapter 7.

3. Formal groups vs. Lie algebras

In Chapter 7 we establish an equivalence between the category of formal groups (over a given $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$) and the category $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$.

3.1.   Assume first that $\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$. Our first step in defining the functors that connect formal groups and Lie algebras in $\mathrm{IndCoh}(X)$ is to set up a kind of 'covariant formal algebraic geometry'.

What we mean by this is that we define a pair of mutually adjoint functors

(3.1) $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \rightleftarrows \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) : \mathrm{Spec}^{\mathrm{inf}} .$

The functor $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$ sends an object $(\mathcal{Y} \xrightarrow{f} X) \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$ to

$$f_*^{\mathrm{IndCoh}}(\omega_{\mathcal{Y}}) \in \mathrm{IndCoh}(X),$$

with the co-commutative co-algebra structure coming from the diagonal morphism $\mathcal{Y} \to \mathcal{Y} \underset{X}{\times} \mathcal{Y}$, and the augmentation from the section $X \to \mathcal{Y}$.

The functor $\mathrm{Spec}^{\mathrm{inf}}$ is defined as the right adjoint of $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$.

We should warn the reader that the situation here, although formally analogous, is not totally parallel to the usual algebraic geometry. In particular, the functor $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$ is *not* fully faithful.

3.2. A basic example of an object in $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$ is a vector group, denoted $\mathrm{Vect}_X(\mathcal{F})$, associated to $\mathcal{F} \in \mathrm{IndCoh}(X)$.

For $(X' \xrightarrow{g} X) \in \mathrm{Ptd}(^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}))$, we have

$$(3.2) \qquad \mathrm{Maps}(X', \mathrm{Vect}_X(\mathcal{F})) = \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathrm{Distr}^+(X'), \mathcal{F}),$$

where

$$\mathrm{Distr}^+(X') := \mathrm{Fib}(g_*^{\mathrm{IndCoh}}(\omega_{X'}) \to \omega_X).$$

One shows that

$$\mathrm{Vect}_X(\mathcal{F}) \simeq \mathrm{Spec}^{\mathrm{inf}}(\mathrm{Sym}(\mathcal{F})).$$

Note that formula (3.2) is parallel to the definition of the affine scheme $\underline{V}$ associated to a finite-dimensional vector space V in algebraic geometry, i.e., $\underline{V} = \mathrm{Spec}(\mathrm{Sym}(V^*))$; namely, we have

$$\mathrm{Hom}(X, \underline{V}) \simeq \Gamma(X, \mathcal{O}_X) \otimes V.$$

3.3. We are now ready to describe the mutually inverse functors

$$(3.3) \qquad \mathrm{Lie} : \mathrm{Grp}(\mathrm{FormMod}_{/X}) \rightleftarrows \mathrm{LieAlg}(\mathrm{IndCoh}(X)) : \exp.$$

The functor Lie is the composition of the functor

$$\mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}) : \mathrm{Grp}(\mathrm{FormMod}_{/X}) \to \mathrm{Grp}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))) =:$$
$$= \mathrm{CocomHopf}(\mathrm{IndCoh}(X))$$

and the functor

$$\mathrm{CocomHopf}(\mathrm{IndCoh}(X)) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X))$$

of (2.3).

3.4. One shows that the composition

$$\mathrm{Grp}(\mathrm{FormMod}_{/X}) \xrightarrow{\mathrm{Lie}} \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \xrightarrow{\mathbf{oblv}_{\mathrm{Lie}}} \mathrm{IndCoh}(X)$$

is the functor

$$\mathrm{Grp}(\mathrm{FormMod}_{/X}) \xrightarrow{\mathbf{oblv}_{\mathrm{Grp}}} \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \xrightarrow{\mathcal{Y} \mapsto T(\mathcal{Y}/X)|_X} \mathrm{IndCoh}(X).$$

I.e., the object of $\mathrm{IndCoh}(X)$ underlying the Lie algebra of $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/X})$ is the tangent space of $\mathcal{H}$ at the origin, as it should be.

3.5. The functor exp is defined as the composition of

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{CocomHopf}(\mathrm{IndCoh}(X)) =$$
$$= \mathrm{Grp}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)))$$

and the functor

$$\mathrm{Grp}(\mathrm{Spec}^{\mathrm{inf}}) : \mathrm{Grp}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))) \to \mathrm{Grp}(\mathrm{FormMod}_{/X}).$$

Knowing the equivalence (3.3), one can interpret in its terms the adjunction (3.1). Namely, it becomes the adjunction (2.1) for the category $\mathbf{O} = \mathrm{IndCoh}(\mathcal{X})$.

3.6.   We note that it follows from Chapter 6, Theorem 4.2.2 that the composed functor

$$\mathbf{oblv}_{\mathrm{Grp}} \circ \exp : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{Ptd}(\mathrm{FormMod}_{/X})$$

is isomorphic to

$$\mathrm{LieAlg}(\mathrm{IndCoh}(X)) \overset{\mathbf{oblv}_{\mathrm{Lie}}}{\longrightarrow} \mathrm{IndCoh}(X) \overset{\mathrm{Vect}_X(-)}{\longrightarrow} \mathrm{Ptd}(\mathrm{FormMod}_{/X}).$$

I.e., the object of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$ underlying a formal group $\mathcal{H}$ is canonically isomorphic to the vector group $\mathrm{Vect}_X(T(\mathcal{H}/X)|_X)$.

Using this fact, one shows that the functors exp, and hence Lie, are compatible with base change with respect to X, and thus give rise to an equivalence

$$\mathrm{Lie} : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \rightleftarrows \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) : \exp$$

for any $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$.

4. Lie algebroids

In Chapter IV.4 we initiate the study of Lie algebroids.

4.1.   Lie algebroids are defined classically as quasi-coherent sheaves with some extra structure, while this structure involves a differential operator of order one. Because the definition of Lie algebroids involves explicit formulas, it is difficult to render it directly to the the world of derived algebraic geometry.

For this reason, we take a different approach and define Lie algebroids via geometry. Namely, we let the category of Lie algebroids $\mathrm{LieAlgbroid}(\mathcal{X})$ on $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$ be, by definition, equivalent to that of formal groupoids over $\mathcal{X}$.

The reason why this definition has a chance to be reasonable is the equivalence (3.3) between formal groups and Lie algebras.

Much of this chapter is devoted to the explanation of why Lie algebroids defined in the above way really behave as Lie algebroids should.

4.2.   We define the forgetful functor

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} : \mathrm{LieAlgbroid}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})},$$

that sends a Lie algebroid $\mathfrak{L}$ to the underlying quasi-coherent sheaf $\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})$, equipped with the anchor map

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}) \to T(\mathcal{X}).$$

We show that this functor is monadic; in particular, it admits a left adjoint, denoted

$$\mathbf{free}_{\mathrm{LieAlgbroid}} : \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \to \mathrm{LieAlgbroid}(\mathcal{X}).$$

We show that the endo-functor

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}$$

of $\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$ has the 'right size', i.e., what one expect from a reasonable definition of Lie algebroids (it has a canonical filtration with the expected form of the associated graded).

4.3.   Thus, we have the equivalences

(4.1) $\mathrm{LieAlgbroid}(\mathcal{X}) \simeq \mathrm{FormGrpoid}(\mathcal{X}) \simeq \mathrm{FormMod}_{\mathcal{X}/}.$

We show that the functor $\mathbf{free}_{\mathrm{LieAlgbroid}}$ translates into the functor of the *square-zero extension*

$$\mathrm{RealSqZ} : \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \to \mathrm{FormMod}_{\mathcal{X}/}.$$

4.4.   The category of Lie algebroids on $\mathcal{X}$ is related to the category $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$ by a pair of adjoint functors

(4.2) $\mathrm{diag} : \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \rightleftarrows \mathrm{LieAlgbroid}(\mathcal{X}) : \mathrm{ker\text{-}anch}.$

The meaning of the functor diag should be clear: a Lie algebra on $\mathcal{X}$ can be viewed as a Lie algebroid with the trivial anchor map. The functor ker-anch sends a Lie algebroid to the kernel of the anchor map.

We show that the adjoint pair (4.2) is also monadic. The corresponding monad

$$\mathrm{LieAlgbroid}(\mathcal{X}) \circ \mathrm{diag}$$

on the category $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$ is given by *semi-direct product* with the inertia Lie algebra $\mathrm{inert}_{\mathcal{X}}$ (the Lie algebra of the inertia group $\mathrm{Inert}_{\mathcal{X}} := \mathcal{X} \underset{\mathcal{X} \times \mathcal{X}}{\times} \mathcal{X}$).

We learned about this way of realizing Lie algebroids from J. Francis.

5. Infinitesimal differential geometry

In Chapter Chapter 9 we develop the ideas from Chapter 8 to set up constructions of differential nature on objects $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$.

5.1.   The key construction in Chapter 9 is that of *deformation to the normal bundle*.

We start with $(\mathcal{X} \to \mathcal{Y}) \in \mathrm{FormMod}_{\mathcal{X}/}$ and we define an $\mathbb{A}^1$-family

$$(\mathcal{X} \times \mathbb{A}^1 \to \mathcal{Y}_{\mathrm{scaled}})$$

of formal moduli problems under $\mathcal{X}$. In doing so, we follow an idea that was suggested to us by J. Lurie.

A crucial piece of structure that $\mathcal{Y}_{\mathrm{scaled}}$ has is that of *left-lax equivariance* with respect to $\mathbb{A}^1$ that acts on itself *by multiplication*.

The structure of equivariance with respect to $\mathbb{G}_m \subset \mathbb{A}^1$ implies that the fibers $\mathcal{Y}_a$ of $\mathcal{Y}_{\mathrm{scaled}}$ at $0 \neq a \in \mathbb{A}^1$ are all canonically isomorphic to $\mathcal{X}$.

The fiber at $0 \in \mathbb{A}^1$ identifies with $\mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1])$, i.e., the formal version of the total space of the normal to $\mathcal{X}$ inside $\mathcal{Y}$.

The latter observation allows to reduce many isomorphism questions regarding formal moduli problems to the simplest situation, when our moduli problem is a vector group $\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})$ for $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$.

5.2. If $\mathcal{X} = X$ is a classical scheme, and $\mathcal{Y}$ is the formal completion of X in Y, where $X \to Y$ is a regular embedding, then $\mathcal{Y}_{\text{scaled}}$ is the completion of $X \times \mathbb{A}^1$ in the usual deformation of Y to the normal cone.

For the final object in the category $\text{FormMod}_{\mathcal{X}/}$, i.e.,

$$\mathcal{X} \to \mathcal{X}_{\text{dR}},$$

the deformation $(\mathcal{X}_{\text{dR}})_{\text{scaled}}$ is the Dolbeault deformation of $\mathcal{X}_{\text{dR}}$ to $\text{Vect}_{\mathcal{X}}(T(\mathcal{X})[1])$.

5.3. The relevance of the $\mathbb{A}^1$ left-lax equivariant family $\mathcal{Y}_{\text{scaled}}$ is the following: functors from $\text{FormMod}_{\mathcal{X}/}$ with values in a DG category $\mathbf{C}$ will automatically upgrade to functors with values in the category

$$\mathbf{C}^{\text{Fil},\geq 0}.$$

This is due to the equivalence

$$\mathbf{C}^{\text{Fil},\geq 0} \simeq (\mathbf{C} \otimes \text{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}},$$

see Chapter 6, Lemma 2.5.5(a).

5.4. As a first application of the deformation $\mathcal{Y} \rightsquigarrow \mathcal{Y}_{\text{scaled}}$ we construct a canonical filtration on the universal enveloping algebra

$$U(\mathfrak{L}) \in \text{AssocAlg}(\text{Funct}_{\text{cont}}(\text{IndCoh}(\mathcal{X}), \text{IndCoh}(\mathcal{X}))$$

of a Lie algebroid $\mathfrak{L}$.

This approach to the filtration on the universal enveloping algebra is natural from the point of view of classical algebraic geometry and smooth schemes: the canonical filtration on the algebra of differential operators is closely related to the Dolbeault deformation.

5.5. Another central construction in chapter Chapter 9 is that of the n-th infinitesimal neighborhood

$$\mathcal{X} \to \mathcal{X}^{(n)} \to \mathcal{Y}$$

for $\mathcal{Y} \in \text{FormMod}_{\mathcal{X}/}$.

Again, this construction is not at all straightforward in the generality in which we consider it: nil-isomorphisms between objects of $\text{PreStk}_{\text{laft-def}}$.

We construct the n-th infinitesimal neighborhood inductively, with $\mathcal{X}^{(n)}$ being a square-zero extension of $\mathcal{X}^{(n-1)}$ by means of $\text{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])$.

In the process of construction of this extension we crucially rely on the deformation

$$\mathcal{Y} \rightsquigarrow \mathcal{Y}_{\text{scaled}}.$$

5.6. We show that the natural map

$$\operatorname*{colim}_{n} \mathcal{X}^{(n)} \to \mathcal{Y}$$

is an isomorphism.

In particular, we obtain that the dualizing sheaf $\omega \in \operatorname{IndCoh}(\mathcal{Y})$ has a canonical filtration whose n-th term is the direct image of $\omega_{\mathcal{X}^{(n)}}$ under $\mathcal{X}^{(n)} \to \mathcal{Y}$.

Translating to the language of Lie algebroids via (4.1), the above filtration can be interpreted as the de Rham resolution of the unit module over a Lie algebroid $\mathfrak{L}$, with the n-th associated graded being the induced module from

$$\operatorname{Sym}^n(\mathbf{oblv}_{\operatorname{LieAlgebroid}}(\mathfrak{L})).$$

For $(\mathcal{X} \to \mathcal{X}_{\operatorname{dR}}) \in \operatorname{FormMod}_{\mathcal{X}/}$ we recover the Hodge filtration on the unit crystal (D-module) $\omega_{\mathcal{X}_{\operatorname{dR}}}$.

6. A simplifying remark

The theory of formal geometry developed in Part II uses in an essential way the theory of ind-coherent sheaves on laft prestacks, in the form of the functor

$$(6.1) \qquad \operatorname{IndCoh}^!_{\operatorname{PreStk}_{\operatorname{laft}}} : (\operatorname{PreStk}_{\operatorname{laft}})^{\operatorname{op}} \to \operatorname{DGCat}_{\operatorname{cont}}.$$

I.e., for the purposes of the present Part, we *do not need* IndCoh as a functor out of the category of correspondences.

In order to construct (6.1) we only need the functor

$$\operatorname{IndCoh}^!_{\operatorname{Sch}_{\operatorname{aft}}} : (\operatorname{Sch}_{\operatorname{aft}})^{\operatorname{op}} \to \operatorname{DGCat}_{\operatorname{cont}},$$

and the latter can be constructed using the shortcut explained in Volume I, Chapter 5, Sect. 4.3.

This is to say that the contents of Part II do not depend on the (heavy) material from Volume I, Parts III and Appendix.

Formal moduli

Introduction

In this Chapter we prove one of the main results of this book: the existence of a well-defined procedure of taking a quotient with respect to a formal groupoid.

0.1. Groupoids and quotients.

0.1.1.   First off, a groupoid in Spc is an object $R^\bullet \in \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}$ that is a Segal space such that all of its 1-morphisms are invertible. We shall say that $R^\bullet$ acts on the space $X = R^0$. Sometimes we abuse the notation and instead of $R^\bullet$ write just the space $R := R^1$.

In other words, a groupoid acting on X is a space R, equipped with a pair of projections

$$
\begin{array}{ccc}
 & R & \\
{}^{p_s}\swarrow & & \searrow^{p_t} \\
X & & X,
\end{array}
$$

and a multiplication map

$$
R \underset{p_t,X,p_s}{\times} R \overset{m}{\to} R
$$

over $X \times X$, satisfying a homotopy-coherent system of associativity conditions, and such that the map

$$
R \underset{p_t,X,p_s}{\times} R \overset{m,\mathrm{id}}{\longrightarrow} R \underset{p_t,X,p_t}{\times} R
$$

is an isomorphism.

0.1.2.   Given a map $X \to Y$ in $\mathrm{PreStk}_{\mathrm{laft}}$, the Čech nerve construction gives rise to a canonically defined groupoid $R^\bullet$ acting on X with

$$
R = X \underset{Y}{\times} X.
$$

The above construction is a functor from the category of spaces under X to that of groupoids acting on X.

This functor admits a fully faithful left adjoint that sends $R^\bullet$ to its geometric realization $Y = |R^\bullet|$. The image of this left adjoint is the full subcategory consisting of those $X \to Y$ that induce a surjection on π_0.

0.1.3.   The notion of groupoid makes sense in arbitrary ∞-category $\mathcal{C}$ with finite limits, see [**Lu1**, Sect. 6.1.2].

Namely, given an object $X \in \mathcal{C}$, a groupoid acting on X is a simplicial object $R^\bullet$ of $\mathcal{C}$ with $R^0 = X$ such that for any $X' \in \mathcal{C}$, the object

$$\mathrm{Maps}_{\mathcal{C}}(X', R^\bullet) \in \mathrm{Spc}^{\mathbf{\Delta}^{\mathrm{op}}}$$

is a groupoid in spaces.

As in the case of $\mathcal{C} = \mathrm{Spc}$, given a map $X \to Y$, we canonically attach to it its Čech nerve, which is a groupoid acting on X.

However, the existence of the left adjoint can only be guaranteed if $\mathcal{C}$ has colimits. This left adjoint will be fully faithful if geometric realizations in $\mathcal{C}$ commute with fiber products.

0.1.4.   Thus, we obtain a well-defined notion of groupoid object $\mathcal{R}^\bullet$ in $\mathrm{PreStk}_{\mathrm{laft}}$ acting on a given $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$.

Let $\mathcal{X} \to \mathcal{Y}$ be a map in $\mathrm{PreStk}_{\mathrm{laft}}$. Taking its Čech nerve, we obtain a groupoid $\mathcal{R}^\bullet$. The assignment

$$\mathcal{R}^\bullet \to |\mathcal{R}^\bullet|$$

provides a fully faithful left adjoint.

0.2. Formal groupoids.

0.2.1.   We now modify our problem: instead of the category $\mathrm{PreStk}_{\mathrm{laft}}$, we now consider the category $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. I.e., we impose the condition that our prestacks *admit deformation theory.* In addition, we will restrict to maps between prestacks that are nil-isomorphisms.

Groupoid objects in this context will be called *formal groupoids*; for a given $\mathcal{X}$ we denote the category of formal groupoids over $\mathcal{X}$ by $\mathrm{FormGrpoid}(\mathcal{X})$.

Starting from $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$ and an object $\mathcal{R}^\bullet \in \mathrm{FormGrpoid}(\mathcal{X})$, it is *not true* that the prestack $|\mathcal{R}^\bullet|$ admits deformation theory. So, the existence of a fully faithful left adjoint to the Čech nerve construction is not so obvious in this case.

However, the main result of this chapter, Theorem 2.3.2 says:

THEOREM 0.2.2.   *For $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$, the Čech nerve construction is an equivalence between the category of $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$ equipped with a nil-isomorphism $\mathcal{X} \to \mathcal{Y}$ and the category $\mathrm{FormGrpoid}(\mathcal{X})$.*

In other words, this theorem says that, given a formal groupoid $\mathcal{R}$ acting on $\mathcal{X}$, there is a well-defined quotient

$$B_{\mathcal{X}}(\mathcal{R}) \in (\mathrm{PreStk}_{\mathrm{laft}})_{\mathcal{X}/},$$

such that $B_{\mathcal{X}}(\mathcal{R})$ *admits deformation theory* and the map $\mathcal{X} \to B_{\mathcal{X}}(\mathcal{R})$ is a nil-isomorphism (it is then automatically inf-schematic).

0.2.3. As a particular case of Theorem 0.2.2 we obtain that the loop functor

$$\mathcal{Y} \mapsto \mathcal{G} := \mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X}$$

defines an equivalence between the category of $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$, equipped with a pair of inf-schematic nil-isomorphisms

$$\mathcal{X} \overset{i}{\to} \mathcal{Y} \overset{s}{\to} \mathcal{X}, \quad s \circ i = \mathrm{id}$$

and that of group-objects in the category prestacks $\mathcal{G}$ equipped with an inf-schematic nil-isomorphism $\mathcal{G} \to \mathcal{X}$. We denote the latter category by $\mathrm{Grp}(\mathrm{Form}_{/\mathcal{X}})$, and refer to its objects as *formal groups over* $\mathcal{X}$.

Thus, to any $\mathcal{G}$ as above, we can attach its classifying prestack $B_{\mathcal{X}}(\mathcal{G})$

$$\mathcal{X} \overset{i}{\to} B_{\mathcal{X}}(\mathcal{G}) \overset{s}{\to} \mathcal{X}$$

where i and s are inf-schematic nil-isomorphisms.

0.3. What else is done in this chapter? Let X be an object of $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$. We introduce several notions of formal moduli problems associated with X, and we relate them to notions developed in [**Lu6**].

0.3.1. A formal moduli problem *over* X is an inf-scheme $\mathcal{Y}$ equipped with a nil-isomorphism $\mathcal{Y} \to X$.

Recall that, by definition, an inf-scheme is a prestack locally almost of finite type. I.e., $\mathcal{Y}$ is encoded by a functor

$$((^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/X})^{\mathrm{op}} \to \mathrm{Spc}.$$

In Proposition 1.2.2, we show that the data of $\mathcal{Y}$ is completely determined by its values on a much smaller category: namely,

$$(^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{nil\text{-}isom\ to}\ X} \subset (^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/X}$$

that consists of those $S \to X$ that are nil-isomorphisms.

Moreover, the condition that $\mathcal{Y}$ admit deformation theory can also be expressed via the resulting functor

$$((^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{nil\text{-}isom\ to}\ X})^{\mathrm{op}} \to \mathrm{Spc}.$$

0.3.2. Note that when $X = \mathrm{pt} = \mathrm{Spec}(k)$, the category $(^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{nil\text{-}isom\ to}\ X}$ is the one opposite to that of connective commutative DG algebras A over k that have finitely many cohomologies, and all of whose cohomologies are finite dimensional, with $H^0(A)$ local.

So, a formal moduli problem over pt is the same as a functor on the category of such algebras, subject to some conditions that guarantee deformation theory.

0.3.3.   Suppose now that $X \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$. In this case, we study the notion of formal moduli problem *under* X. By definition, this is an inf-scheme $\mathcal{Y}$, equipped with a nil-isomorphism

$$X \to \mathcal{Y}.$$

We show that the data of such $\mathcal{Y}$, viewed as a functor $({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}$, is recovered from its values on a smaller category, namely the category

$$({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\text{nil-isom from } X} \subset ({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{X/},$$

consisting of those $X \to S$ that are nil-isomorphisms.

0.3.4.   Note that when $\mathcal{X} = \mathrm{pt}$, for an object $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ to be a formal moduli problem under $\mathcal{X}$ simply means that $\mathcal{Y}$ is an inf-scheme with ${}^{\mathrm{red}}\mathcal{Y} = \mathrm{pt}$.

I.e., formal moduli problems under pt are the same as formal moduli problems over pt.

1. Formal moduli problems

In this section we introduce the notions of formal moduli problem *over* and *under* a given prestack $\mathcal{X}$.

Let us note the following substantial difference between our set-up and that of [**Lu6**] (in which the case $\mathcal{X} = \mathrm{pt}$):

In the context of *loc.cit.* a formal moduli problem is a functor on the category of connective finite-dimensional commutative DG algebras over k, whose 0-th cohomology is local.

By contrast, our formal moduli problems (for $\mathcal{X} = \mathrm{pt}$) are objects $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$ with ${}^{\mathrm{red}}\mathcal{Y} = \mathrm{pt}$, so they can be evaluated on connective commutative finite-dimensional DG algebras over k. The two notions are related by the procedures of restriction and left Kan extension; the fact that these two procedures are inverses of one another is the consequence of Chapter 2, Corollary 4.4.6.

1.1. Formal moduli problems *over* a prestack. Unlike [**Lu6**], we define the category of formal moduli problems over a given $\mathcal{X}$ to be a full subcategory in $(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}$. The equivalence of this definition and the one in *loc. cit.* will be established in Sect. 1.2.

1.1.1.   Let us fix $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$. We define

$$\mathrm{FormMod}_{/\mathcal{X}} \subset (\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}$$

to be the full subcategory of spanned by those $\mathcal{Y} \to \mathcal{X}$, for which the above map is:

- inf-schematic (see Chapter 2, Definition 3.1.5 for what this means);
- a nil-isomorphism (i.e., ${}^{\mathrm{red}}\mathcal{Y} \to {}^{\mathrm{red}}\mathcal{X}$ is an isomorphism).

We shall refer to $\mathrm{FormMod}_{/\mathcal{X}}$ as the category of *formal moduli problems over* $\mathcal{X}$.

1.1.2.   Let $\mathcal{X}'$ be an object of $\mathrm{FormMod}_{/\mathcal{X}}$. We will use the notation

$$\mathrm{FormMod}_{\mathcal{X}'//\mathcal{X}}$$

to denote the category

$$(\mathrm{FormMod}_{/\mathcal{X}})_{\mathcal{X}'/}.$$

1.1.3.   The following results easily from the definitions:

LEMMA 1.1.4. *Let $\mathcal{Y} \to \mathcal{X}$ be a map in* $\mathrm{PreStk}_{\mathrm{laft}}$. *Then* $\mathcal{Y} \in \mathrm{FormMod}_{/\mathcal{X}}$
if and only if for every $S \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$, *the prestack* $S \underset{\mathcal{X}}{\times} \mathcal{Y}$ *is an infscheme and*
${}^{\mathrm{red}}(S \underset{\mathcal{X}}{\times} \mathcal{Y}) \to {}^{\mathrm{red}}S$ *is an isomorphism.*

1.1.5.   The following will be useful:

LEMMA 1.1.6. *The functor*
$$\mathrm{FormMod}_{/\mathcal{X}} \to \lim_{(Z,x)\in(({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{X}})^{\mathrm{op}}} \mathrm{FormMod}_{/Z}$$
is an equivalence.

1.2. Situation over an affine scheme. In this subsection we will assume that
that $\mathcal{X} = X \in \mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}}$. We will show that a formal moduli problem over X, viewed
as a functor
$$(({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/X})^{\mathrm{op}} \to \mathrm{Spc},$$
is completely determined by its value on those $Z \in ({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/X}$ for which ${}^{\mathrm{red}}Z \to$
${}^{\mathrm{red}}X$ is an isomorphism.

Note that when $X = \mathrm{pt}$, the category $(({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/X})^{\mathrm{op}}$ is the same as that
of connective finite-dimensional commutative DG algebras over k, whose 0-th co-
homology is local. This brings us in contact with the defintion of formal moduli
problems in [**Lu6**].

1.2.1.   We have:

PROPOSITION 1.2.2.

(a) *Every* $\mathcal{Y} \in \mathrm{FormMod}_{/X}$, *viewed as a functor*
$$(({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/X})^{\mathrm{op}} \to \mathrm{Spc},$$
is the left Kan extension of its restriction to the full subcategory

(1.1) $(({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ to}\ X})^{\mathrm{op}} \subset (({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/X})^{\mathrm{op}}.$

(b) *Let* $\mathcal{Y}_{\mathrm{nil\text{-}isom}}$ *be a presheaf on the category* $({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ to}\ X}$, *satisfying:*
   - $\mathcal{Y}_{\mathrm{nil\text{-}isom}}({}^{\mathrm{red}}X) = *$;
   - *For a push-out diagram* $S_1 \underset{S}{\sqcup} S'$ *in* $({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ to}\ X}$, *where* $S \to S'$
     has a structure of square-zero extension, the resulting map
     $$\mathcal{Y}_{\mathrm{nil\text{-}isom}}(S_1 \underset{S}{\sqcup} S') \to \mathcal{Y}_{\mathrm{nil\text{-}isom}}(S_1) \underset{\mathcal{Y}_{\mathrm{nil\text{-}isom}}(S)}{\times} \mathcal{Y}_{\mathrm{nil\text{-}isom}}(S')$$
     is an isomorphism.

   Then if
   $$\mathcal{Y} \in (\mathrm{PreStk}_{\mathrm{laft}})_{/X}$$
   denotes the left Kan extension of $\mathcal{Y}_{\mathrm{nil\text{-}isom}}$ *under* (1.1), *then* $\mathcal{Y} \in \mathrm{FormMod}_{/X}$.

(c) *The assignments*
$$\mathcal{Y} \mapsto \mathcal{Y}|_{({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ to}\ X}} \quad \text{and} \quad \mathcal{Y}_{\mathrm{nil\text{-}isom}} \mapsto \mathcal{Y}$$
are mutually inverse equivalences of categories.

PROOF. Point (a) follows from Chapter 2, Corollary 4.3.4.

Point (b) follows from Chapter 2, Proposition 4.4.5.

Point (c) follows from Chapter 2, Corollary 4.4.6.   □

1.2.3.   The following assertion will be used extensively in Chapter 7:

COROLLARY 1.2.4.   *For* $\mathcal{Y} \in \mathrm{FormMod}_{/X}$, *the map*

$$\operatorname*{colim}_{(Z,f)} f_*^{\mathrm{IndCoh}}(\omega_Z) \to \omega_{\mathcal{Y}}$$

is an isomorphism, where the colimit is taken over the category

$$((^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ to}\ X})_{/\mathcal{Y}}.$$

PROOF. By Proposition 1.2.2, the functor

$$((^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ to}\ X})_{/\mathcal{Y}} \to (^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$$

is cofinal. Hence, the restriction functor

$$\mathrm{IndCoh}(\mathcal{Y}) \to \lim_{(Z,f)} \mathrm{IndCoh}(Z)$$

is an isomorphism, where the limit is taken over the category $((^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ to}\ X})_{/\mathcal{Y}}$.

By Chapter 3, Corollary 4.3.4 and Volume I, Chapter 1, Proposition 2.5.7, we obtain that the functors $f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Z) \to \mathrm{IndCoh}(\mathcal{Y})$ define an equivalence

$$\operatorname*{colim}_{(Z,f)} \mathrm{IndCoh}(Z) \to \mathrm{IndCoh}(\mathcal{Y}),$$

where the colimit is taken with respect to the $(\mathrm{IndCoh}, *)$-direct image functors.

In particular, we obtain that

$$\omega_{\mathcal{Y}} \simeq \operatorname*{colim}_{(Z,f)} f_*^{\mathrm{IndCoh}} \circ f^!(\omega_{\mathcal{Y}}) \simeq \operatorname*{colim}_{(Z,f)} f_*^{\mathrm{IndCoh}}(\omega_Z),$$

as required.

□

1.3. Formal moduli problems *under* a prestack. In this subsection we consider a prestack

$$\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}.$$

We will consider another paradigm for formal moduli problems, by looking at prestacks *under* $\mathcal{X}$.

1.3.1.   We define the category $\mathrm{FormMod}_{\mathcal{X}/}$ to be the full subcategory of $(\mathrm{PreStk}_{\mathrm{laft}})_{\mathcal{X}/}$ spanned by those $\mathcal{X} \to \mathcal{Y}$, for which:

   - $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$;
   - The map $\mathcal{X} \to \mathcal{Y}$ is a nil-isomorphism.

Note that since in the above definition, the map $\mathcal{X} \to \mathcal{Y}$ is automatically inf-schematic, and so realizes $\mathcal{X}$ as an object of $\mathrm{FormMod}_{/\mathcal{Y}}$.

1.3.2.   Let $\mathcal{X}'$ be an object of $\mathrm{FormMod}_{\mathcal{X}/}$. Note that the category

$$(\mathrm{FormMod}_{\mathcal{X}/})_{/\mathcal{X}'}$$

identifies with

$$\mathrm{FormMod}_{\mathcal{X}//\mathcal{X}'}$$

from Sect. 1.1.2.

1.3.3.   Note that when $\mathcal{X} = \mathrm{pt}$, there is no difference between $\mathrm{FormMod}_{\mathcal{X}/}$ and $\mathrm{FormMod}_{/\mathcal{X}}$.

1.4. Formal moduli problems *under* an affine scheme. In this subsection we specialize to the case when

$$\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}.$$

We will show that a formal moduli problem $\mathcal{Y}$ under X, viewed as a functor

$$({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc},$$

is completely determined by its value on the category of affine schemes Z, equipped with a nil-isomorphism $X \to Z$.

1.4.1.   Let

$$({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\text{nil-isom from } X} \subset ({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{X/}$$

be the full subcategory formed by those $f : X \to Z$, for which f is a nil-isomorphism.

For a prestack $\mathcal{Y}$ under X, consider the functor

$$\mathcal{Y}|_{({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\text{nil-isom from } X}} \underset{\mathrm{Maps}(X,\mathcal{Y})}{\times} * : ({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\text{nil-isom from } X} \to \mathrm{Spc}$$

that sends $X \to S$ to

$$\mathrm{Maps}(S, \mathcal{Y}) \underset{\mathrm{Maps}(X,\mathcal{Y})}{\times} *.$$

We claim:

PROPOSITION 1.4.2.
(a) *For $\mathcal{Y} \in \mathrm{FormMod}_{X/}$, viewed as a functor*

$$({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc},$$

the map

$$\mathrm{LKE}_{(({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\text{nil-isom from } X})^{\mathrm{op}} \to ({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})^{\mathrm{op}}} (\mathcal{Y}|_{({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\text{nil-isom from } X}} \underset{\mathrm{Maps}(X,\mathcal{Y})}{\times} *) \to \mathcal{Y}$$

is an isomorphism.

(b) *Let $\mathcal{Y}_{\text{nil-isom}}$ be a presheaf on $({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\text{nil-isom from } X}$, satisfying:*
   - *$\mathcal{Y}_{\text{nil-isom}}(X) = *$;*
   - *For a push-out diagram $S_1 \underset{S}{\sqcup} S'$ in $({}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\text{nil-isom from } X}$, where $S \to S'$ has a structure of square-zero extension, the resulting map*

$$\mathcal{Y}_{\text{nil-isom}}(S_1 \underset{S}{\sqcup} S') \to \mathcal{Y}_{\text{nil-isom}}(S_1) \underset{\mathcal{Y}_{\text{nil-isom}}(S)}{\times} \mathcal{Y}_{\text{nil-isom}}(S')$$

   is an isomorphism.

Then if $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ denotes the left Kan extension of $\mathcal{Y}_{\mathrm{nil\text{-}isom}}$ under

$$((^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{nil\text{-}isom\ from}\ X})^{\mathrm{op}} \to (^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}},$$

then the canonical map $X \to \mathcal{Y}$ makes $\mathcal{Y}$ into an object of $\mathrm{FormMod}_{X/}$.

(c) *The assignments*

$$\mathcal{Y} \mapsto \mathcal{Y}|_{(^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{nil\text{-}isom\ from}\ X}} \underset{\mathrm{Maps}(X,\mathcal{Y})}{\times} * \ \text{and}\ \mathcal{Y}_{\mathrm{nil\text{-}isom}} \mapsto \mathcal{Y}$$

are mutually inverse equivalences of categories.

PROOF. Applying Chapter 2, Corollary 4.3.4, Proposition 4.4.5 and Corollary 4.4.6, it is enough to evaluate the functor $\mathcal{Y}$ as in the proposition on the subcategory

$$(^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}) \underset{^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\times} \{^{\mathrm{red}}X\}.$$

The assertion of the proposition follows now from the fact that the forgetful functor

$$(^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{nil\text{-}isom\ from}\ X} \to (^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}) \underset{^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}{\times} \{^{\mathrm{red}}X\}$$

admits a left adjoint, given by

$$S \mapsto S \underset{^{\mathrm{red}}X}{\sqcup} X.$$

$\square$

1.4.3.   As a corollary, we obtain:

COROLLARY 1.4.4. *For $\mathcal{Y} \in \mathrm{FormMod}_{X/}$, the map*

$$\underset{(Z,f)}{\mathrm{colim}}\ f_*^{\mathrm{IndCoh}}(\omega_Z) \to \omega_\mathcal{Y}$$

is an isomorphism, where the colimit is taken over the category

$$((^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ from}\ X})_{/\mathcal{Y}}.$$

PROOF. Same as that of Corollary 1.2.4. $\square$

1.5. The pointed case. For $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ we consider the category

$$\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) = \mathrm{FormMod}_{\mathcal{X}//\mathcal{X}}$$

of pointed objects in $\mathrm{FormMod}_{/\mathcal{X}}$.

1.5.1.   By definition, $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ is the category of diagrams

$$(\pi : \mathcal{Y} \rightleftarrows \mathcal{X} : s), \quad \pi \circ s = \mathrm{id}$$

with the map π being an inf-schematic nil-isomorphism.

Note also that if $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$, then

$$\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \simeq (\mathrm{Ptd}(\mathrm{FormMod}_{\mathcal{X}/}))_{/\mathcal{X}}.$$

Combining Propositions 1.2.2 and 1.4.2, we obtain:

COROLLARY 1.5.2. *For* $X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ *we have:*

(a) *Any* $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$, *viewed as a functor*

$$(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/X})^{\mathrm{op}} \to \mathrm{Spc},$$

receives an isomorphism from the left Kan extension along

$$(\mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\text{nil-isom to }X}))^{\mathrm{op}} \to (({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/X})^{\mathrm{op}}$$

of

$$\mathcal{Y}_{\mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\text{nil-isom to }X})} \underset{\mathrm{Maps}(X,\mathcal{Y})}{\times} *.$$

(b) *Let* $\mathcal{Y}_{\text{nil-isom}}$ *be a presheaf on* $\mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\text{nil-isom to }X})$, *satisfying:*

- $\mathcal{Y}_{\text{nil-isom}}(X) = *$;
- *For a push-out diagram* $S_1 \underset{S}{\sqcup} S'$ *in* $\mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\text{nil-isom to }X})$, *where* $S \to S'$ *has a structure of square-zero extension, the resulting map*

  $$\mathcal{Y}_{\text{nil-isom}}(S_1 \underset{S}{\sqcup} S') \to \mathcal{Y}_{\text{nil-isom}}(S_1) \underset{\mathcal{Y}_{\text{nil-isom}}(S)}{\times} \mathcal{Y}_{\text{nil-isom}}(S')$$

  is an isomorphism.

  Then if $\mathcal{Y} \in (\mathrm{PreStk}_{\mathrm{laft}})_{/X}$ *denotes the left Kan extension of* $\mathcal{Y}_{\text{nil-isom}}$ *along*

  $$(\mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\text{nil-isom to }X}))^{\mathrm{op}} \to (({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/X})^{\mathrm{op}},$$

then the canonical map $X \to \mathcal{Y}$ *makes* $\mathcal{Y}$ *into an object of* $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$.

(c) *The assignments*

$$\mathcal{Y} \mapsto \mathcal{Y}|_{\mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\text{nil-isom to }X})} \underset{\mathrm{Maps}(X,\mathcal{Y})}{\times} * \ and \ \mathcal{Y}_{\text{nil-isom}} \mapsto \mathcal{Y}$$

are mutually inverse equivalences of categories.

REMARK 1.5.3. The informal meaning of this corollary is that in order to 'know' an object $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$ as a prestack, it is is enough to test it on affine schemes S, equipped with a nil-isomorphism to X *and* a section of this nil-isomorphisms.

1.5.4.   As in the case of Corollary 1.2.4, from Corollary 1.5.2, we obtain:

COROLLARY 1.5.5. *For* $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$, *the map*

$$\underset{(Z,f)}{\mathrm{colim}}\, f_*^{\mathrm{IndCoh}}(\omega_Z) \to \omega_{\mathcal{Y}}$$

is an isomorphism, where the colimit is taken over the category

$$(\mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}})_{\text{nil-isom to }X}))_{/\mathcal{Y}}.$$

1.6. Formal groups. In this subsection we let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\mathrm{laft}}$, and we introduce formal groups over $\mathcal{X}$ as group-objects in the category of formal moduli problems.

In Chapter 7 we will show that the category of formal groups identifes with that of Lie algebras in $\mathrm{IndCoh}(\mathcal{X})$, thereby generalizing the main result in [**Lu6**].

1.6.1.   Let $\mathrm{Monoid}(\mathrm{FormMod}_{/\mathcal{X}})$ be the category of monoid-objects in $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$, and let

$$\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \subset \mathrm{Monoid}(\mathrm{FormMod}_{/\mathcal{X}})$$

be the full subcategory spanned by group-like objects.

LEMMA 1.6.2.   *The inclusion* $\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \subset \mathrm{Monoid}(\mathrm{FormMod}_{/\mathcal{X}})$ *is an equivalence.*

PROOF.   We need to show that for $\mathcal{H} \in \mathrm{Monoid}(\mathrm{FormMod}_{/\mathcal{X}})$ the map

$$\mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{H} \to \mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{H}, \quad (h_1, h_2) \mapsto (h_1, h_1 \cdot h_2)$$

is an isomorphism.

This follows from Chapter 1, Corollary 8.3.6, applied to the Cartesian square

$$
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\ h \mapsto (1,h)\ } & \mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{H} \\
{\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle (h_1,h_2) \mapsto (h_1, h_1 \cdot h_2)} \\
\mathcal{H} & \xrightarrow{\ h \mapsto (1,h)\ } & \mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{H}.
\end{array}
$$

$\square$

1.6.3.   We have a naturally defined functor

$$(1.2) \qquad \Omega_{\mathcal{X}} : \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}), \quad \mathcal{Y} \mapsto \mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X}.$$

In Sect. 2.3.4 we will prove:

THEOREM 1.6.4.   *The functor* $\Omega_{\mathcal{X}}$ *of* (1.2) *is an equivalence.*

In what follows we shall denote by $B_{\mathcal{X}}$ the functor

$$\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}),$$

inverse to $\Omega_{\mathcal{X}}$.

1.6.5.   Combining Theorem 1.6.4 with Chapter 7, Corollary 3.6.3 (which will be proved independently), we obtain:

COROLLARY 1.6.6.   *The category* $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ *contains sifted colimits, and the functor*

$$(\mathcal{X} \to \mathcal{Y} \to \mathcal{X}) \mapsto T(\mathcal{X}/\mathcal{Y}) : \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{IndCoh}(\mathcal{X})$$

preserves sifted colimits.

REMARK 1.6.7.   Note, however, that the forgetful functor

$$\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{Ptd}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}})$$

does *not* preserve sifted colimits.

1.6.8.   Assume for a moment that $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Consider the category $\mathrm{FormMod}_{\mathcal{X}/}$. We note that even before we knew that the category $\mathrm{FormMod}_{\mathcal{X}/}$ contains sifted colimits, we could conclude that forgetful functor

$$(\mathcal{X} \to \mathcal{Y} \to \mathcal{X}) \mapsto (\mathcal{X} \to \mathcal{Y}) : \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{FormMod}_{\mathcal{X}/}$$

preserves colimits. This follows from the fact that the above functor admits a right adjoint, given by

$$(\mathcal{X} \to \mathcal{Y}') \mapsto (\mathcal{X} \to \mathcal{X} \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{Y}').$$

2. Groupoids

In this section we introduce the notion of formal groupoid over a given object $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$. We show that if $\mathcal{X}$ admits deformation theory, then there is a well-defined procedure of taking a quotient by a formal groupoid, that produces another object of $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$.

2.1. Digression: groupoids and groups over spaces. In this subsection we review the definition of the notion of *groupoid* acting on a space in the category Spc.

2.1.1.   Given a space X, recall the category $\mathrm{Grpoid}(X)$ of groupoids acting on X (see [**Lu1**], Sect. 6.1.2). By definition, $\mathrm{Grpoid}(X)$ is a full subcategory in the category of simplicial spaces, equipped with an identification of the space of 0-simplices with X. A simplicial object $R^\bullet$ belongs to $\mathrm{Grpoid}(X)$ if the following two conditions are satisfied:

- For every $n \geq 2$, the map

$$R^n \to R^1 \underset{X}{\times} ... \underset{X}{\times} R^1,$$

  given by the product of the maps

$$[1] \to [n], \quad 0 \mapsto i, 1 \mapsto i + 1, \quad i = 0, ..., n - 1,$$

  is an isomorphism. [1]
- The map $R^2 \to R^1 \underset{X}{\times} R^1$, given by the product of the maps $[1] \to [2]$

$$0 \mapsto 0, 1 \mapsto 1 \text{ and } 0 \mapsto 0, 1 \mapsto 2,$$

  is an isomorphism.

We shall symbolically depict a groupoid via a diagram

(2.1)

$$
\begin{array}{ccc}
 & R & \\
p_s \swarrow & & \searrow p_t \\
X & & X,
\end{array}
$$

while properly we should be thinking about the entire simplicial object $R^\bullet$ of Spc with $R^0 = X$ and $R^1 = R$.

The category $\mathrm{Grpoid}(X)$ contains an initial object, the identity groupoid, where all $R^i = X$. We shall denote it by diag_X.

The category $\mathrm{Grpoid}(X)$ also contains a final object, namely $X \times X$.

---

[1]If we impose just this condition, the corresponding category is that of *Segal objects* (a.k.a. *category-objects*) acting on X, denoted $\mathrm{Seg}(X)$.

2.1.2.   The following assertion will be used repeatedly:

LEMMA 2.1.3. *The forgetful functor* $\mathrm{Grpoid}(X) \to \mathrm{Spc}$ *that sends a groupoid to* R^n *for any n preserves sifted colimits.*

PROOF. Follows from the fact that if I is a sifted index category and

$$i \mapsto R_i^\bullet$$

is an I family of objects of $\mathrm{Grpoid}(X)$, the map

$$\operatorname*{colim}_i \left(R_i^1 \underset{X}{\times} ... \underset{X}{\times} R_i^1 \right) \to \left(\operatorname*{colim}_i R_i^1 \right) \underset{X}{\times} ... \underset{X}{\times} \left(\operatorname*{colim}_i R_i^1 \right)$$

is an isomorphism.

$\square$

2.1.4.   Given a groupoid R acting on X we can consider the quotient space

$$X/R := |R^\bullet|,$$

which receives a natural map from X:

$$X = R^0 \to |R^\bullet| \to X/R.$$

Vice versa, given a space Y under X, we construct the groupoid over X by $R := X \underset{Y}{\times} X$, i.e., $R^\bullet$ is the Čech nerve of the above map $X \to Y$.

It is clear that the two functors

$$\mathrm{Grpoid}(X) \rightleftarrows \mathrm{Spc}_{X/}$$

are adjoint to one another.

We have:

LEMMA 2.1.5. *The above two functors define equivalences between the category* $\mathrm{Grpoid}(X)$ *and the full subcategory* $\mathrm{Spc}_{X/,\mathrm{surj}}$ *of* $\mathrm{Spc}_{X/}$ *spanned by objects* $i : X \to Y$, *for which the map* i *is surjective on* π_0.

2.1.6.   Given a space X, consider the category

$$\mathrm{Grp}(\mathrm{Spc}_{/X}),$$

of group-objects in the category of spaces over X. We have:

$$\mathrm{Grp}(\mathrm{Spc}_{/X}) \simeq \mathrm{Grpoid}(X)_{/\mathrm{diag}_X},$$

where $\mathrm{diag}_X \in \mathrm{Grpoid}(X)$ is the identity groupoid.

Consider also the category $\mathrm{Ptd}(\mathrm{Spc}_{/X})$, which is the same as the category of of retraction diagrams

(2.2) $i : X \rightleftarrows Y : s, \ s \circ i \simeq \mathrm{id}_X .$

We will also use the notation

$$\mathrm{Spc}_{X//X}$$

for the above category.

We have a natural functor

(2.3) $\mathrm{Grp}(\mathrm{Spc}_{/X}) \to \mathrm{Ptd}(\mathrm{Spc}_{/X}),$

given by

$$G \mapsto B_X(G),$$

where $B_X(G)$ is the relative classifying space over X.

We also have the adjoint loop functor

$$\Omega_X : \mathrm{Ptd}(\mathrm{Spc}_{/X}) \to \mathrm{Grp}(\mathrm{Spc}_{/X}).$$

LEMMA 2.1.7. *The functors B_X and Ω_X define an equivalence between* $\mathrm{Grp}(\mathrm{Spc}_{/X})$ *and the full subcategory* $\mathrm{Ptd}(\mathrm{Spc}_{/X})_{\mathrm{isom}}$ *of* $\mathrm{Ptd}(\mathrm{Spc}_{/X})$, *spanned by those objects*

$$i : X \rightleftarrows Y : s$$

for which the map i is an isomorphism on π_0.

Note that the condition on an object $(i : X \rightleftarrows Y : s) \in \mathrm{Ptd}(\mathrm{Spc}_{/X})$ to belong to

$$\mathrm{Ptd}(\mathrm{Spc}_{/X})_{\mathrm{isom}} \subset \mathrm{Ptd}(\mathrm{Spc}_{/X})$$

is equivalent to the map i being a surjection on π_0.

2.2. Groupoids in formal geometry. In this subsection we render the set-up of Sect. 2.1 into the context of algebraic geometry.

2.2.1.   The definitions of Sect. 2.1 carry over automatically to the algebro-geometric setting. For $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$, we consider the categories

$$(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}, \ \mathrm{Ptd}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}), \ \mathrm{Grp}((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}), \ \mathrm{Grpoid}_{\mathrm{laft}}(\mathcal{X}), \ \text{ and } \ \mathrm{Seg}_{\mathrm{laft}}(\mathcal{X})$$

and their full subcategories

$$\mathrm{FormMod}_{/\mathcal{X}}, \ \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}), \ \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}), \ \mathrm{FormGrpoid}(\mathcal{X}), \ \text{ and } \ \mathrm{FormSeg}(\mathcal{X})$$

formed by objects that are formal as prestacks over $\mathcal{X}$.

Note, however, that as in Lemma 1.6.2, one shows that the inclusion

$$\mathrm{FormGrpoid}(\mathcal{X}) \hookrightarrow \mathrm{FormSeg}(\mathcal{X})$$

is an equivalence.

2.2.2. *Examples.* Let $\mathcal{X} \to \mathcal{Y}$ be a map between objects of $\mathrm{PreStk}_{\mathrm{laft}}$. To it we attach the object of $\mathrm{Grpoid}_{\mathrm{laft}}(\mathcal{X})$, namely, the Čech nerve of this map. Thus, the corresponding

$$\mathcal{R} \rightrightarrows \mathcal{X}$$

is given by

$$\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X}.$$

If the above map $\mathcal{X} \to \mathcal{Y}$ is an inf-schematic nil-isomorphism, then $\mathcal{R} \in \mathrm{FormGrpoid}(\mathcal{X})$.

2.2.3.   As in Lemma 2.1.3, we obtain:

COROLLARY 2.2.4. *The category* $\mathrm{FormGrpoid}(\mathcal{X})$ *contains sifted colimits, and the functors*

$$\mathrm{FormGrpoid}(\mathcal{X}) \to \mathrm{FormMod}_{\mathcal{X}/}, \quad \mathrm{FormGrpoid}(\mathcal{X}) \rightrightarrows \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$$

and

$$\mathrm{FormGrpoid}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X})$$

that send a groupoid to

$$(\mathcal{X} \overset{\mathrm{unit}}{\to} \mathcal{R}), \quad (\mathcal{X} \underset{p_s}{\overset{\mathrm{unit}}{\rightleftarrows}} \mathcal{R}), \quad (\mathcal{X} \underset{p_t}{\overset{\mathrm{unit}}{\rightleftarrows}} \mathcal{R}),$$

and $T(\mathcal{X}/\mathcal{R})$, *respectively, commute with sifted colimits.*

PROOF. Let I be a filtered index category and

$$i \mapsto \mathcal{R}_i^{\bullet}$$

be an I-family of objects of $\mathrm{FormGrpoid}(\mathcal{X})$. It follows from Corollary 1.6.6, Sect. 1.6.8 and Chapter 1, Proposition 8.3.2 that for any n, the colimit

$$\underset{i}{\mathrm{colim}}\,(\mathcal{R}_i^1 \underset{\mathcal{X}}{\times} ... \underset{\mathcal{X}}{\times} \mathcal{R}_i^1)$$

exists and the map

$$\underset{i}{\mathrm{colim}}\,(\mathcal{R}_i^1 \underset{\mathcal{X}}{\times} ... \underset{\mathcal{X}}{\times} \mathcal{R}_i^1) \to (\underset{i}{\mathrm{colim}}\,\mathcal{R}_i^1) \underset{\mathcal{X}}{\times} ... \underset{\mathcal{X}}{\times} (\underset{i}{\mathrm{colim}}\,\mathcal{R}_i^1)$$

is an isomorphism.

□

2.2.5. *Ind-coherent sheaves equivariant with respect to a groupoid.* Let $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$; let $\mathcal{R}$ be an object of $\mathrm{FormGrpoid}(\mathcal{X})$, and let $\mathcal{R}^{\bullet}$ be the corresponding simplicial object of $\mathrm{PreStk}_{\mathrm{laft}}$. We define the category

$$\mathrm{IndCoh}(\mathcal{X})^{\mathcal{R}}$$

of *ind-coherent sheaves equivariant with respect to a* $\mathcal{R}$ to be

$$\mathrm{Tot}(\mathrm{IndCoh}(\mathcal{R}^{\bullet})).$$

By Chapter 3, Proposition 3.3.3(b), we have:

PROPOSITION 2.2.6. *Let* $\mathcal{R}$ *be the formal groupoid corresponding to a map* $\mathcal{X} \to \mathcal{Y}$ *in* $\mathrm{PreStk}_{\mathrm{laft}}$, *which is an inf-schematic nil-isomorphism* $\mathcal{X} \to \mathcal{Y}$. *Then the pullback functor defines an equivalence*

$$\mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{X})^{\mathcal{R}}.$$

2.3. Taking the quotient by a formal groupoid. In this subsection we state one of the main results of this book: namely that in the world of prestacks admitting deformation theory there is a well-defined procedure of taking the quotient by a formal groupoid.

2.3.1.   Assume that $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Recall the category $\mathrm{FormMod}_{\mathcal{X}/}$, see Sect. 1.3.1. We have a naturally defined functor

(2.4) $\mathrm{FormMod}_{\mathcal{X}/} \to \mathrm{FormGrpoid}(\mathcal{X})$,

namely, $\mathcal{Y} \mapsto \mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X}$, see Sect. 2.2.2.

The main result of this section is the following:

THEOREM 2.3.2. *The functor* (2.4) *is an equivalence.*

2.3.3. *An example.* Let $\mathcal{X} \to \mathcal{Y}$ be a map in $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Consider the formal completion $\mathcal{Y}^{\wedge}_{\mathcal{X}}$ of $\mathcal{Y}$ along $\mathcal{X}$, i.e.,

$$\mathcal{Y}^{\wedge}_{\mathcal{X}} := \mathcal{X}_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}.$$

Then the map $\mathcal{X} \to \mathcal{Y}^{\wedge}_{\mathcal{X}}$ defines an object of $\mathrm{FormMod}_{\mathcal{X}/}$.

Consider the groupoid

$$\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X},$$

(see Sect. 2.2.2) and its formal completion along the diagonal map,

$$(\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X})^{\wedge}.$$

It is easy to see that

$$(\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X})^{\wedge} \simeq \mathcal{X} \underset{\mathcal{Y}^{\wedge}_{\mathcal{X}}}{\times} \mathcal{X},$$

where the latter is an object of $\mathrm{FormGrpoid}(\mathcal{X})$ by Sect. 2.2.2.

Thus, the formal completion $\mathcal{Y}^{\wedge}_{\mathcal{X}}$ can be recovered from $(\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X})^{\wedge}$ by taking the functor inverse to that in Theorem 2.3.2.

2.3.4.   Note that Theorem 2.3.2 implies Theorem 1.6.4:

PROOF. To prove Theorem 1.6.4 we can assume that $\mathcal{X} \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$. In particular, we can assume that $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Then the required assertion follows from Theorem 2.3.2 by noting that

$$\mathrm{Grp}(\mathrm{FormMod}_{\mathcal{X}}) = \mathrm{FormGrpoid}(\mathcal{X})_{/\,\mathrm{diag}_{\mathcal{X}}},$$

and

$$\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) = (\mathrm{FormMod}_{\mathcal{X}/})_{/\mathcal{X}}.$$

$\square$

2.3.5.   As another formal consequence of Theorem 2.3.2, combined with Corollary 2.2.4, we obtain:

COROLLARY 2.3.6. *The category* $\mathrm{FormMod}_{\mathcal{X}/}$ *contains sifted colimits, and the functor*

$$\mathrm{FormMod}_{\mathcal{X}/} \to \mathrm{IndCoh}(\mathcal{X}), \quad (\mathcal{X} \to \mathcal{Y}) \mapsto T(\mathcal{X}/\mathcal{Y})$$

commutes with sifted colimits.

We emphasize again that the forgetful functor

$$\mathrm{FormMod}_{\mathcal{X}/} \to (\mathrm{PreStk}_{\mathrm{laft}})_{\mathcal{X}/}$$

does *not* commute with sifted colimits.

However, from Chapter 1, Corollary 7.2.8, we obtain:

COROLLARY 2.3.7. *The forgetful functor*

$$\mathrm{FormMod}_{\mathcal{X}/} \to (\mathrm{PreStk}_{\mathrm{laft}})_{\mathcal{X}/}$$

commutes with filtered *colimits*.

2.4. Constructing the classifying space of a groupoid. In this subsection we will begin the proof of Theorem 2.3.2. In fact, we will explicitly construct the inverse functor.

2.4.1. For $\mathcal{R} \in \mathrm{FormGrpoid}(\mathcal{X})$ we define an object $B_{\mathcal{X}}(\mathcal{R}) \in \mathrm{PreStk}_{\mathrm{laft}}$ as follows:

For $Z \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, we let $\mathrm{Maps}(Z, B_{\mathcal{X}}(\mathcal{R}))$ be the groupoid consisting of the following data:

$$\{(\widetilde{\mathcal{Z}} \to Z) \in \mathrm{FormMod}_{/Z},\ \widetilde{\mathcal{Z}} \to \mathcal{X},\ \text{a map of groupoids}\ \widetilde{\mathcal{Z}} \underset{Z}{\times} \widetilde{\mathcal{Z}} \to \mathcal{R}\},$$

where we require that the diagram

$$
\begin{array}{ccc}
\widetilde{\mathcal{Z}} \underset{Z}{\times} \widetilde{\mathcal{Z}} & \longrightarrow & \mathcal{R} \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{Z}} & \longrightarrow & \mathcal{X}
\end{array}
$$

be Cartesian, where the vertical arrows are either of the projections.

2.4.2. We have a tautological map $\mathcal{X} \to B_{\mathcal{X}}(\mathcal{R})$ that sends $Z \to \mathcal{X}$ to

$$\widetilde{\mathcal{Z}} := Z \underset{\mathcal{X}}{\times} \mathcal{R},$$

where the fiber product $Z \underset{\mathcal{X}}{\times} \mathcal{R}$ is formed using the map $p_1 : \mathcal{R} \to \mathcal{X}$, and the map $Z \underset{\mathcal{X}}{\times} \mathcal{R} \to \mathcal{X}$ corresponds to $p_2 : \mathcal{R} \to \mathcal{X}$.

2.4.3. Let us show that the map $\mathcal{X} \to B_{\mathcal{X}}(\mathcal{R})$ makes $\mathcal{X}$ into an object of $\mathrm{FormMod}_{/B_{\mathcal{X}}(\mathcal{R})}$. Indeed, for a given map $Z \to B_{\mathcal{X}}(\mathcal{R})$, the fiber product

$$Z \underset{B_{\mathcal{X}}(\mathcal{R})}{\times} \mathcal{X}$$

identifies with $\widetilde{\mathcal{Z}}$.

The latter observation also implies that

$$(2.5) \qquad\qquad \mathcal{X} \underset{B_{\mathcal{X}}(\mathcal{R})}{\times} \mathcal{X} \simeq \mathcal{R}.$$

2.4.4. We claim that it suffices to show that the object $B_{\mathcal{X}}(\mathcal{R})$ belongs to $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Indeed, let us assume this for the moment and conclude the proof of the theorem.

First, (2.5) implies that the construction

$$(2.6) \qquad\qquad \mathcal{R} \mapsto B_{\mathcal{X}}(\mathcal{R})$$

is a right inverse to the functor (2.4). In particular, the functor (2.4) is essentially surjective.

For $\mathcal{Y} \in \mathrm{FormMod}_{\mathcal{X}/}$ we have a tautological map

$$(2.7) \qquad\qquad \mathcal{Y} \to B_{\mathcal{X}}(\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X}),$$

given by $(\mathcal{X} \to \mathcal{Y}) \in \mathrm{FormMod}_{/\mathcal{Y}}$, which becomes an isomorphism after applying the functor (2.4). Hence, by Chapter 1, Proposition 8.3.2, the map (2.7) is an

isomorphism. Hence, the construction (2.6) is also the left inverse of the functor (2.4).

We will now show that the functor (2.4) is fully faithful, thereby finishing the proof of Theorem 2.3.2. (The caveat here is that it is not clear a priori that the construction (2.6) is a functor[2].)

2.4.5.   Given $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathrm{FormMod}_{\mathcal{X}/}$, we will explicitly construct an inverse to the map

$$(2.8) \qquad \mathrm{Maps}_{\mathrm{FormMod}_{\mathcal{X}/}}(\mathcal{Y}_1, \mathcal{Y}_2) \to \mathrm{Maps}_{\mathrm{FormGrpoid}(\mathcal{X})}(\mathcal{R}_1, \mathcal{R}_2),$$

where $\mathcal{R}_i = \mathcal{X} \underset{\mathcal{Y}_i}{\times} \mathcal{X}$. Note that we already know that $\mathcal{Y}_i \simeq B_{\mathcal{X}}(\mathcal{R}_i)$.

First, it follows from the construction (2.6) that for $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft-def}}$, the groupoid of maps $\mathcal{Z} \to B_{\mathcal{X}}(\mathcal{R})$ admits the same description as in Sect. 2.4.1 with Z replaced by $\mathcal{Z}$.

Thus, given a point in $\mathrm{Maps}_{\mathrm{FormGrpoid}(\mathcal{X})}(\mathcal{R}_1, \mathcal{R}_2)$, we need to produce an object $\widetilde{B}_{\mathcal{X}}(\mathcal{R}_1) \in \mathrm{FormGrpoid}_{/B_{\mathcal{X}}(\mathcal{R}_1)}$, a map $\widetilde{B}_{\mathcal{X}}(\mathcal{R}_1) \to \mathcal{X}$ and a map of groupoids

$$\widetilde{B}_{\mathcal{X}}(\mathcal{R}_1) \underset{B_{\mathcal{X}}(\mathcal{R}_1)}{\times} \widetilde{B}_{\mathcal{X}}(\mathcal{R}_1) \to \mathcal{R}_2,$$

making the corresponding diagrams Cartesian.

Note, however, that $\mathcal{R}_1 \underset{\mathcal{X}}{\times} \mathcal{R}_2$ can be viewed as a groupoid acting on $\mathcal{R}_2$. We set

$$\widetilde{B}_{\mathcal{X}}(\mathcal{R}_1) := B_{\mathcal{R}_2}(\mathcal{R}_1 \underset{\mathcal{X}}{\times} \mathcal{R}_2).$$

It is easy to check that $\widetilde{B}_{\mathcal{X}}(\mathcal{R}_1)$ has all the required pieces of structure. Moreover, it follows from the construction that the resulting map

$$\mathrm{Maps}_{\mathrm{FormGrpoid}(\mathcal{X})}(\mathcal{R}_1, \mathcal{R}_2) \to \mathrm{Maps}_{\mathrm{FormMod}_{\mathcal{X}/}}(\mathcal{Y}_1, \mathcal{Y}_2)$$

is indeed the inverse of (2.8).

2.5. Verification of deformation theory. In this subsection we will prove that the object $B_{\mathcal{X}}(\mathcal{R}) \in \mathrm{PreStk}_{\mathrm{laft}}$ constructed in Sect. 2.4.1, admits deformation theory.

2.5.1.   Let Z be an object of $Z \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, equipped with a map to $B_{\mathcal{X}}(\mathcal{R})$. We will now construct a certain object of $\mathrm{Pro}(\mathrm{QCoh}(Z)^-)_{\mathrm{laft}}$, which we will later identify with the pro-cotangent space to $B_{\mathcal{X}}(\mathcal{R})$ at our given point $Z \to B_{\mathcal{X}}(\mathcal{R})$.

Consider the Čech nerve $\widetilde{\mathcal{Z}}^{\bullet}$ of the corresponding map $\widetilde{\mathcal{Z}} \to Z$, and consider the resulting map of simplicial prestacks

$$\widetilde{\mathcal{Z}}^{\bullet} \to \mathcal{R}^{\bullet}.$$

Let

$$T^*(\widetilde{\mathcal{Z}}^{\bullet}/\mathcal{R}^{\bullet}) \in \mathrm{Tot}\left(\mathrm{Pro}(\mathrm{QCoh}(\widetilde{\mathcal{Z}}^{\bullet})^-)^{\mathrm{fake}}_{\mathrm{laft}}\right)$$

be the corresponding relative pro-cotangent complex (see Chapter 1, Sect. 4.3.1), which receives a canonically defined map from the pullback of $T^*(Z)$.

_____

[2]We are grateful to Y. Zhao for pointing this out to us.

By nil-descent for $\mathrm{Pro}(\mathrm{QCoh}(-)^-)_{\mathrm{laft}}^{\mathrm{fake}}$ with respect to $\widetilde{\mathcal{Z}} \to Z$ (see Chapter 3, Corollary 3.3.5), we obtain that $T^*(\widetilde{\mathcal{Z}}^\bullet/\mathcal{R}^\bullet)$ gives rise to a canonically defined object, denoted,

$$'T^*(Z/B_{\mathcal{X}}(\mathcal{R})) \in \mathrm{Pro}(\mathrm{QCoh}(Z)^-)_{\mathrm{laft}},$$

which receives a map from $T^*(Z)$. Set

$$'T^*(B_{\mathcal{X}}(\mathcal{R}))|_Z := \mathrm{Fib}\left(T^*(Z) \to {'T^*}(Z/B_{\mathcal{X}}(\mathcal{R}))\right).$$

We will show that the above object

$$'T^*(B_{\mathcal{X}}(\mathcal{R}))|_Z \in \mathrm{Pro}(\mathrm{QCoh}(Z)^-)_{\mathrm{laft}},$$

identifies with the pro-cotangent space of $B_{\mathcal{X}}(\mathcal{R})$ at the above point $Z \to B_{\mathcal{X}}(\mathcal{R})$.

2.5.2.   We need to show that, given a square-zero extension $Z \hookrightarrow Z'$, corresponding to

$$\gamma : T^*(Z) \to \mathcal{I}[1], \quad \mathcal{I} \in \mathrm{Coh}(Z)^{\leq 0},$$

the groupoid of extensions of the initial map $Z \to B_{\mathcal{X}}(\mathcal{R})$ to a map $Z' \to B_{\mathcal{X}}(\mathcal{R})$, identifies canonically with groupoid of factorizations of γ as

$$T^*(Z) \to {'T^*}(Z/B_{\mathcal{X}}(\mathcal{R}))|_Z \to \mathcal{I}[1].$$

This will show that $B_{\mathcal{X}}(\mathcal{R})$ admits pro-cotangent spaces that are indeed identified with ones constructed in Sect. 2.5.1, and that $B_{\mathcal{X}}(\mathcal{R})$ is infinitesimally cohesive. The fact that $B_{\mathcal{X}}(\mathcal{R})$ admits a pro-cotangent complex (i.e., that the formation of pro-cotangent spaces is compatible with pullback) will follow from the construction in Sect. 2.5.1.

2.5.3.   For $Z \hookrightarrow Z'$ as above, by Chapter 1, Proposition 10.3.5, the datum of a prestack $\mathcal{Z}' \to Z'$, equipped with a Cartesian diagram

$$
\begin{array}{ccc}
\widetilde{\mathcal{Z}} & \longrightarrow & \widetilde{\mathcal{Z}}' \\
\downarrow & & \downarrow \\
Z & \longrightarrow & Z'
\end{array}
$$

is equivalent to that of a map $T^*(\widetilde{\mathcal{Z}}) \to \mathcal{I}|_{\widetilde{\mathcal{Z}}}[1]$ (in the category $\mathrm{Pro}(\mathrm{QCoh}(\widetilde{\mathcal{Z}})^-)_{\mathrm{laft}}^{\mathrm{fake}}$), and a homotopy between the composition

$$T^*(Z)|_{\widetilde{\mathcal{Z}}} \to T^*(\widetilde{\mathcal{Z}}) \to \mathcal{I}|_{\widetilde{\mathcal{Z}}}[1]$$

and $\gamma|_{\widetilde{\mathcal{Z}}}$. Moreover, by Chapter 1, Proposition 10.4.2, such $\mathcal{Z}'$ is automatically an inf-scheme.

The same discussion applies to each term of the Čech nerve $\widetilde{\mathcal{Z}}^\bullet$.

Furthermore, by Chapter 1, Proposition 10.2.6 the datum of a compatible system of maps from the Čech nerve $\widetilde{\mathcal{Z}}'^\bullet$ to $\mathcal{R}^\bullet$, extending the initial system $\widetilde{\mathcal{Z}}^\bullet \to \mathcal{R}^\bullet$, is equivalent to a compatible system of factorizations of the resulting maps

$$T^*(\widetilde{\mathcal{Z}}^\bullet) \to \mathcal{I}|_{\widetilde{\mathcal{Z}}\bullet}[1]$$

as

$$T^*(\widetilde{\mathcal{Z}}^\bullet) \to T^*(\widetilde{\mathcal{Z}}^\bullet/\mathcal{R}^\bullet) \to \mathcal{I}|_{\widetilde{\mathcal{Z}}\bullet}[1].$$

2.5.4. Hence, we obtain that the datum of extension of the initial map $Z \to B_{\mathcal{X}}(\mathcal{R})$ to a map $Z' \to B_{\mathcal{X}}(\mathcal{R})$ is equivalent to that of a compatible family of maps

$$T^*(\widetilde{\mathcal{Z}}^\bullet/\mathcal{R}^\bullet) \to \mathcal{I}|_{\widetilde{\mathcal{Z}}^\bullet}[1],$$

and homotopies between

$$T^*(Z)|_{\widetilde{\mathcal{Z}}^\bullet} \to T^*(\widetilde{\mathcal{Z}}^\bullet/\mathcal{R}^\bullet) \to \mathcal{I}|_{\widetilde{\mathcal{Z}}^\bullet}[1]$$

and $\gamma|_{\widetilde{\mathcal{Z}}^\bullet}$.

By nil-descent for $\mathrm{Pro}(\mathrm{QCoh}(-)^-)^{\mathrm{fake}}_{\mathrm{laft}}$ with respect to $\widetilde{\mathcal{Z}} \to Z$, the latter datum is equivalent to that of factorizations of γ as

$$T^*(Z) \to {}'T^*(Z/B_{\mathcal{X}}(\mathcal{R}))|_Z \to \mathcal{I}[1],$$

as desired.

Lie algebras and co-commutative co-algebras

Introduction

0.1. Why does this chapter exist? Only a small portion of this chapter consists of original mathematics: if anything, it would be Theorem 6.1.2 (that expresses the functor of universal enveloping Lie algebra in terms of the Chevalley functor), and perhaps also Theorem 2.9.4 (that computes primitives in 'fake' co-free co-algebras).

Our main intention in writing this chapter was to provide a reference point for Chapter 7, where we will study the relation between moduli problems and Lie algebras.

0.1.1. The main actors in our study of Lie algebras will be the pair of mutually adjoint functors

$$(0.1) \qquad \mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathbf{O}) \rightleftarrows \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) : \mathrm{coChev}^{\mathrm{enh}}$$

that connect Lie algebras and augmented co-commutative co-algebras in a given symmetric monoidal category $\mathbf{O}$. (In our applications in the subsequent chapters we will take $\mathbf{O} = \mathrm{IndCoh}(\mathcal{X})$, where $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$, equipped with the $\overset{!}{\otimes}$ symmetric monoidal structure.)

The difficulty here (and what makes the game interesting) is that the above functors are *not* fully faithful, but they are close to being such.

For example, we conjecture that the unit and the co-unit of the adjunction

$$\mathrm{Id} \to \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Chev}^{\mathrm{enh}} \ \text{ and } \ \mathrm{Chev}^{\mathrm{enh}} \circ \mathrm{coChev}^{\mathrm{enh}} \to \mathrm{Id}$$

become isomorphisms when evaluated on the essential image of $\mathrm{coChev}^{\mathrm{enh}}$ and $\mathrm{Chev}^{\mathrm{enh}}$, respectively.

We will now describe the two main results of this chapter.

0.1.2. One is Theorem 4.2.4, which is a particular case of the more general Theorem 2.9.4. It says that the unit of the adjunction

$$\mathrm{Id} \to \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Chev}^{\mathrm{enh}}$$

is an isomorphism, when evaluated on any trivial Lie algebra.

As a consequence we deduce (see Theorem 4.4.6) that if we precompose the Chevalley functor with the loop functor

$$\Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{Grp}(\mathrm{LieAlg}(\mathbf{O})),$$

and view the result as a functor

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomBialg}(\mathbf{O}),$$

the latter will be fully faithful.

A key observation here is that for a Lie algebra $\mathfrak{h}$, if we view $\Omega_{\mathrm{Lie}}(\mathfrak{h})$ again as a mere Lie algebra (i.e., disregard the Lie algebra structure), then it will be canonically trivialized (see Proposition 1.7.2). The latter result is true for any operad[1].

0.1.3.   The second main result of this chapter is Theorem 6.1.2:

It says that the functor

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomBialg}(\mathbf{O}),$$

considered above identifies canonically with the functor that assigns to a Lie algebra its universal enveloping algebra, considered as a co-commutative Hopf algebra.

0.2. What else is done in this chapter?

0.2.1.   In Sect. 1 we give an overview of the general theory of algebras over operads.

We show that for a given operad $\mathcal{P}$, a $\mathcal{P}$-algebra B can be canonically lifted to *non-negatively filtered* $\mathcal{P}$-algebra B^{Fil}, such that its associated graded is trivial. This construction implies that many functors from the category of $\mathcal{P}$-algebras admit filtered versions, whose associated graded is easy to control.

In addition, we prove the above-mentioned fact that the loop functor followed by the forgetful functor on the category of $\mathcal{P}$-algebras canonically produces *trivial $\mathcal{P}$-algebras*. As an application we give a simple proof of the fact that the stabilization of the category of $\mathcal{P}$-algebras (in a symmetric monoidal DG category $\mathbf{O}$) identifies with $\mathbf{O}$ itself.

0.2.2.   In Sect. 2 we review the theory of Koszul duality between algebras over an operad and co-algebras over the Koszul dual operad.

One of the key points is that there are two *inequivalent* notions of co-algebra over a co-operad. One is the usual notion of co-algebra (which in the example of the co-commutative co-operad corresponds to augmented co-commutative co-algebras). And another is that of *ind-nilpotent* co-algebra. There is a naturally defined functor (denoted $\mathbf{res}^{\star \to *}$) from the category of the latter (denoted $\mathcal{Q}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O})$) to the category of the former (denoted $\mathcal{Q}\text{-}\mathrm{Coalg}(\mathbf{O})$), and we conjecture that this functor is fully faithful.

The forgetful functor $\mathbf{oblv}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}} : \mathcal{Q}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O}) \to \mathbf{O}$ admits a right adjoint, denoted $\mathbf{cofree}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}}$. Composing with the functor

$$\mathbf{res}^{\star \to *} : \mathcal{Q}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O}) \to \mathcal{Q}\text{-}\mathrm{Coalg}(\mathbf{O}),$$

we obtain the functor that we denote by

$$\mathbf{cofree}_{\mathcal{Q}}^{\mathrm{fake}} : \mathbf{O} \to \mathcal{Q}\text{-}\mathrm{Coalg}(\mathbf{O}).$$

For example, for $\mathcal{Q} = \mathrm{Cocom}^{\mathrm{aug}}$, the functor $\mathbf{cofree}_{\mathcal{Q}}^{\mathrm{fake}}$ is the functor of symmetric co-algebra $V \mapsto \mathrm{Sym}(V)$.

---

[1] In this generality we learned this fact, along with its proof, from M. Kontsevich.

If we knew that the functor $\mathbf{res}^{\star \to *}$ was fully faithful, we would know that for $V, W \in \mathbf{O}$ the composite map

$$(0.2) \quad \mathrm{Maps}_{\mathbf{O}}(W, V) \simeq \mathrm{Maps}_{\mathbf{O}}(\mathbf{oblv}_{\mathcal{Q}}^{\text{ind-nilp}} \circ \mathbf{triv}_{\mathcal{Q}}^{\text{ind-nilp}}(W), V) \simeq$$

$$\simeq \mathrm{Maps}_{\mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})}(\mathbf{triv}_{\mathcal{Q}}^{\text{ind-nilp}}(W), \mathbf{cofree}_{\mathcal{Q}}^{\text{ind-nilp}}(V)) \to$$

$$\to \mathrm{Maps}_{\mathcal{Q}\text{-Coalg}(\mathbf{O})}(\mathbf{res}^{\star \to *} \circ \mathbf{triv}_{\mathcal{Q}}^{\text{ind-nilp}}(W), \mathbf{res}^{\star \to *} \circ \mathbf{cofree}_{\mathcal{Q}}^{\text{ind-nilp}}(V)) \simeq$$

$$\simeq \mathrm{Maps}_{\mathcal{Q}\text{-Coalg}(\mathbf{O})}(\mathbf{triv}_{\mathcal{Q}}(W), \mathbf{cofree}_{\mathcal{Q}}^{\text{fake}}(V))$$

is an isomorphism.

Unfortunately, we do not know whether $\mathbf{res}^{\star \to *}$ is fully faithful. However, we prove, and this is one of the key technical assertions, that for a certain class of co-operads (that includes $\mathrm{Cocom}^{\text{aug}}$ and $\mathrm{Coassoc}^{\text{aug}}$) that map in (0.2) is an isomorphism. This is Theorem 2.9.4.

0.2.3. In Sect. 3 we specialize the context of Koszul duality to the case of associative algebras.

0.2.4. In Sect. 4 we prove Theorem 4.4.6, mentioned above, which says that the functor

$$\mathrm{Grp}(\mathrm{coChev}^{\text{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomBialg}(\mathbf{O}),$$

is fully faithful.

We study the functor

$$\mathrm{CocomBialg}(\mathbf{O}) \xrightarrow{\mathrm{Monoid}(\mathrm{coChev}^{\text{enh}})} \mathrm{Monoid}(\mathrm{LieAlg}(\mathbf{O})) \simeq \mathrm{Grp}(\mathrm{LieAlg}(\mathbf{O})) \xrightarrow{B_{\mathrm{Lie}}} \mathrm{LieAlg}(\mathbf{O}),$$

right adjoint to $\mathrm{Grp}(\mathrm{coChev}^{\text{enh}}) \circ \Omega_{\mathrm{Lie}}$.

We show that it fits into a commutative diagram

$$
\begin{array}{ccc}
\mathrm{CocomBialg}(\mathbf{O}) & \xrightarrow{\mathbf{oblv}_{\mathrm{Assoc}}} & \mathrm{CocomCoalg}^{\text{aug}}(\mathbf{O}) \\
{\scriptstyle B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\text{enh}})} \downarrow & & \downarrow {\scriptstyle \mathrm{Prim}_{\mathrm{Cocom}^{\text{aug}}}} \\
\mathrm{LieAlg}(\mathbf{O}) & \xrightarrow{\mathbf{oblv}_{\mathrm{Lie}}} & \mathbf{O}.
\end{array}
$$

I.e., we obtain that when we apply the functor

$$\mathrm{Prim}_{\mathrm{Cocom}^{\text{aug}}} : \mathrm{CocomCoalg}^{\text{aug}}(\mathbf{O}) \to \mathbf{O}$$

to an object of $\mathrm{CocomBialg}(\mathbf{O})$, the result has a natural structure of Lie algebra.

This can be regarded as an 'ultimate explanation' of why the tangent space to a Lie group at the origin has a structure of Lie algebra (one that does not use explicit formulas).

0.2.5. In Sect. 5 we recall the basic constructions associated with the functor of universal enveloping algebra of a Lie algebra.

In Sect. 6 we prove the second main result of this chapter, described in Sect. 0.1.3 above.

In Sect. 7 we give an interpretation of an equivalence

$$\mathfrak{h}\text{-mod} \simeq U(\mathfrak{h})\text{-mod}$$

(here $\mathfrak{h}$ is a Lie algebra) in terms of the incarnation of $U^{\mathrm{Hopf}}(\mathfrak{h})$ as

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}).$$

0.2.6.   In Sect. A we prove Theorem 2.9.4 described in Sect. 0.2.1.

In Sect. B, we prove the PBW theorem in the setting of higher algebra.

0.2.7.   In Sect. C we address the following issue: co-commutative bialgebras can be defined in two ways: as associative algebras in the category of co-commutative co-algebras or as co-commutative co-algebras in the category of associative algebras.

In the setting of higher algebra it is not obvious that these two definitions lead to the same object. However, in Proposition C.1.3 we prove that they in fact do.

1. Algebras over operads

In this section, we review the general theory of algebras over operads.

For the purposes of this chapter, we will regard operads as algebras in the category of symmetric sequences. We review this notion in Sect. 1.1.

In this section, we also review the notions of filtered and graded objects in a DG category. We show that algebras over operads have a canonical filtration and, as a result, various functors on the category of algebras over an operad obtain canonical filtrations.

Finally, for an operad $\mathcal{P}$, we consider group objects in the category of $\mathcal{P}$-algebras. We show that the underlying $\mathcal{P}$-algebra of a group object in the category of $\mathcal{P}$-algebras is *canonically* a trivial $\mathcal{P}$-algebra.

1.1. Operads and algebras. In this subsection we introduce operads and algebras over them (in a given DG category).

1.1.1.   Let Vect^Σ denote the category of symmetric sequences. As a DG category, we have:

$$\mathrm{Vect}^\Sigma := \prod_{n \geq 1} \mathrm{Rep}(\Sigma_n),$$

i.e., consists of objects

$$\mathcal{P} := \{\mathcal{P}(n) \in \mathrm{Rep}(\Sigma_n), \ n \geq 1\}.$$

The category Vect^Σ has a canonical symmetric monoidal structure such that it is the free symmetric monoidal DG category on a single object. It follows by the $(\infty, 2)$-categorical Yoneda lemma Volume I, Chapter 11, Proposition 6.3.7 that Vect^Σ is the category of endomorphisms of the functor

$$\mathrm{DGCat}^{\text{2-Cat,SymMon}}_{\mathrm{cont}} \to \textbf{1-Cat}$$

Hence, the category Vect^Σ is endowed with another natural (non-symmetric) monoidal structure, called the *composition monoidal structure*, corresponding to composition of functors. The unit object

$$\mathbf{1}_{\mathrm{Vect}^\Sigma} \in \mathrm{Vect}^\Sigma$$

is the one given by

$$\mathbf{1}_{\mathrm{Vect}^\Sigma}(1) = k, \quad \mathbf{1}_{\mathrm{Vect}^\Sigma}(n) = 0 \text{ for } n > 1.$$

Let $\mathbf{O}$ be a symmetric monoidal DG category. The category $\mathbf{O}$ is then a module category for Vect^Σ (with the composition monoidal structure). Explicitly, given an object $\mathcal{P} \in \mathrm{Vect}^\Sigma$ and $V \in \mathbf{O}$, the action of $\mathcal{P}$ on V is given by the formula

$$\mathcal{P} \star V := \bigoplus_{n \geq 1} \left(\mathcal{P}(n) \otimes V^{\otimes n} \right)_{\Sigma_n}.$$

1.1.2. A (unital) operad is by definition a unital associative algebra in Vect^Σ with respect to the composition monoidal structure.

Convention: Unless explicitly stated otherwise, we will only consider operads $\mathcal{P}$, for which the unit map defines an isomorphism $k \to \mathcal{P}(1)$. In particular, such operads, viewed as associative algebras in Vect^Σ, are automatically augmented.

1.1.3. For an operad $\mathcal{P} \in \mathrm{AssocAlg}(\mathrm{Vect}^\Sigma)$, the category of $\mathcal{P}\text{-Alg}(\mathbf{O})$ of $\mathcal{P}$-algebras in $\mathbf{O}$ is by definition the category $\mathcal{P}\text{-mod}(\mathbf{O})$.

We shall denote by

$$\mathbf{free}_\mathcal{P} : \mathbf{O} \rightleftarrows \mathcal{P}\text{-Alg}(\mathbf{O}) : \mathbf{oblv}_\mathcal{P}$$

the resulting pair of adjoint functors.

The functor $\mathbf{oblv}_\mathcal{P}$ is conservative, and being a right adjoint, it preserves limits.

The composite functor $\mathbf{oblv}_\mathcal{P} \circ \mathbf{free}_\mathcal{P}$ is given by

$$V \mapsto \mathcal{P} \star V = \bigoplus_{n \geq 1} \left(\mathcal{P}(n) \otimes V^{\otimes n} \right)_{\Sigma_n}.$$

In particular, it preserves *sifted* colimits. Thus, the monad on $\mathbf{O}$, defined by $\mathcal{P}$, preserves sifted colimits. Hence, the forgetful functor $\mathbf{oblv}_\mathcal{P}$ also preserves *sifted* colimits.

1.1.4. The augmentation on $\mathcal{P}$ gives rise to a functor

$$\mathbf{triv}_\mathcal{P} : \mathbf{O} \to \mathcal{P}\text{-Alg}(\mathbf{O}),$$

which is a right inverse on $\mathbf{oblv}_\mathcal{P}$.

1.1.5. We will consider the following operads: $\mathrm{Assoc}^{\mathrm{aug}}$, $\mathrm{Com}^{\mathrm{aug}}$ and Lie. By definition

$$\mathrm{Assoc}^{\mathrm{aug}}(n) = k^{\Sigma_n}, \quad \mathrm{Com}^{\mathrm{aug}}(n) = k.$$

By definition

$$\mathrm{Assoc}^{\mathrm{aug}}\text{-Alg}(\mathbf{O}) =: \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \text{ and } \mathrm{Com}^{\mathrm{aug}}\text{-Alg}(\mathbf{O}) =: \mathrm{ComAlg}^{\mathrm{aug}}(\mathbf{O})$$

are the categories of unital augmented (equivalently, non-unital) associative and commutative algebras in $\mathbf{O}$, respectively[2].

We will also consider the operad Lie; this is the classical Lie operad, where we set by definition $\mathrm{Lie}(1) = k$. We have

$$\mathrm{Lie}\text{-Alg}(\mathbf{O}) =: \mathrm{LieAlg}(\mathbf{O});$$

this the category of Lie algebras in $\mathbf{O}$.

---

[2]Note that in the interpretation as augmented algebras, the forgetful functor $\mathbf{oblv}_\mathcal{P}$ corresponds to taking the augmentation ideal.

1.2. Tensoring a $\mathcal{P}$-algebra by a commutative algebra. Let $\mathfrak{h}$ be a Lie algebra and A is a commutative algebra. Then the vector space $\mathfrak{h} \otimes A$ has a canonical structure of a Lie algebra given by $[h_1 \otimes a_1, h_2 \otimes a_2] = [h_1, h_2] \otimes (a_1 \cdot a_2)$.

In this subsection, we describe the following generalization of this construction. Let $\mathbf{O}$ be a symmetric monoidal category, and let A be a commutative algebra in $\mathbf{O}$. For an operad $\mathcal{P}$, let B be a $\mathcal{P}$-algebra in $\mathcal{O}$. We will show that the object $A \otimes B$ has a canonical structure of a $\mathcal{P}$-algebra.

REMARK 1.2.1. This construction has the following generalization (which we will not need in the sequel). The category of operads has a symmetric monoidal structure characterized by the property that if A is a $\mathcal{P}$-algebra and B is a $\mathcal{Q}$-algebra then $A \otimes B$ is a $(\mathcal{P} \otimes \mathcal{Q})$-algebra. The commutative operad is the unit object for this symmetric monoidal structure.

1.2.2. Let $\Phi : \mathbf{O} \to \mathbf{O}'$ be a homomorphism of symmetric monoidal DG categories. Then Φ induces a (strict) functor between module categories for the monoidal category Vect^Σ.

In particular, for any operad $\mathcal{P}$, the functor Φ induces a functor

(1.1) $\Phi : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}')$

that makes the diagrams

(1.2)

$$
\begin{array}{ccc}
\mathcal{P}\text{-Alg}(\mathbf{O}) & \xrightarrow{\;\;\Phi\;\;} & \mathcal{P}\text{-Alg}(\mathbf{O}') \\
{\scriptstyle \mathbf{oblv}_\mathcal{P}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{oblv}_\mathcal{P}} \\
\mathbf{O} & \xrightarrow{\;\;\Phi\;\;} & \mathbf{O}'
\end{array}
$$

and

(1.3)

$$
\begin{array}{ccc}
\mathcal{P}\text{-Alg}(\mathbf{O}) & \xrightarrow{\;\;\Phi\;\;} & \mathcal{P}\text{-Alg}(\mathbf{O}') \\
{\scriptstyle \mathbf{free}_\mathcal{P}}\big\uparrow & & \big\uparrow{\scriptstyle \mathbf{free}_\mathcal{P}} \\
\mathbf{O} & \xrightarrow{\;\;\Phi\;\;} & \mathbf{O}'
\end{array}
$$

commute.

1.2.3. Consider the right adjoint[3] Φ^R of Φ (where we view the latter as a functor between mere DG categories).

The functor Φ^R has a natural structure of *right-lax functor* of module categories over Vect^Σ. In particular, it induces a functor

$$\Phi^R : \mathcal{P}\text{-Alg}(\mathbf{O}') \to \mathcal{P}\text{-Alg}(\mathbf{O}),$$

right adjoint to (1.1).

By passing to right adjoints in (1.3), we obtain a commutative diagram

(1.4)

$$
\begin{array}{ccc}
\mathcal{P}\text{-Alg}(\mathbf{O}) & \xleftarrow{\;\;\Phi^R\;\;} & \mathcal{P}\text{-Alg}(\mathbf{O}') \\
{\scriptstyle \mathbf{oblv}_\mathcal{P}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{oblv}_\mathcal{P}} \\
\mathbf{O} & \xleftarrow{\;\;\Phi^R\;\;} & \mathbf{O}'.
\end{array}
$$

———————

[3]Here we do not even need to require that this right adjoint be continuous.

1.2.4. Let now A be a commutative algebra in $\mathbf{O}$. Set $\mathbf{O}' := A\text{-mod}(\mathbf{O})$. The composition

$$\Phi^R \circ \Phi : \mathbf{O} \to \mathbf{O}$$

is the functor of tensor product by A.

By the above, this functor admits a natural structure of *right-lax functor* of module categories over Vect^Σ. In particular, we obtain a well-defined functor

$$A \otimes - : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}),$$

that makes the diagram

$$
\begin{array}{ccc}
\mathcal{P}\text{-Alg}(\mathbf{O}) & \xrightarrow{\ A\otimes-\ } & \mathcal{P}\text{-Alg}(\mathbf{O}') \\
\mathbf{oblv}_{\mathcal{P}} \downarrow & & \downarrow \mathbf{oblv}_{\mathcal{P}} \\
\mathbf{O} & \xrightarrow{\ A\otimes-\ } & \mathbf{O}'
\end{array}
$$

commute.

1.2.5. Note that the construction

$$(1.5) \qquad\qquad A \rightsquigarrow A \otimes - : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O})$$

is functorial in A, so we obtain a functor

$$\mathrm{ComAlg}(\mathbf{O}) \times \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}).$$

For the sequel, we note the following:

LEMMA 1.2.6. *The functor* (1.5) *commutes with finite limits in each variable.*

PROOF. It is enough to prove the assertion after applying the functor $\mathbf{oblv}_{\mathcal{P}}$, and then it becomes obvious, because the functor

$$- \otimes - : \mathbf{O} \times \mathbf{O} \to \mathbf{O}$$

commutes with finite limits in each variable.

$\square$

1.3. Digression: filtered and graded objects. In this subsection we will make a digression and fix some notation pertaining to filtered and graded objects in a DG category.

1.3.1. For a DG category $\mathbf{C}$, we let $\mathbf{C}^{\mathrm{Fil}}$ (resp., $\mathbf{C}^{\mathrm{Fil},\geq 0}$, $\mathbf{C}^{\mathrm{Fil},\leq 0}$) denote the category of filtered (resp., non-negatively filtered, non-positively filtered) objects. By definition,

$$\mathbf{C}^{\mathrm{Fil}} := \mathrm{Funct}(\mathbb{Z}, \mathbf{C}), \quad \mathbf{C}^{\mathrm{Fil},\geq 0} := \mathrm{Funct}(\mathbb{Z}^{\geq 0}, \mathbf{C}), \quad \mathbf{C}^{\mathrm{Fil},\leq 0} := \mathrm{Funct}(\mathbb{Z}^{\leq 0}, \mathbf{C}),$$

where $\mathbb{Z}$ is viewed as an ordered set and hence a category.

We have the natural restriction functors

$$\mathbf{C}^{\mathrm{Fil},\geq 0} \leftarrow \mathbf{C}^{\mathrm{Fil}} \to \mathbf{C}^{\mathrm{Fil},\leq 0}.$$

The above functors both admit left adjoints, given by left Kan extension. The essential image of $\mathbf{C}^{\mathrm{Fil},\geq 0}$ in $\mathbf{C}^{\mathrm{Fil}}$ consists of functors sending the negative integers to 0. Then essential image of $\mathbf{C}^{\mathrm{Fil},\leq 0}$ in $\mathbf{C}^{\mathrm{Fil}}$ consists of functors that take the constant value on $\mathbb{Z}^{\geq 0}$. Thus, we obtain the usual embeddings

$$\mathbf{C}^{\mathrm{Fil},\geq 0} \hookrightarrow \mathbf{C}^{\mathrm{Fil}} \hookleftarrow \mathbf{C}^{\mathrm{Fil},\leq 0}.$$

The functor of 'forgetting the filtration'

$$\mathbf{oblv}_{\mathrm{Fil}} : \mathbf{C}^{\mathrm{Fil}} \to \mathbf{C}$$

is by definition the functor

$$\operatorname*{colim}_{\mathbb{Z}} : \mathrm{Funct}(\mathbb{Z}, \mathbf{C}) \to \mathbf{C}.$$

1.3.2.   Consider also the category

$$\mathbf{C}^{\mathrm{gr}} := \mathbf{C}^{\mathbb{Z}},$$

and its subcategories

$$\mathbf{C}^{\mathrm{gr},\geq 0} \subset \mathbf{C}^{\mathrm{gr}} \supset \mathbf{C}^{\mathrm{gr},\leq 0}.$$

We have the functor of forgetting the grading $\mathbf{oblv}_{\mathrm{gr}} : \mathbf{C}^{\mathrm{gr}} \to \mathbf{C}$, given by

$$\underset{\mathbb{Z}}{\oplus} : \mathbf{C}^{\mathbb{Z}} \to \mathbf{C}.$$

For $n \in \mathbb{Z}$ we let

$$(\deg = n) : \mathbf{C} \to \mathbf{C}^{\mathrm{gr}}$$

the functor that creates an object concentrated in degree n. Sometimes, we will also use the notation

$$V^{\mathrm{deg=n}} := (\deg = n)(V).$$

1.3.3.   We have a canonically defined functor

(1.6) $$(\mathrm{gr} \to \mathrm{Fil}) : \mathbf{C}^{\mathrm{gr}} \to \mathbf{C}^{\mathrm{Fil}},$$

given by left Kan extension along

(1.7) $$\mathbb{Z}^{\mathrm{Spc}} \to \mathbb{Z}.$$

(I.e., the target $\mathbb{Z}$ is considered as a category with respect to its natural order, while the source copy is considered as a groupoid.)

Explicitly, if an object of $\mathbf{C}^{\mathrm{gr}}$ is given by $n \rightsquigarrow V_n$, the corresponding object of $\mathbf{C}^{\mathrm{Fil}}$ is given by

$$n \rightsquigarrow \underset{n' \leq n}{\oplus} V_{n'}.$$

The functor $(\mathrm{gr} \to \mathrm{Fil})$ admits a right adjoint, denoted Rees, given by restriction along (1.7).

1.3.4.   We now consider the functor of associated graded

$$\mathrm{ass\text{-}gr} : \mathbf{C}^{\mathrm{Fil}} \to \mathbf{C}^{\mathrm{gr}},$$

given by

$$n \mapsto \mathrm{coFib}(V_{n-1} \to V_n).$$

It is a basic (and obvious) fact that the functor ass-gr is *conservative when restricted to* $\mathbf{C}^{\mathrm{Fil},\geq 0}$.

We have the following (evident) isomorphism of endo-functors of $\mathbf{C}^{\mathrm{gr}}$:

$$\mathrm{ass\text{-}gr} \circ (\mathrm{gr} \to \mathrm{Fil}) \simeq \mathrm{Id}.$$

1.3.5.   The above constructions are functorial with respect to $\mathbf{C}$. In particular, if $\mathbf{O}$ is a (symmetric) monoidal category, then so are $\mathbf{O}^{\mathrm{Fil}}$ and $\mathbf{O}^{\mathrm{gr}}$, and each of the functors

$$\text{ass-gr} : \mathbf{O}^{\mathrm{Fil}} \to \mathbf{O}^{\mathrm{gr}}, \ (\mathrm{gr} \to \mathrm{Fil}) : \mathbf{O}^{\mathrm{gr}} \to \mathbf{O}^{\mathrm{Fil}} \text{ and } (\deg = 0) : \mathbf{O} \to \mathbf{O}^{\mathrm{gr}}$$

has a natural (symmetric) monoidal structure.

1.4. Adding a filtration. Suppose that A is an *augmented* associative algebra. In this case, A has a canonical filtration given by $A_n = 0$ for $n < 0$, $A_0 = k$ and $A_n = A$ for $n \geq 1$. The corresponding associated graded algebra is given by the square zero extension (i.e. trivial augmented associative algebra) $k \oplus A^+$, where A^+ is the augmentation ideas of A.

In this subsection, we describe a generalization of this construction. Namely, we show that any[4] $\mathcal{P}$-algebra has a canonical lift to a *filtered* $\mathcal{P}$-algebra such that the associated graded is the trivial $\mathcal{P}$-algebra. Roughly speaking, at the level of the corresponding Rees algebras, this construction amounts to scaling all the operations to zero.

This is a technically important tool as it allows to reduce many statements about $\mathcal{P}$-algebras to trivial $\mathcal{P}$-algebras.

1.4.1.   Consider the commutative algebra $A := k \oplus k$; we endow it with an augmentation, given by projection on the first copy of k. We also endow it with a non-negative filtration by setting

$$A_n = \left\{ \begin{array}{l} A \text{ for } n \geq 1 \\ k \text{ for } n = 0. \end{array} \right.$$

By Sect. 1.2, we can regard the assignment

$$B \mapsto A \otimes B$$

as a functor

$$\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}^{\mathrm{Fil}, \geq 0}).$$

Using the augmentation on A, we obtain a natural transformation

$$A \otimes B \to B.$$

Here we abuse the notation slightly, and denote simply by B the object of $\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}^{\mathrm{Fil}, \geq 0})$ that should properly be denoted by $(\mathrm{gr} \to \mathrm{Fil})(B^{\deg=0})$.

1.4.2.   We define the functor

$$\mathrm{AddFil} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}^{\mathrm{Fil}, \geq 0})$$

by:

$$B \mapsto \mathrm{Fib}(A \otimes B \to B) := (A \otimes B) \underset{B}{\times} \{0\}.$$

Sometimes, we will also use the notation

$$B^{\mathrm{Fil}} := \mathrm{AddFil}(B).$$

---

[4]Recall our conventions for operads!

1.4.3.   Since $\mathbf{oblv}_{\mathrm{Fil}}(A) = k \times k$, by Lemma 1.2.6, we obtain an isomorphism of functors:

$$\mathbf{oblv}_{\mathrm{Fil}}(A \otimes B) \simeq B \times B.$$

From here, we obtain that the isomorphism of functors

$$\mathbf{oblv}_{\mathrm{Fil}} \circ \mathrm{AddFil} \simeq \mathrm{Id}.$$

So, the assignment

$$B \rightsquigarrow B^{\mathrm{Fil}}$$

can be viewed as a canonical lift of $B \in \mathcal{P}\text{-Alg}(\mathbf{O})$ to an object of $\mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{Fil},\geq 0})$.

1.4.4.   The following diagram commutes by construction:

$$
\begin{array}{ccc}
\mathcal{P}\text{-Alg}(\mathbf{O}) & \xrightarrow{\mathrm{AddFil}} & \mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{Fil},\geq 0}) \\
{\scriptstyle\mathbf{triv}_{\mathcal{P}}} \uparrow & & {\scriptstyle\mathbf{triv}_{\mathcal{P}}} \uparrow \\
\mathbf{O} \xrightarrow{\mathrm{deg}=1} \mathbf{O}^{\mathrm{gr},\geq 0} & \xrightarrow{(\mathrm{gr} \to \mathrm{Fil})} & \mathbf{O}^{\mathrm{Fil},\geq 0}.
\end{array}
$$

The following diagram also commutes:

$$
\begin{array}{ccc}
\mathcal{P}\text{-Alg}(\mathbf{O}) & \xrightarrow{\mathrm{AddFil}} & \mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{Fil},\geq 0}) \\
{\scriptstyle\mathbf{oblv}_{\mathcal{P}}} \downarrow & & {\scriptstyle\mathbf{oblv}_{\mathcal{P}}} \downarrow \\
\mathbf{O} \xrightarrow{\mathrm{deg}=1} \mathbf{O}^{\mathrm{gr},\geq 0} & \xrightarrow{(\mathrm{gr} \to \mathrm{Fil})} & \mathbf{O}^{\mathrm{Fil},\geq 0}.
\end{array}
$$

1.4.5.   We now claim:

PROPOSITION 1.4.6.   *The functor*

$$\mathrm{ass\text{-}gr} \circ \mathrm{AddFil} : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{gr},\geq 0})$$

is canonically isomorphic to $\mathbf{triv}_{\mathcal{P}} \circ (\mathrm{deg} = 1) \circ \mathbf{oblv}_{\mathcal{P}}$, *i.e.,*

$$\mathcal{P}\text{-Alg}(\mathbf{O}) \xrightarrow{\mathbf{oblv}_{\mathcal{P}}} \mathbf{O} \xrightarrow{(\mathrm{deg}=1)} \mathbf{O}^{\mathrm{gr},\geq 0} \xrightarrow{\mathbf{triv}_{\mathcal{P}}} \mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{gr},\geq 0}).$$

Let us repeat the statement of Proposition 1.4.6 in words. It says that for $B \in \mathcal{P}\text{-Alg}(\mathbf{O})$, the associated graded of B^{Fil} is canonically trivial.

PROOF.   We need to show that the functor $\mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{gr},\geq 0})$, given by

$$B \mapsto \mathrm{Fib}(\mathrm{ass\text{-}gr}(A) \otimes B \to B)$$

is canonically isomorphic to

$$B \mapsto \mathbf{triv}_{\mathcal{P}}(\mathbf{oblv}_{\mathcal{P}}(B)^{\mathrm{deg}=1}).$$

We will deduce this from a particular property of the canonical action of Vect^{Σ} on $\mathbf{O}$ from Sect. 1.1.1:

Note that $\mathrm{ass\text{-}gr}(A) \simeq k[\epsilon]/\epsilon^2$, where $\mathrm{deg}(\epsilon) = 1$. Consider the functor

(1.8) $$V \mapsto \mathrm{Fib}(k[\epsilon]/\epsilon^2 \otimes V^{\mathrm{deg}=0} \to V^{\mathrm{deg}=0}), \quad \mathbf{O} \to \mathbf{O}^{\mathrm{gr},\geq 0},$$

as a right-lax functor of modules categories over Vect^{Σ}.

For a symmetric monoidal category $\mathbf{O}'$ let us denote by $\mathbf{O}'_{\mathrm{triv}}$ the same DG category (i.e., $\mathbf{O}'$), but equipped with the *trivial* action of Vect^{Σ}, i.e., the action

that factors through the projection on the degree 1 component $\mathrm{Vect}^\Sigma \to \mathrm{Vect}$. Note that the identity functor on $\mathbf{O}'$ can be made into a right-lax functor of modules categories over Vect^Σ for both

$$(1.9) \qquad\qquad\qquad\qquad \mathbf{O}' \to \mathbf{O}'_{\mathrm{triv}}$$

and

$$(1.10) \qquad\qquad\qquad\qquad \mathbf{O}'_{\mathrm{triv}} \to \mathbf{O}'.$$

With these notations, the observation is that the functor (1.8) canonically factors as a composition

$$\mathbf{O} \overset{(1.9)}{\longrightarrow} \mathbf{O}_{\mathrm{triv}} \overset{(\mathrm{deg}=1)}{\longrightarrow} (\mathbf{O}^{\mathrm{gr},\geq 0})_{\mathrm{triv}} \overset{(1.10)}{\longrightarrow} \mathbf{O}^{\mathrm{gr},\geq 0}.$$

Indeed, this follows for reasons of degree since the functor (1.8) sends V to $V^{\mathrm{deg}=1}$.

$\square$

1.5. Filtered objects arising from $\mathcal{P}$-algebras. The construction in this subsection expresses the following idea: many functors from the category of $\mathcal{P}$-algebras in $\mathbf{O}$ to $\mathbf{O}$ itself automatically lift to functors with values in the category of filtered objects in $\mathbf{O}$.

The typical examples of this phenomenon that we will consider are the functors of universal envelope or Chevalley complex of a Lie algebra (see Sect. 2.5 for the latter example).

1.5.1.   Let $\mathcal{C}$ be a functor

$$\mathrm{DGCat}^{\mathrm{SymMon}} \to 1\text{-}\mathrm{Cat},$$

and let Φ be a natural transformation

$$\mathbf{O} \quad \rightsquigarrow \quad \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \overset{\Phi|_{\mathbf{O}}}{\to} \mathcal{C}(\mathbf{O}).$$

We observe that the natural transformation Φ automatically upgrades to a natural transformation, denoted Φ^{Fil},

$$\mathbf{O} \quad \rightsquigarrow \quad \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathcal{C}(\mathbf{O}^{\mathrm{Fil},\geq 0}).$$

Indeed, we let $\Phi^{\mathrm{Fil}}|_{\mathbf{O}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathcal{C}(\mathbf{O}^{\mathrm{Fil},\geq 0})$ be the composition

$$\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \overset{\mathrm{AddFil}}{\longrightarrow} \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}^{\mathrm{Fil},\geq 0}) \overset{\Phi|_{\mathbf{O}^{\mathrm{Fil},\geq 0}}}{\longrightarrow} \mathcal{C}(\mathbf{O}^{\mathrm{Fil},\geq 0}).$$

1.5.2.   Note that we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) & \overset{\Phi^{\mathrm{Fil}}|_{\mathbf{O}}}{\longrightarrow} & \mathcal{C}(\mathbf{O}^{\mathrm{Fil},\geq 0}) \\
{\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{Fil}}} \\
\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) & \overset{\Phi|_{\mathbf{O}}}{\longrightarrow} & \mathcal{C}(\mathbf{O}).
\end{array}
$$

The next diagram commutes due to Proposition 1.4.6:

$$
\begin{array}{ccc}
\mathcal{P}\text{-Alg}(\mathbf{O}) & \xrightarrow{\ \Phi^{\mathrm{Fil}}|_{\mathbf{O}}\ } & \mathcal{C}(\mathbf{O}^{\mathrm{Fil},\geq 0}) \\
{\scriptstyle \mathbf{oblv}_{\mathcal{P}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ass\text{-}gr}} \\
\mathbf{O} & & \mathcal{C}(\mathbf{O}^{\mathrm{gr},\geq 0}) \\
{\scriptstyle (\deg=1)}\big\downarrow & & \big\uparrow{\scriptstyle \Phi|_{\mathbf{O}^{\mathrm{gr},\geq 0}}} \\
\mathbf{O}^{\mathrm{gr},\geq 0} & \xrightarrow{\ \mathbf{triv}_{\mathcal{P}}\ } & \mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{gr},\geq 0})
\end{array}
$$

(1.11)

In addition, we have the following commutative diagram

$$
\begin{array}{ccc}
\mathbf{O} & \xrightarrow{\ \mathbf{triv}_{\mathcal{P}}\ } & \mathcal{P}\text{-Alg}(\mathbf{O}) \\
{\scriptstyle (\deg=1)}\big\downarrow & & \big\downarrow{\scriptstyle \Phi^{\mathrm{Fil}}|_{\mathbf{O}}} \\
\mathbf{O}^{\mathrm{gr},\geq 0} & & \mathcal{C}(\mathbf{O}^{\mathrm{Fil},\geq 0}) \\
{\scriptstyle \mathbf{triv}_{\mathcal{P}}}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{gr}\to\mathrm{Fil}} \\
\mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{gr},\geq 0}) & \xrightarrow{\ \Phi|_{\mathbf{O}^{\mathrm{gr},\geq 0}}\ } & \mathcal{C}(\mathbf{O}^{\mathrm{gr},\geq 0}).
\end{array}
$$

1.6. Group objects in the category of $\mathcal{P}$-algebras. In this subsection we show that the category of $\mathcal{P}$-algebras has the feature that the functors of taking the loop space and the classifying space of a group-object are mutually inverse equivalences of categories.

1.6.1. Consider the categories

$$
\mathrm{Grp}(\mathcal{P}\text{-Alg}(\mathbf{O})) \subset \mathrm{Monoid}(\mathcal{P}\text{-Alg}(\mathbf{O})).
$$

We claim:

LEMMA 1.6.2. *The inclusion* $\mathrm{Grp}(\mathcal{P}\text{-Alg}(\mathbf{O})) \subset \mathrm{Monoid}(\mathcal{P}\text{-Alg}(\mathbf{O}))$ *is an equivalence.*

PROOF. The inclusion $\mathrm{Grp}(\mathbf{C}) \subset \mathrm{Monoid}(\mathbf{C})$ is an equivalence for any pointed category $\mathbf{C}$, for which a map $\mathbf{c}_1 \to \mathbf{c}_2$ is an isomorphism whenever $\mathbf{c}_1 \underset{\mathbf{c}_2}{\times} * \to *$ is:

Namely, recall that a monoid object $\mathbf{c} \in \mathbf{C}$ is a group object if and only if the map

$$
(p_1, m) : \mathbf{c}_1 := \mathbf{c} \times \mathbf{c} \to \mathbf{c} \times \mathbf{c} =: \mathbf{c}_2
$$

is an isomorphism. However, if $\mathbf{C}$ is pointed, the canonical map $* \to \mathbf{c}$ is the unit; therefore, the natural map $\mathbf{c}_1 \underset{\mathbf{c}_2}{\times} * \to *$ is an isomorphism.

$\square$

1.6.3.   Consider now the pair of adjoint functors:

(1.12) $B_{\mathcal{P}} : \mathrm{Grp}(\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O})) \rightleftarrows \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) : \Omega_{\mathcal{P}}.$

We claim:

PROPOSITION 1.6.4. *The functors (1.12) are mutually inverse equivalences.*

PROOF. We have to show that the natural transformations

$$\mathrm{Id} \to \Omega_{\mathcal{P}} \circ B_{\mathcal{P}} \text{ and } B_{\mathcal{P}} \circ \Omega_{\mathcal{P}} \to \mathrm{Id}$$

are isomorphisms.

It is enough to show that the resulting natural transformations

$$\mathbf{oblv}_{\mathcal{P}} \to \mathbf{oblv}_{\mathcal{P}} \circ \Omega_{\mathcal{P}} \circ B_{\mathcal{P}} \text{ and } \mathbf{oblv}_{\mathcal{P}} \circ B_{\mathcal{P}} \circ \Omega_{\mathcal{P}} \to \mathbf{oblv}_{\mathcal{P}}$$

are isomorphisms.

The following diagram commutes tautologically

$$
\begin{array}{ccc}
\mathrm{Grp}(\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O})) & \xleftarrow{\ \Omega_{\mathcal{P}}\ } & \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \\
{\scriptstyle \mathbf{oblv}_{\mathcal{P}} \circ \mathbf{oblv}_{\mathrm{Grp}}} \downarrow & & \downarrow {\scriptstyle \mathbf{oblv}_{\mathcal{P}}} \\
\mathbf{O} & \xleftarrow{\ [-1]\ } & \mathbf{O},
\end{array}
$$

because the functor $\mathbf{oblv}_{\mathcal{P}}$ commutes with limits.

The next diagram, obtained from one above by passing to left adjoints along the horizontal arrows,

$$
\begin{array}{ccc}
\mathrm{Grp}(\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O})) & \xrightarrow{\ B_{\mathcal{P}}\ } & \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \\
{\scriptstyle \mathbf{oblv}_{\mathcal{P}} \circ \mathbf{oblv}_{\mathrm{Grp}}} \downarrow & & \downarrow {\scriptstyle \mathbf{oblv}_{\mathcal{P}}} \\
\mathbf{O} & \xrightarrow{\ [1]\ } & \mathbf{O}
\end{array}
$$

also commutes, because $\mathbf{oblv}_{\mathcal{P}}$ commutes with sifted colimits.

This implies the required assertion.

$\square$

1.7. Forgetting the group structure. In this subsection we show the following: if we consider a group-object of the category of $\mathcal{P}$-algebras, and forget the group structure, then the resulting $\mathcal{P}$-algebra is *canonically* trivial.

1.7.1.   We will prove:

PROPOSITION 1.7.2. *The composite functor*

$$\mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathcal{P}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O})$$

is canonically isomorphic to

$$\mathbf{triv}_{\mathcal{P}} \circ [-1] \circ \mathbf{oblv}_{\mathcal{P}}.$$

Combining with Proposition 1.6.4, we obtain:

COROLLARY 1.7.3. *The functor*

$$\mathbf{oblv}_{\mathrm{Grp}} : \mathrm{Grp}(\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O})) \to \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O})$$

is canonically isomorphic to

$$\mathbf{triv}_{\mathcal{P}} \circ \mathbf{oblv}_{\mathcal{P}} \circ \mathbf{oblv}_{\mathrm{Grp}}.$$

The rest of this subsection is devoted to the proof of Proposition 1.7.2. The idea of the proof, explained to us by M. Kontsevich, is to interpret the composite functor $\mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathcal{P}}$ as tensor product by a certain (non-unital) commutative algebra.

1.7.4. *Step 1.* Consider the commutative augmented algebra A in Vect equal to

$$\mathbf{triv}_{\mathrm{Com}^{\mathrm{aug}}}(k[-1]).$$

I.e., $A = k[-1] \oplus k$, with the multiplication on $k[-1]$ is trivial. We claim that that there exists a canonical isomorphism of endo-functors of $\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O})$

$$\mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathcal{P}}(B) \simeq \mathrm{Fib}(A \otimes B \to B).$$

Indeed, take $A' = k \times k$, so that $A := k \underset{A'}{\times} k$. By Lemma 1.2.6, the pullback diagram of commutative algebras

(1.13)
$$
\begin{array}{ccc}
A & \longrightarrow & k \\
\downarrow & & \downarrow \\
k & \longrightarrow & A'
\end{array}
$$

gives rise to a pullback diagram in $\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O})$,

$$
\begin{array}{ccc}
A \otimes B & \longrightarrow & B \\
\downarrow & & \downarrow{\scriptstyle \mathrm{diag}} \\
B & \xrightarrow{\;\mathrm{diag}\;} & B \times B,
\end{array}
$$

functorially in $B \in \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O})$.

The projection on the second factor defines an augmentation $A' \to k$, thereby allowing to view (1.13) as a diagram *over* k. From here we obtain a pullback diagram

$$
\begin{array}{ccc}
\mathrm{Fib}(A \otimes B \to B) & \longrightarrow & \mathrm{Fib}(B \to B) \\
\downarrow & & \downarrow \\
\mathrm{Fib}(B \to B) & \longrightarrow & \mathrm{Fib}(B \times B \to B),
\end{array}
$$

i.e., we obtain a pullback diagram

$$
\begin{array}{ccc}
\mathrm{Fib}(A \otimes B \to B) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & B,
\end{array}
$$

as desired.

1.7.5. *Step 2.* Thus, to prove Proposition 1.7.2 we need to establish a canonical isomorphism of functors

$$(1.14) \qquad \mathrm{Fib}(A \otimes B \to B) \simeq \mathbf{triv}_{\mathcal{P}}(\mathbf{oblv}_{\mathcal{P}}(B)[-1]), \quad B \in \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}).$$

This repeats the argument of Proposition 1.4.6.

1.8. Stabilization. In this subsection we use Proposition 1.7.2 to give a simple proof of the fact that the stabilization of the category of $\mathcal{P}$-algebras (in a symmetric monoidal DG category $\mathbf{O}$) identifies with $\mathbf{O}$ itself.

1.8.1. For an ∞-category $\mathbf{C}$, let $\mathrm{ComMonoid}(\mathbf{C})$ denote the category of commutative monoids in $\mathbf{C}$, see Volume I, Chapter 1, Sect. 3.3.3.

Recall also that if $\mathbf{C}$ is stable, the forgetful functor

$$\mathrm{ComMonoid}(\mathbf{C}) \to \mathbf{C}$$

is an equivalence.

1.8.2. Let $\mathbf{O}$ be a symmetric monoidal DG category. We regard it a mere ∞-category, and consider the corresponding category $\mathrm{ComMonoid}(\mathbf{O})$. Since $\mathbf{O}$ is stable, by the above we have $\mathrm{ComMonoid}(\mathbf{O}) \simeq \mathbf{O}$.

Since the functor $\mathbf{triv}_{\mathcal{P}}$ commutes with limits (and, in particular, products), it induces a functor

$$(1.15) \qquad \mathbf{O} \simeq \mathrm{ComMonoid}(\mathbf{O}) \overset{\mathrm{ComMonoid}(\mathbf{triv}_{\mathcal{P}})}{\longrightarrow} \mathrm{ComMonoid}(\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O})).$$

We will prove:

PROPOSITION 1.8.3. *The functor* (1.15) *is an equivalence.*

The above proposition can be reformulated as a commutative diagram

$$
\begin{array}{ccc}
\mathbf{O} & \xrightarrow{\ \mathbf{triv}_{\mathcal{P}}\ } & \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \\
{\scriptstyle\sim}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{Id}} \\
\mathrm{ComMonoid}(\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O})) & \xrightarrow{\ \mathbf{oblv}_{\mathrm{ComMonoid}}\ } & \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}).
\end{array}
$$

Hence, we obtain:

COROLLARY 1.8.4. *The functor*

$$\mathrm{coPrim}_{\mathcal{P}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathbf{O},$$

left adjoint to $\mathbf{triv}_{\mathcal{P}}$ *(see Sect. 2.4.1 below), identifies* $\mathbf{O}$ *with the stabilization of* $\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O})$.

PROOF OF PROPOSITION 1.8.3. Since the functor $\mathbf{oblv}_{\mathcal{P}}$ preserves limits (and, in particular, products), it induces a functor

$$\mathrm{ComMonoid}(\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O})) \overset{\mathrm{ComMonoid}(\mathbf{oblv}_{\mathcal{P}})}{\longrightarrow} \mathrm{ComMonoid}(\mathbf{O}).$$

We claim that the functors $\mathrm{ComMonoid}(\mathbf{oblv}_{\mathcal{P}})$ and $\mathrm{ComMonoid}(\mathbf{triv}_{\mathcal{P}})$ are inverses of each other.

The fact that the composition $\mathrm{ComMonoid}(\mathbf{oblv}_{\mathcal{P}}) \circ \mathrm{ComMonoid}(\mathbf{triv}_{\mathcal{P}})$ is isomorphic to the identity functor follows from the fact that $\mathbf{oblv}_{\mathcal{P}} \circ \mathbf{triv}_{\mathcal{P}} \simeq \mathrm{Id}$.

To prove that the other composition is isomorphic to the identity functor, we proceed as follows. Recall that for any ∞-category $\mathbf{C}$, the forgetful functor

$$\text{ComMonoid}(\text{Monoid}(\mathbf{C})) \xrightarrow{\text{ComMonoid}(\mathbf{oblv}_{\text{Monoid}})} \text{ComMonoid}(\mathbf{C})$$

is an equivalence, see [**Lu2**, Theorem 5.1.2.2].

Hence, it suffices to construct an isomorphism between the composition

$$\text{ComMonoid}(\text{Monoid}(\mathcal{P}\text{-Alg}(\mathbf{O}))) \xrightarrow{\text{ComMonoid}(\mathbf{oblv}_{\text{Monoid}})} \text{ComMonoid}(\mathcal{P}\text{-Alg}(\mathbf{O})) \rightarrow$$

$$\xrightarrow{\text{ComMonoid}(\mathbf{oblv}_{\mathcal{P}})} \text{ComMonoid}(\mathbf{O}) \xrightarrow{\text{ComMonoid}(\mathbf{triv}_{\mathcal{P}})} \text{ComMonoid}(\mathcal{P}\text{-Alg}(\mathbf{O}))$$

and

$$\text{ComMonoid}(\text{Monoid}(\mathcal{P}\text{-Alg}(\mathbf{O}))) \xrightarrow{\text{ComMonoid}(\mathbf{oblv}_{\text{Monoid}})} \text{ComMonoid}(\mathcal{P}\text{-Alg}(\mathbf{O})).$$

However, this follows by applying ComMonoid to the isomorphism between

$$\text{Monoid}(\mathcal{P}\text{-Alg}(\mathbf{O}))) \xrightarrow{\mathbf{oblv}_{\text{Monoid}}} \mathcal{P}\text{-Alg}(\mathbf{O}) \xrightarrow{\mathbf{oblv}_{\mathcal{P}}} \mathbf{O} \xrightarrow{\mathbf{triv}_{\mathcal{P}}} \mathcal{P}\text{-Alg}(\mathbf{O})$$

and

$$\text{Monoid}(\mathcal{P}\text{-Alg}(\mathbf{O})) \xrightarrow{\mathbf{oblv}_{\text{Monoid}}} \mathcal{P}\text{-Alg}(\mathbf{O}),$$

the latter given by Corollary 1.7.3.

$\square$

1.8.5.   We can use Proposition 1.7.2 also to describe the co-stabilization of $\mathcal{P}\text{-Alg}(\mathbf{O})$, i.e., the stabilization of $\mathcal{P}\text{-Alg}(\mathbf{O})^{\text{op}}$.

PROPOSITION 1.8.6.   *The suspension functor* $\Sigma_{\mathcal{P}}$ *on* $\mathcal{P}\text{-Alg}(\mathbf{O})$ *identifies with*

$$\mathbf{free}_{\mathcal{P}} \circ [1] \circ \text{coPrim}_{\mathcal{P}},$$

where $\text{coPrim}_{\mathcal{P}}$ *is as in Sect. 2.4.1.*

PROOF.   Follows by adjunction from Proposition 1.7.2.

$\square$

COROLLARY 1.8.7.   *The functor*

$$(\mathbf{free}_{\mathcal{P}})^{\text{op}} : \mathbf{O}^{\text{op}} \rightarrow (\mathcal{P}\text{-Alg}(\mathbf{O}))^{\text{op}}$$

identifies $\mathbf{O}^{\text{op}}$ *with the stabilization of* $(\mathcal{P}\text{-Alg}(\mathbf{O}))^{\text{op}}$.

PROOF.   We have to show that the functor $(\mathbf{free}_{\mathcal{P}})^{\text{op}}$ identifies $\mathbf{O}^{\text{op}}$ with the category of *spectrum objects* in $(\mathcal{P}\text{-Alg}(\mathbf{O}))^{\text{op}}$, i.e., with the category of sequences

$$A_0 \simeq \Omega(A_1), \ A_1 \simeq \Omega(A_2)..., \quad A_i \in (\mathcal{P}\text{-Alg}(\mathbf{O}))^{\text{op}},$$

where Ω is the loop functor on $(\mathbf{free}_{\mathcal{P}})^{\text{op}}$, i.e., when we regard A_i as $\mathcal{P}$-algebras in $\mathbf{O}$, we have

$$A_0 \simeq \Sigma_{\mathcal{P}}(A_1), \ A_1 \simeq \Sigma_{\mathcal{P}}(A_2)...$$

We claim that any such sequence is canonically of the form

$$(1.16) \qquad\qquad A_i = \mathbf{free}_{\mathcal{P}} \circ [-i] \circ \text{coPrim}_{\mathcal{P}}(A_0).$$

Indeed, it follows from Proposition 1.8.6 that for every $i \geq 0$ we have

$$(1.17) \qquad\qquad A_i \simeq \mathbf{free}_{\mathcal{P}} \circ [1] \circ \text{coPrim}_{\mathcal{P}}(A_{i+1}),$$

hence
$$\mathrm{coPrim}_{\mathcal{P}}(A_{i+1}) \simeq \mathrm{coPrim}_{\mathcal{P}}(A_i)[-1],$$

and hence

(1.18) $$\mathrm{coPrim}_{\mathcal{P}}(A_{i+1}) \simeq \mathrm{coPrim}_{\mathcal{P}}(A_0)[-(i+1)].$$

Combining (1.18) and (1.17) we obtain (1.16).

$\square$

2. Koszul duality

In this section we review the general theory of Koszul duality that relates algebras over an operad with co-algebras over the Koszul dual co-operads.

The main point of this section is that there are two *inequivalent* notions of co-algebra over a co-operad: the usual one, and what we call a *ind-nilpotent co-algebra*. There is a forgetful functor from the latter to the former, which we conjecture to be fully faithful.

The Koszul duality functors naturally connect $\mathcal{P}$-algebras and ind-nilpotent co-algebras for $\mathcal{P}^\vee$. We propose some conjectures to the effect of what fully-faithfulness properties we can expect from the Koszul duality functors.

2.1. Co-operads. In this subsection we introduce the notion of co-operad. There are no surprises here, but there will be some when we will consider the corresponding notion of co-algebra over a co-operad.

2.1.1.   By a *co-operad* we shall mean a co-associative co-algebra object in Vect^Σ.

As in the case of operads (see Sect. 1.1.2), we will only consider co-operads $\mathcal{Q}$ for which the co-unit defines an isomorphism $\mathcal{Q}(1) \to k$. (In particular, all our co-operads are augmented.)

2.1.2.   Let $\mathrm{Vect}^\Sigma_{\mathrm{f.d.}} \subset \mathrm{Vect}^\Sigma$ be the full subcategory spanned by those objects $\mathcal{P}$, for which $\mathcal{P}(n) \in \mathrm{Vect}$ is finite-dimensional in each cohomological degree for every n.

Term-wise dualization $\mathcal{P} \mapsto \mathcal{P}^*$ defines a monoidal equivalence

$$(\mathrm{Vect}^\Sigma_{\mathrm{f.d.}})^{\mathrm{op}} \to \mathrm{Vect}^\Sigma_{\mathrm{f.d.}} .$$

In particular, it defines an anti-equivalence between the subcategories of operads and co-operads that belong to $\mathrm{Vect}^\Sigma_{\mathrm{f.d.}}$.

2.1.3.   We set
$$\mathrm{Coassoc}^{\mathrm{aug}} := (\mathrm{Assoc}^{\mathrm{aug}})^*.$$

This is the co-operad responsible for unital and augmented (or, equivalently, non-unital) co-associative co-algebras.

We set
$$\mathrm{Cocom}^{\mathrm{aug}} := (\mathrm{Com}^{\mathrm{aug}})^*.$$

This is the co-operad responsible for unital and augmented (or, equivalently, non-unital) co-commutative co-algebras.

2.2. Ind-nilpotent co-algebras over a co-operad. It turns out that there are two (and, outside of characteristic 0, four) *inequivalent* notions of co-algebra over a given co-operad. In this subsection we will study one of them: the notion of ind-nilpotent co-algebra.

2.2.1.　Recall the action of Vect^Σ on $\mathbf{O}$, considered in Sect. 1.1.1.

Let $\mathcal{Q}$ be a co-operad. By definition, the category

$$\mathcal{Q}\text{-Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O})$$

is that of $\mathcal{Q}$-comodules in $\mathbf{O}$ with respect to the $\star$-action.

REMARK 2.2.2. Modules for the above monad should be more properly called 'ind-nilpotent co-algebras with *divided powers*', see [**FraG**, Sect. 3.5]. However, we shall omit the reference to divided powers from the notation because we are working over a field of characteristic zero.

2.2.3.　We have the pair of adjoint functors

$$\mathbf{oblv}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}} : \mathcal{Q}\text{-Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O}) \rightleftarrows \mathbf{O} : \mathbf{cofree}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}},$$

with $\mathbf{oblv}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}}$ being co-monadic. In particular, $\mathbf{oblv}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}}$ is conservative, preserves all colimits and totalizations of $\mathbf{oblv}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}}$-split co-simplicial objects.

2.2.4.　The augmentation on $\mathcal{Q}$ gives rise to a functor

$$\mathbf{triv}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}} : \mathbf{O} \to \mathcal{Q}\text{-Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O}),$$

right inverse to $\mathbf{oblv}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}}$.

2.3. The Koszul dual (co)-operad. In this subsection we introduce the Koszul duality functor that relates operads and co-operads.

2.3.1.　Let $\mathbf{O}'$ be a (not necessarily symmetric) monoidal category with limits and colimits. We will assume that the monoidal operation on $\mathbf{O}'$ commutes with *sifted* colimits in each variable (but not not necessarily all colimits).

In this case we have a pair of mutually adjoint functors

$$(2.1) \qquad \mathrm{Bar}^{\mathrm{enh}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}') \rightleftarrows \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}') : \mathrm{coBar}^{\mathrm{enh}},$$

referred to as Koszul duality, see Sect. 3.2.7 below.

(Note, however, that since the monoidal operation on $\mathbf{O}'$ is not assumed to commute with coproducts, augmented associatove/co-associative algebras/co-algebras in $\mathbf{O}'$ are *not* the same as algebras/co-algebras over the $\mathrm{AssocAlg}^{\mathrm{aug}}$-operad.)

We apply this to $\mathbf{O}' := \mathrm{Vect}^\Sigma$. In this case, the resulting functors

$$\mathrm{Bar}^{\mathrm{enh}} : \mathrm{Operads} \rightleftarrows \mathrm{coOperads} : \mathrm{coBar}^{\mathrm{enh}}$$

are mutually inverse equivalences. One can prove this by adapting the argument of [**FraG**, Proposition 4.1.2].

2.3.2.   Let $\mathcal{P}$ be an operad. We denote

$$\mathcal{P}^\vee := \mathrm{Bar}^{\mathrm{enh}}(\mathcal{P}),$$

and for a co-operad $\mathcal{Q}$ we denote

$$\mathcal{Q}^\vee := \mathrm{coBar}^{\mathrm{enh}}(\mathcal{Q}).$$

If $\mathcal{P} \in \mathrm{Vect}^\Sigma_{\mathrm{f.d.}}$, then $\mathcal{P}^\vee$ has the same property, and vice versa.

2.3.3.   It is known that

$$(\mathrm{Coassoc}^{\mathrm{aug}})^\vee \simeq \mathrm{Assoc}^{\mathrm{aug}}[-1],$$

and hence

$$(\mathrm{Assoc}^{\mathrm{aug}})^\vee \simeq \mathrm{Coassoc}^{\mathrm{aug}}[1].$$

It is also known that

$$(\mathrm{Cocom}^{\mathrm{aug}})^\vee \simeq \mathrm{Lie}[-1],$$

and hence

$$\mathrm{Lie}^\vee \simeq \mathrm{Cocom}^{\mathrm{aug}}[1].$$

2.4. Koszul duality functors. In this subsection we will define the operation central for this section (and the entire chapter): the functors of Koszul duality that relate algebras over an operad to co-algebras over the Koszul dual co-operad.

The exposition here follows closely [**FraG**, Sect. 3].

2.4.1.   Let $\mathcal{P}$ be an operad. Recall the functor

$$\mathbf{triv}_\mathcal{P} : \mathbf{O} \to \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}).$$

It preserves limits (since its composition with the conservative limit-preserving functor $\mathbf{oblv}_\mathcal{P}$ does). Hence, by the Adjoint Functor Theorem, it admits a left adjoint:

$$\mathrm{coPrim}_\mathcal{P} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathbf{O}.$$

By adjunction,

$$\mathrm{coPrim}_\mathcal{P} \circ \mathbf{free}_\mathcal{P} \simeq \mathrm{Id}.$$

2.4.2.   We calculate the functor $\mathrm{coPrim}_\mathcal{P}$ as the Bar-construction of the augmented associative algebra $\mathcal{P}$ in $\mathrm{AssocAlg}(\mathrm{Vect}^\Sigma)$ acting on a $\mathcal{P}$-module in $\mathbf{O}$:

$$\mathrm{coPrim}_\mathcal{P} \simeq \mathrm{Bar}(\mathcal{P}, -).$$

It follows from the definition of the Koszul duality functor that for an operad $\mathcal{P}$ we have a canonical isomorphism of co-monads acting on $\mathbf{O}$:

$$\mathrm{coPrim}_\mathcal{P} \circ \mathbf{triv}_\mathcal{P} \simeq \mathcal{P}^\vee \star -,$$

see [**FraG**, Lemma 3.3.4] (or Sect. 7.1.3 later in this chapter).

2.4.3.   Hence, we obtain that the functor $\mathrm{coPrim}_{\mathcal{P}}$ canonically lifts to a functor

$$\mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathcal{Q}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O}),$$

where $\mathcal{Q} = \mathcal{P}^{\vee}$, so that

$$\mathrm{coPrim}_{\mathcal{P}} \simeq \mathbf{oblv}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}} \circ \mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp}},$$

and

(2.2) $$\mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp}} \circ \mathbf{triv}_{\mathcal{P}} \simeq \mathbf{cofree}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}},$$

see [**FraG**, Corollary 3.3.5 and Sect. 3.3.6].

We also have:

(2.3) $$\mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp}} \circ \mathbf{free}_{\mathcal{P}} \simeq \mathbf{triv}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}}$$

2.5. A digression: the filtered version. In this subsection we observe that the functors $\mathrm{coPrim}_{\mathcal{P}}$ and $\Phi = \mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp}}$ naturally admit filtered versions.

2.5.1.   In the context of Sect. 1.5 let us take

(i) $\mathcal{C}(\mathbf{O}) = \mathbf{O}$, $\Phi = \mathrm{coPrim}_{\mathcal{P}}$.

(ii) $\mathcal{C}(\mathbf{O}) = \mathcal{P}^{\vee}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O})$, $\Phi = \mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp}}$.

2.5.2.   Thus, we obtain that the functor

$$\mathrm{coPrim}_{\mathcal{P}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathbf{O}$$

canonically lifts to a functor

$$\mathrm{coPrim}_{\mathcal{P}}^{\mathrm{Fil}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathbf{O}^{\mathrm{Fil},\geq 0},$$

and the functor

$$\mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathcal{P}^{\vee}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O})$$

canonically lifts to a functor

$$\mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp,Fil}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathcal{P}^{\vee}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O}^{\mathrm{Fil},\geq 0})$$

so that

$$\mathrm{coPrim}_{\mathcal{P}}^{\mathrm{Fil}} \simeq \mathbf{oblv}_{\mathcal{P}^{\vee}}^{\mathrm{ind\text{-}nilp}} \circ \mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp,Fil}} .$$

We have a canonical isomorphism

(2.4) $$\mathrm{ass\text{-}gr} \circ \mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp,Fil}} \simeq \mathbf{cofree}_{\mathcal{P}^{\vee}}^{\mathrm{ind\text{-}nilp}} \circ (\deg = 1) \circ \mathbf{oblv}_{\mathcal{P}},$$

as functors $\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathcal{P}^{\vee}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O}^{\mathrm{gr},\geq 0})$.

REMARK 2.5.3.   One can show that the composite functor

$$\mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp,Fil}} \circ \mathbf{triv}_{\mathcal{P}} : \mathbf{O} \to \mathcal{P}^{\vee}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O}^{\mathrm{Fil},\geq 0})$$

identifies canonically with

$$(\mathrm{gr} \to \mathrm{Fil}) \circ \mathbf{cofree}_{\mathcal{P}^{\vee}}^{\mathrm{ind\text{-}nilp}} \circ (\deg = 1).$$

2.6. The adjoint Koszul duality functors. In this subsection we describe the construction of the adjoint Koszul duality functor: it goes from the category of (ind-nilpotent) co-algebras over a given co-operad $\mathcal{Q}$ to the category of algebras over the Koszul dual of $\mathcal{Q}$.

2.6.1.   Let $\mathcal{Q}$ be a co-operad. The functor $\mathbf{triv}_{\mathcal{Q}}^{\text{ind-nilp}}$ preserves colimits, since its composition with $\mathbf{oblv}_{\mathcal{Q}}^{\text{ind-nilp}}$ does. Hence, by the Adjoint Functor Theorem, it admits a right adjoint

$$\mathrm{Prim}_{\mathcal{Q}}^{\text{ind-nilp}} : \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O}) \to \mathbf{O}.$$

By adjunction, $\mathrm{Prim}_{\mathcal{Q}}^{\text{ind-nilp}} \circ \mathbf{cofree}_{\mathcal{Q}}^{\text{ind-nilp}} \simeq \mathrm{Id}$.

2.6.2.   We calculate the functor $\mathrm{Prim}_{\mathcal{P}}^{\text{ind-nilp}}$ as the coBar-construction of the augmented co-associative co-algebra $\mathcal{Q}$ in $\mathrm{CoassocCoalg}(\mathrm{Vect}^{\Sigma})$ acting on a $\mathcal{Q}$-comodule in $\mathbf{O}$:

$$\mathrm{Prim}_{\mathcal{Q}}^{\text{ind-nilp}} \simeq \mathrm{coBar}(\mathcal{Q}, -).$$

In addition, for a co-operad $\mathcal{Q}$, we have a canonical *morphism* (but not an isomorphism) of monads

(2.5) $$(\mathcal{Q}^{\vee} \star -) \to \mathrm{Prim}_{\mathcal{Q}}^{\text{ind-nilp}} \circ \mathbf{triv}_{\mathcal{Q}}^{\text{ind-nilp}},$$

see [**FraG**, Lemma 3.3.9].

2.6.3.   Hence, we obtain that the functor $\mathrm{Prim}_{\mathcal{Q}}^{\text{ind-nilp}}$ canonically lifts to a functor

$$\mathrm{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}} : \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}),$$

where $\mathcal{P} = \mathcal{Q}^{\vee}$, so that

$$\mathrm{Prim}_{\mathcal{Q}}^{\text{ind-nilp}} \simeq \mathbf{oblv}_{\mathcal{P}} \circ \mathrm{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}},$$

see [**FraG**, Corollary 3.3.11], and

$$\mathrm{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}} \circ \mathbf{cofree}_{\mathcal{Q}}^{\text{ind-nilp}} \simeq \mathbf{triv}_{\mathcal{P}}.$$

The map (2.5) gives rise to a natural transformation of functors $\mathbf{O} \to \mathcal{P}\text{-Alg}(\mathbf{O})$, namely,

$$\mathbf{free}_{\mathcal{P}} \to \mathrm{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}} \circ \mathbf{triv}_{\mathcal{Q}}^{\text{ind-nilp}}.$$

2.6.4.   Furthermore, according to [**FraG**, Corollary 3.3.13] the functors

(2.6) $$\mathrm{coPrim}_{\mathcal{P}}^{\text{enh,ind-nilp}} : \mathcal{P}\text{-Alg}(\mathbf{O}) \rightleftarrows \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O}) : \mathrm{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}}$$

are mutually adjoint.

2.6.5.   The following is part of [**FraG**, Conjecture 3.4.5]:

CONJECTURE 2.6.6.   *The functor*

$$\mathrm{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}} : \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O})$$

is fully faithful.

In the sequel, we will relate Conjecture 2.6.6 to several other plausible conjectures, see Sect. 2.11.

REMARK 2.6.7.   Added in November 2021: It turns out that Conjecture 2.6.6 is false. Namely, the co-unit map

$$\mathrm{coPrim}_{\mathcal{P}}^{\text{enh,ind-nilp}} \circ \mathrm{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}} \to \mathrm{Id}$$

fails to be an isomorphism for $\mathcal{P} = \mathrm{Com}^{\text{aug}}$ (and so $\mathcal{Q}$ is the shifted Lie operad), when evaluated on

$$A := \mathbf{triv}_{\mathcal{Q}}^{\text{ind-nilp}}(V),$$

where V is an infinite-dimensional vector space. We are grateful to J. Lurie for pointing this out to us.

Indeed, in this case $\mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh,ind\text{-}nilp}}(A)$ is the completed polynomial algebra on the vector space V as generators, and its classical cotangent space has a non-trivial kernel when mapping to V.

2.7. (Usual) co-algebras over a co-operad. In this subsection we will define another notion of co-algebra over a given co-operad. It is this notion that in the case of $\mathrm{CoAssoc}^{\mathrm{aug}}$ (resp., $\mathrm{Cocom}^{\mathrm{aug}}$) recovers co-associative co-algebras (resp., co-commutative co-algebras).

2.7.1.   We have another *right-lax* action of Vect^{Σ} on $\mathbf{O}$, given by
$$\mathcal{P} * V = \prod_{n \geq 1} \left(\mathcal{P}(n) \otimes V^{\otimes n} \right)^{\Sigma_n}.$$

2.7.2.   For a co-operad $\mathcal{Q}$, the category $\mathcal{Q}\text{-}\mathrm{Coalg}(\mathbf{O})$ of augmented $\mathcal{Q}$-co-algebras is that of $\mathcal{Q}$-co-modules in $\mathbf{O}$ with respect to the $*$-action.

REMARK 2.7.3. Note, however, that since the $*$-action of Vect^{Σ} on $\mathbf{O}$ is only right-lax, the functor $\mathbf{O} \to \mathbf{O}$, defined by $\mathcal{Q}$, is *not* a co-monad.

2.7.4.   For example, for $\mathcal{Q} = \mathrm{Coassoc}^{\mathrm{aug}}$, we obtain the usual category $\mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O})$ of co-unital augmented (or, equivalently, non co-unital) co-associative co-algebras.

Similarly, for $\mathcal{Q} = \mathrm{Cocom}^{\mathrm{aug}}$, we obtain the usual category $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$ of co-unital augmented (or, equivalently, non co-unital) co-commutative co-algebras.

2.7.5.   We let
$$\mathbf{oblv}_{\mathcal{Q}} : \mathcal{Q}\text{-}\mathrm{Coalg}(\mathbf{O}) \to \mathbf{O}$$
denote the corresponding forgetful functor.

The functor $\mathbf{oblv}_{\mathcal{Q}}$ is conservative and preserves all colimits (in fact, one can show that $\mathbf{oblv}_{\mathcal{Q}}$ admits a right adjoint, but it is not easy to describe this right adjoint explicitly).

In addition, it is known that the functor $\mathbf{oblv}_{\mathcal{Q}}$ commutes with totalizations of $\mathbf{oblv}_{\mathcal{Q}}$-split co-simplicial objects.

REMARK 2.7.6. From the above it follows that the functor
$$\mathbf{oblv}_{\mathcal{Q}} : \mathcal{Q}\text{-}\mathrm{Coalg}(\mathbf{O}) \to \mathbf{O}$$
is co-monadic. Yet, as was noted in Remark 2.7.3, the corresponding endo-functor of $\mathbf{O}$ is *not* the one, given by the $*$-action of $\mathcal{Q}$.

2.7.7.   The augmentation on $\mathcal{Q}$ defines the functor
$$\mathbf{triv}_{\mathcal{Q}} : \mathbf{O} \to \mathcal{Q}\text{-}\mathrm{Coalg}(\mathbf{O}),$$
right inverse to $\mathbf{oblv}_{\mathcal{Q}}$.

The functor $\mathbf{triv}_{\mathcal{Q}}$ preserves colimits, since its composition with $\mathbf{oblv}_{\mathcal{Q}}$ does. Hence, the functor $\mathbf{triv}_{\mathcal{Q}}$ admits a right adjoint
$$\mathrm{Prim}_{\mathcal{Q}} : \mathcal{Q}\text{-}\mathrm{Coalg}(\mathbf{O}) \to \mathbf{O}.$$

In Sect. A.2 we will describe the functor $\mathrm{Prim}_{\mathcal{Q}}$ a little more explicitly.

2.8. Relation between two types of co-algebras. In this subsection we will study the relationship between the notions of co-algebra over a co-operad and that of ind-nilpotent co-algebra.

2.8.1. Note that we have the following natural transformation between the two right-lax actions of Vect^Σ on $\mathbf{O}$:

$$\tag{2.7} \mathcal{P} \star V \to \mathcal{P} * V.$$

REMARK 2.8.2. Note that the natural transformation (2.7) involves the operation of averaging with respect to symmetric groups, see [**FraG**, Sect. 3.5.5].

2.8.3. The natural transformation (2.7) gives rise to the forgetful functor

$$\tag{2.8} \mathbf{res}^{\star\to*} : \mathcal{Q}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O}) \to \mathcal{Q}\text{-}\mathrm{Coalg}(\mathbf{O}).$$

We propose[5]:

CONJECTURE 2.8.4. *The functor*

$$\mathbf{res}^{\star\to*} : \mathcal{Q}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O}) \to \mathcal{Q}\text{-}\mathrm{Coalg}(\mathbf{O})$$

of (2.8) *is fully faithful.*

2.8.5. We have:

$$\tag{2.9} \mathbf{oblv}_\mathcal{Q} \circ \mathbf{res}^{\star\to*} \simeq \mathbf{oblv}_\mathcal{Q}^{\mathrm{ind\text{-}nilp}}, \quad \mathcal{Q}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O}) \to \mathbf{O}$$

and

$$\tag{2.10} \mathbf{triv}_\mathcal{Q} \simeq \mathbf{res}^{\star\to*} \circ \mathbf{triv}_\mathcal{Q}^{\mathrm{ind\text{-}nilp}}, \quad \mathbf{O} \to \mathcal{Q}\text{-}\mathrm{Coalg}(\mathbf{O}).$$

We shall denote

$$\mathbf{cofree}_\mathcal{Q}^{\mathrm{fake}} := \mathbf{res}^{\star\to*} \circ \mathbf{cofree}_\mathcal{Q}^{\mathrm{ind\text{-}nilp}} : \mathbf{O} \to \mathcal{Q}\text{-}\mathrm{Coalg}(\mathbf{O}).$$

2.8.6. Let $\mathcal{P} := \mathcal{Q}^\vee$ be the Koszul dual operad. We denote

$$\mathrm{coPrim}_\mathcal{P}^{\mathrm{enh}} := \mathbf{res}^{\star\to*} \circ \mathrm{coPrim}_\mathcal{P}^{\mathrm{enh,ind\text{-}nilp}}, \quad \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \to \mathcal{Q}\text{-}\mathrm{Coalg}(\mathbf{O}).$$

By (2.2), we have

$$\tag{2.11} \mathrm{coPrim}_\mathcal{P}^{\mathrm{enh}} \circ \mathbf{triv}_\mathcal{P} \simeq \mathbf{cofree}_\mathcal{Q}^{\mathrm{fake}}$$

and by (2.3), we have

$$\tag{2.12} \mathrm{coPrim}_\mathcal{P}^{\mathrm{enh}} \circ \mathbf{free}_\mathcal{P} \simeq \mathbf{triv}_\mathcal{Q}.$$

---

[5]In [**FraG**, Remark 3.5.3] it was erroneously stated that the authors knew how to prove this statement. Unfortunately, this turned out not be the case.

2.8.7.   It follows from (2.9) that the functor $\mathbf{res}^{\star\to*}$ commutes with colimits. Hence, it admits a right adjoint, denoted $(\mathbf{res}^{\star\to*})^R$.

We define

$$(2.13) \qquad \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh}} := \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh,ind\text{-}nilp}} \circ (\mathbf{res}^{\star\to*})^R : \mathcal{Q}\text{-Coalg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}).$$

By adjunction, the functors

$$\mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh}} : \mathcal{P}\text{-Alg}(\mathbf{O}) \rightleftarrows \mathcal{Q}\text{-Coalg}(\mathbf{O}) : \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh}}$$

form an adjoint pair.

By passing to right adjoints in (2.10), we obtain an isomorphism:

$$(2.14) \qquad\qquad \mathrm{Prim}_{\mathcal{Q}} \simeq \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}} \circ (\mathbf{res}^{\star\to*})^R,$$

and applying the definition of $\mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh}}$

$$\mathbf{oblv}_{\mathcal{Q}} \circ \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh}} \simeq \mathrm{Prim}_{\mathcal{Q}}.$$

2.8.8.   We propose the following variant of Conjecture 2.6.6:

CONJECTURE 2.8.9.

(a) *The unit of the adjunction*

$$\mathrm{Id} \to \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh}} \circ \mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh}}$$

is an isomorphism, when evaluated on objects lying in the essential image of the functor $\mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh}}$.

(b) *The co-unit of the adjunction*

$$\mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh}} \circ \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh}} \to \mathrm{Id}$$

is an isomorphism, when evaluated on objects lying in the essential image of the functor $\mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh}}$.

REMARK 2.8.10.   Added in November 2021: just like Conjecture 2.6.6, the same counterexample disproves point (b) of Conjecture 2.8.9.

As of now, point (a) of Conjecture 2.8.9 still stands, but we are highly dubious of its validity.

2.9. Calculation of primitives. In this subsection we will be concerned with the functor

$$\mathrm{Prim}_{\mathcal{Q}} \circ \mathbf{cofree}_{\mathcal{Q}}^{\mathrm{fake}} : \mathbf{O} \to \mathbf{O},$$

where we recall that $\mathbf{cofree}_{\mathcal{Q}}^{\mathrm{fake}}$ is the functor

$$\mathbf{res}^{\star\to*} \circ \mathbf{cofree}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}} : \mathbf{O} \to \mathcal{Q}\text{-Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O}).$$

2.9.1.   Consider the unit of the adjunction

$$\mathrm{Id} \to (\mathbf{res}^{\star \to *})^R \circ \mathbf{res}^{\star \to *}.$$

Composing with $\mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh,ind\text{-}nilp}}$ and pre-composing with $\mathbf{cofree}_{\mathcal{Q}}^{\mathrm{fake}}$, we obtain a natural transformation

$$\mathbf{triv}_{\mathcal{P}} \simeq \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh,ind\text{-}nilp}} \circ \mathbf{cofree}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}} \to$$

$$\to \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh,ind\text{-}nilp}} \circ (\mathbf{res}^{\star \to *})^R \circ \mathbf{res}^{\star \to *} \circ \mathbf{cofree}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}} \simeq \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh}} \circ \mathbf{cofree}_{\mathcal{Q}}^{\mathrm{fake}},$$

where $\mathcal{P} := \mathcal{Q}^{\vee}$. I.e., we have a natural transformation:

$$(2.15) \qquad \mathbf{triv}_{\mathcal{P}} \to \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh}} \circ \mathbf{cofree}_{\mathcal{Q}}^{\mathrm{fake}}, \quad \mathbf{O} \to \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}).$$

Composing further with the forgetful functor $\mathbf{oblv}_{\mathcal{P}}$

$$(2.16) \qquad \mathrm{Id} \to \mathrm{Prim}_{\mathcal{Q}} \circ \mathbf{cofree}_{\mathcal{Q}}^{\mathrm{fake}},$$

as endo-functors of $\mathbf{O}$.

2.9.2.   The following conjecture follows tautologically from Conjecture 2.8.4:

CONJECTURE 2.9.3. *Then the natural transformation* (2.16) *is an isomorphism.*

Since the functor $\mathbf{oblv}_{\mathcal{P}}$ is conservative, Conjecture 2.9.3 is equivalent to the natural transformation (2.15) being an isomorphism.

In Sect. A we will prove:

THEOREM 2.9.4. *Conjecture 2.9.3 holds if the co-operad $\mathcal{Q}$ is such that $\mathcal{Q}$ and $\mathcal{Q}^{\vee}[1]$ are both classical and finite-dimensional.*

2.10. Some implications. In this subsection we will assume that Conjecture 2.9.3 holds for a given co-operad $\mathcal{Q}$ (in particular, it applies to $\mathcal{Q} := \mathrm{Cocom}^{\mathrm{aug}}$ and $\mathcal{Q} := \mathrm{Coassoc}^{\mathrm{aug}}$), and derive some corollaries.

2.10.1.   Note that the fact that the natural transformation (2.16) is an isomorphism can be reformulated as saying that the functor $\mathbf{res}^{\star \to *}$ induces an isomorphism

$$(2.17) \quad \mathrm{Maps}_{\mathcal{Q}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O})} \left(\mathbf{triv}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}}(V), \mathbf{cofree}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}}(W) \right) \to$$

$$\to \mathrm{Maps}_{\mathcal{Q}\text{-}\mathrm{Coalg}(\mathbf{O})} \left(\mathbf{triv}_{\mathcal{Q}}(V), \mathbf{cofree}_{\mathcal{Q}}^{\mathrm{fake}}(W) \right)$$

is an isomorphism for any $V, W \in \mathbf{O}$.

2.10.2.   We claim:

PROPOSITION 2.10.3. *The functor $\mathbf{res}^{\star \to *}$ defines an isomorphism*

$$\mathrm{Maps}_{\mathcal{Q}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O})} \left(\mathbf{triv}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}}(V), A \right) \to \mathrm{Maps}_{\mathcal{Q}\text{-}\mathrm{Coalg}(\mathbf{O})} \left(\mathbf{triv}_{\mathcal{Q}}(V), \mathbf{res}^{\star \to *}(A) \right)$$

for any $V \in \mathbf{O}$ and $A \in \mathcal{Q}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O})$.

PROOF. For the proof we will need the following lemma:

LEMMA 2.10.4. *The functor $\mathbf{res}^{\star \to *}$ preserves totalizations of co-simplicial objects that are $\mathbf{oblv}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}}$-split.*

PROOF. Follows from the combination of the following three facts:

(1) the functor $\mathbf{oblv}_{\mathcal{Q}}^{\text{ind-nilp}}$ commutes with totalizations of $\mathbf{oblv}_{\mathcal{Q}}^{\text{ind-nilp}}$-split co-implicial objects;

(2) the functor $\mathbf{res}^{\star\to\ast}$ sends $\mathbf{oblv}_{\mathcal{Q}}^{\text{ind-nilp}}$-split co-simplicial objects to co-simplicial objects that are $\mathbf{oblv}_{\mathcal{Q}}$-split;

(3) the functor $\mathbf{oblv}_{\mathcal{Q}}$ commutes with totalizations of $\mathbf{oblv}_{\mathcal{Q}}$-split co-implicial objects. $\square$

Now, the assertion of the proposition follows from the fact that every object

$$A \in \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})$$

can be written as such a totalization as in Lemma 2.10.4, whose terms are objects of the form $\mathbf{cofree}_{\mathcal{Q}}^{\text{ind-nilp}}(W)$ for $W \in \mathbf{O}$. $\square$

COROLLARY 2.10.5.

(a) *The natural transformation* $\text{Prim}_{\mathcal{Q}}^{\text{ind-nilp}} \to \text{Prim}_{\mathcal{Q}} \circ \mathbf{res}^{\star\to\ast}$ *is an isomorphism.*

(b) $\text{Prim}_{\mathcal{Q}}^{\text{enh,ind-nilp}} \to \text{Prim}_{\mathcal{Q}}^{\text{enh}} \circ \mathbf{res}^{\star\to\ast}$ *is an isomorphism.*

2.10.6.   As another corollary of Proposition 2.10.3, we obtain:

COROLLARY 2.10.7. *The functor* $\mathbf{res}^{\star\to\ast}$ *defines an isomorphism*

$$\text{Maps}_{\mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})}(A', A) \to \text{Maps}_{\mathcal{Q}\text{-Coalg}(\mathbf{O})}(\mathbf{res}^{\star\to\ast}(A'), \mathbf{res}^{\star\to\ast}(A))$$

for any A' *lying in the essential image of the functor* $\text{coPrim}_{\mathcal{P}}^{\text{enh,ind-nilp}}$, *where* $\mathcal{P} := \mathcal{Q}^{\vee}$.

PROOF. Follows from the fact that any object of $\mathcal{P}\text{-Alg}(\mathbf{O})$ can be written as a colimit of ones of the form $\mathbf{free}_{\mathcal{P}}(V)$, while

$$\text{coPrim}_{\mathcal{P}}^{\text{enh,ind-nilp}} \circ \mathbf{free}_{\mathcal{P}}(V) \simeq \mathbf{triv}_{\mathcal{Q}}^{\text{ind-nilp}}(V).$$

$\square$

2.11. Some implications between the conjectures. In this subsection we continue to assume that $\mathcal{Q}$ is such that Conjecture 2.9.3 holds. We will prove that Conjecture 2.6.6 implies Conjectures 2.8.4 and 2.8.9.

2.11.1.   First, we claim:

THEOREM 2.11.2. *Conjecture 2.6.6 (for the co-operad* $\mathcal{Q}$*) implies Conjecture 2.8.4.*

PROOF. Taking into account Corollary 2.10.7, it suffices to know that the functor

$$\text{coPrim}_{\mathcal{P}}^{\text{enh,ind-nilp}} : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})$$

is essentially surjective. However, the latter follows from Conjecture 2.6.6. $\square$

2.11.3. Next, we claim:

THEOREM 2.11.4. *Conjecture 2.6.6 (for the co-operad $\mathcal{Q}$) implies Conjecture 2.8.9.*

PROOF. For point (b) of Conjecture 2.8.9, we claim that a stronger statement follows from Conjecture 2.6.6. Namely, we claim that the natural transformation

$$\mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh}} \circ \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh}} \to \mathrm{Id}$$

is an isomorphism on the essential image of $\mathbf{res}^{\star \to *}$. Indeed, the composition

$$\mathbf{res}^{\star \to *} \circ \mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp}} \circ \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh,ind\text{-}nilp}} \simeq \mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh}} \circ \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh,ind\text{-}nilp}} \to$$
$$\to \mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh}} \circ \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh}} \circ \mathbf{res}^{\star \to *} \to \mathbf{res}^{\star \to *}$$

equals the natural transformation obtained from the co-unit of the adjunction

$$\mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp}} \circ \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh,ind\text{-}nilp}} \to \mathrm{Id}$$

by composing with $\mathbf{res}^{\star \to *}$. Hence, it is an isomorphism, by assumption.

Now, the second arrow in the above composition is an isomorphism by Corollary 2.10.5(b). Hence, so is the third arrow.

The unit of the adjunction

$$\mathrm{Id} \to \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh}} \circ \mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh}}$$

identifies with the composition

$$\mathrm{Id} \to \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh,ind\text{-}nilp}} \circ \mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp}} \to$$
$$\to \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh,ind\text{-}nilp}} \circ (\mathbf{res}^{\star \to *})^{R} \circ \mathbf{res}^{\star \to *} \circ \mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp}} \simeq \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh}} \circ \mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh}}.$$

Now, since we already know that Conjecture 2.6.6 implies Conjecture 2.8.4, it suffices to show that the map

$$\mathrm{Id} \to \mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh,ind\text{-}nilp}} \circ \mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh,ind\text{-}nilp}}$$

is an isomorphism on the essential image of $\mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh,ind\text{-}nilp}}$. However, this is a formal consequence of the fact that $\mathrm{Prim}_{\mathcal{Q}}^{\mathrm{enh,ind\text{-}nilp}}$ is fully faithful.

□

3. Associative algebras

In this section we specialize the notions from Sects. 1 and 2 to the case of the associative operads, and point out some specifics.

In particular, we will see that the (augmented) associative operad is self Koszul-dual and we will give more explicit descriptions of the Koszul duality functors between augmented associative algebras and co-algebras.

3.1. Associative algebras and co-algebras. In this subsection we recall some basic concepts related to the notion of associative algebra in a given monoidal category.

3.1.1.   Let $\mathbf{O}$ be a monoidal category. We let $\mathrm{AssocAlg}(\mathbf{O})$ denote the category of unital associative algebras in $\mathbf{O}$. We let $\mathbf{oblv}_{\mathrm{Assoc}}$ denote the forgetful functor $\mathrm{AssocAlg}(\mathbf{O}) \to \mathbf{O}$. The functor $\mathbf{oblv}_{\mathrm{Assoc}}$ is conservative and commutes with limits.

Since $\mathbf{1_O} \in \mathbf{O}$ is the initial object in $\mathrm{AssocAlg}(\mathbf{O})$, the functor $\mathbf{oblv}_{\mathrm{Assoc}}$ canonically factors as

$$\mathrm{AssocAlg}(\mathbf{O}) \to \mathbf{O}_{\mathbf{1_O}/} \to \mathbf{O}.$$

We will denote the resulting functor $\mathrm{AssocAlg}(\mathbf{O}) \to \mathbf{O}_{\mathbf{1_O}/}$ by $\mathbf{oblv}_{\mathrm{Assoc},\mathbf{1}/}$.

3.1.2.   Assume that $\mathbf{O}$ admits colimits, and that the monoidal operation preserves *sifted* colimits in each variable. Then the category $\mathrm{AssocAlg}(\mathbf{O})$ also admits colimits, and the functor $\mathbf{oblv}_{\mathrm{Assoc}}$ commutes *sifted* colimits, see [**Lu2**, Proposition 3.2.3.1].

Moreover, in this case $\mathbf{oblv}_{\mathrm{Assoc}}$ admits a left adjoint, denoted

$$\mathbf{free}_{\mathrm{Assoc}} : \mathbf{O} \to \mathrm{AssocAlg}(\mathbf{O}).$$

When the monoidal operation on $\mathbf{O}$ commutes with coproducts, the composition $\mathbf{oblv}_{\mathrm{Assoc}} \circ \mathbf{free}_{\mathrm{Assoc}}$ is canonically isomorphic to the functor

$$V \mapsto \underset{n \geq 0}{\sqcup} V^{\otimes n},$$

see [**Lu2**, Proposition 4.1.1.14].

REMARK 3.1.3. Note that the adjoint pair

$$\mathbf{free}_{\mathrm{Assoc}} : \mathbf{O} \rightleftarrows \mathrm{AssocAlg}(\mathbf{O}) : \mathbf{oblv}_{\mathrm{Assoc}}$$

does *not* fit into the paradigm of algebras over operads as defined in Sect. 1.1.2. This is because in our definition of operads we did not allow 0-ary operations.

3.1.4.   Assume that $\mathbf{O}$ is *symmetric* monoidal. In this case, the category $\mathrm{AssocAlg}(\mathbf{O})$ has a natural symmetric monoidal structure (given by tensor product) and the functor $\mathbf{oblv}_{\mathrm{Assoc}}$ is naturally symmetric monoidal, see Volume I, Chapter 1, Sect. 3.3.5.

Since the initial object of $\mathrm{AssocAlg}(\mathbf{O})$, i.e., $\mathbf{1_O}$, is the unit of $\mathrm{AssocAlg}(\mathbf{O})$ with respect to its symmetric monoidal structure, the identity functor on $\mathrm{AssocAlg}(\mathbf{O})$ has a natural right-lax symmetric monoidal structure, when considered as a functor from $\mathrm{AssocAlg}(\mathbf{O})$ equipped with the tensor product structure to $\mathrm{AssocAlg}(\mathbf{O})$ equipped with the co-Cartesian symmetric monoidal structure.

I.e., we have a compatible system of natural transformations:

$$(3.1) \qquad\qquad\qquad A_1 \sqcup ... \sqcup A_n \to A_1 \otimes ... \otimes A_n,$$

given as the coproduct of the maps

$$A_i \simeq \mathbf{1_O} \otimes ... \otimes A_i \otimes ... \otimes \mathbf{1_O} \to A_1 \otimes ... \otimes A_n.$$

3.1.5.   Let $\mathbf{O} = \mathbf{C}$ be a category with finite limits, viewed as a symmetric monoidal category with respect to the Cartesian symmetric monoidal structure. In this case we have, by definition,

$$\mathrm{AssocAlg}(\mathbf{C}) = \mathrm{Monoid}(\mathbf{C}).$$

3.1.6. Let $\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$ denote the category $(\mathrm{AssocAlg}(\mathbf{O}))_{/\mathbf{1_O}}$. This is the category of augmented associative algebras on $\mathbf{O}$. The category $\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$ has several forgetful functors, denoted

$$\mathbf{oblv}_{\mathrm{Assoc}}, \ \mathbf{oblv}_{\mathrm{Assoc},\mathbf{1}/}, \ \mathbf{oblv}_{\mathrm{Assoc},/\mathbf{1}}, \ \mathbf{oblv}_{\mathrm{Assoc},\mathbf{1}//\mathbf{1}},$$

with values in

$$\mathbf{O}, \ \mathbf{O}_{\mathbf{1_O}/}, \ \mathbf{O}_{/\mathbf{1_O}}, \ \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}},$$

respectively.

3.1.7. In this sub-subsection, we shall assume that $\mathbf{O}$ is a symmetric monoidal DG category. (In particular, the monoidal operation on $\mathbf{O}$ commutes with all colimits.)

We have a canonical equivalence

$$(3.2) \qquad\qquad \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \simeq \mathrm{Assoc}^{\mathrm{aug}}\text{-}\mathrm{Alg}(\mathbf{O}),$$

where the latter is the category of algebras over the $\mathrm{Assoc}^{\mathrm{aug}}$ operad. Thus, we obtain yet another forgetful functor

$$\mathbf{oblv}_{\mathrm{AssocAlg},+} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O},$$

equal in the notion of Sect. 1.1.3 to $\mathbf{oblv}_{\mathrm{Assoc}^{\mathrm{aug}}}$. It equals the composition of $\mathbf{oblv}_{\mathrm{Assoc},\mathbf{1}//\mathbf{1}}$ with the functor $\mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}} \to \mathbf{O}$, inverse to the equivalence

$$(3.3) \qquad\qquad V \mapsto \mathbf{1_O} \oplus V, \quad \mathbf{O} \to \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}},$$

i.e. it is given by the fiber of the augmentation map $V \to \mathbf{1_O}$.

The functor $\mathbf{free}_{\mathrm{Assoc}}$ is naturally isomorphic to the composition

$$\mathbf{O} \overset{\mathbf{free}_{\mathrm{AssocAlg}^{\mathrm{aug}}}}{\longrightarrow} \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{AssocAlg}(\mathbf{O}),$$

where the second arrow is the forgetful functor.

3.1.8. By reversing the arrows, we obtain the corresponding definitions and pieces of notation of co-associative co-algebras.

3.1.9. The following observation will be used repeatedly. Let $\mathbf{O} = \mathbf{C}$ be as in Sect. 3.1.5. Then the forgetful functor

$$\mathbf{oblv}_{\mathrm{Coassoc}} : \mathrm{Coassoc}(\mathbf{C}) \to \mathbf{C}$$

is an equivalence, see [**Lu2**, Proposition 2.4.3.9].

Informally, every object $\mathbf{c}$ of $\mathbf{C}$ canonically lifts to one in $\mathrm{Coassoc}(\mathbf{C})$ via the diagonal map

$$\mathbf{c} \to \mathbf{c} \times \mathbf{c}.$$

3.2. The Bar construction. In this section we let $\mathbf{O}$ be a monoidal category with limits and colimits.

We will review the general Bar-construction that relates augmented associative algebras and co-algebras in $\mathbf{O}$.

3.2.1.   We have a canonically defined functor

$$\mathrm{Bar}^{\bullet} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}},$$

see [**Lu2**, Sect. 5.2.2].

The functor $\mathrm{Bar}^{\bullet}$ lifts to a functor

$$\mathrm{Bar}^{\bullet}_{\mathbf{1}//\mathbf{1}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to (\mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}})_{\mathbf{1_O}//\mathbf{1_O}},$$

where $\mathbf{1_O} \in \mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}}$ is the constant simplicial object with value $\mathbf{1_O}$.

If the monoidal structure on $\mathbf{O}$ is symmetric, then the above functors have a natural symmetric monoidal structure.

3.2.2.   We define the functors

$$\mathrm{Bar}_{\mathbf{1}//\mathbf{1}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}} \quad \text{and} \quad \mathrm{Bar} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}$$

to be the compositions of $\mathrm{Bar}^{\bullet}_{\mathbf{1}//\mathbf{1}}$ (resp., $\mathrm{Bar}^{\bullet}$) with the functor of colimit over $\mathbf{\Delta}^{\mathrm{op}}$ (a.k.a, geometric realization)

$$\mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathbf{O}.$$

If the monoidal structure on $\mathbf{O}$ is symmetric, then the symmetric monoidal structure on $\mathrm{Bar}^{\bullet}_{\mathbf{1}//\mathbf{1}}$ (resp., $\mathrm{Bar}^{\bullet}$) induces one on $\mathrm{Bar}_{\mathbf{1}//\mathbf{1}}$ (resp., Bar).

3.2.3.   The functor $\mathrm{Bar}_{\mathbf{1}//\mathbf{1}}$ can be also thought of as follows:

We have a naturally defined functor

$$\mathbf{triv}_{\mathrm{Assoc}^{\mathrm{aug}}} : \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}} \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}).$$

The functor $\mathrm{Bar}_{\mathbf{1}//\mathbf{1}}$ is the left adjoint of the composition

$$\mathbf{triv}_{\mathrm{Assoc}^{\mathrm{aug}}} \circ \Omega_{\mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}}}.$$

3.2.4.   Suppose for a moment that the monoidal structure on $\mathbf{O}$ is Cartesian. Then

$$\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) = \mathrm{Monoid}(\mathbf{O}),$$

and the corresponding functor

$$\mathrm{Bar}^{\bullet} : \mathrm{Monoid}(\mathbf{O}) \to \mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}}$$

is fully faithful, see [**Lu2**, Proposition 4.1.2.6].

Its essential image consists of those simplicial objects $n \mapsto V^n$, for which the maps for every n the maps

$$[1] \to [n], \quad (\{0\} \mapsto \{i-1\}, \{1\} \mapsto \{i\}), \quad i = 1, ..., n$$

define an isomorphism

$$V^n \to (V^1)^{\times n}.$$

The functor Bar identifies with the classifying space functor

$$B : \mathrm{Monoid}(\mathbf{O}) \to \mathbf{O}.$$

3.2.5. A key feature of the functor

$$\mathrm{Bar}_{\mathbf{1}//\mathbf{1}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}}$$

is that it canonically lifts to a functor

$$\mathrm{Bar}^{\mathrm{enh}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}),$$

i.e.,

$$\mathrm{Bar}_{\mathbf{1}//\mathbf{1}} \simeq \mathbf{oblv}_{\mathrm{Coassoc},\mathbf{1}//\mathbf{1}} \circ \mathrm{Bar}^{\mathrm{enh}},$$

see [**Lu2**, Theorem 5.2.2.17].

If $\mathbf{O}$ is symmetric, then the functor $\mathrm{Bar}^{\mathrm{enh}}$ also acquires a left-lax symmetric monoidal structure, extending that on $\mathrm{Bar}_{\mathbf{1}//\mathbf{1}}$. This structure is strict if the monoidal operation on $\mathbf{O}$ preserves colimits.

3.2.6. Reversing the arrows, we obtain the corresponding functors

$$\mathrm{coBar}^{\bullet}_{\mathbf{1}//\mathbf{1}} : \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}) \to (\mathbf{O}^{\mathbf{\Delta}})_{\mathbf{1_O}//\mathbf{1_O}},$$

$$\mathrm{coBar}^{\bullet} : \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}^{\mathbf{\Delta}},$$

$$\mathrm{coBar}_{\mathbf{1}//\mathbf{1}} : \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}},$$

$$\mathrm{coBar} : \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O},$$

and

$$\mathrm{coBar}^{\mathrm{enh}} : \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}).$$

3.2.7. It is another basic fact that the functors

$$(3.4) \qquad \mathrm{Bar}^{\mathrm{enh}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \rightleftarrows \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}) : \mathrm{coBar}^{\mathrm{enh}}$$

form an adjoint pair.

In general, neither of the functors (3.4) is fully faithful.

3.3. Koszul duality functors: associative case. In this subsection we let $\mathbf{O}$ be a symmetric monoidal DG category.

We will specialize the paradigm of Koszul duality functors

$$\mathrm{coPrim}^{\mathrm{enh}}_{\mathcal{P}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \rightleftarrows \mathcal{P}^{\vee}\text{-}\mathrm{Coalg}(\mathbf{O}) : \mathrm{Prim}^{\mathrm{enh}}_{\mathcal{P}^{\vee}}$$

to the case $\mathcal{P} = \mathrm{Assoc}^{\mathrm{aug}}$.

3.3.1. According to Sect. 3.2, we have a canonically defined functor

$$\mathrm{Bar}_{\mathbf{1}//\mathbf{1}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}}.$$

Let

$$\mathrm{Bar}_{/\mathbf{1}}, \ \mathrm{Bar}_{\mathbf{1}/}$$

and Bar_{+} denote the composition of $\mathrm{Bar}_{\mathbf{1}//\mathbf{1}}$ with the functors

$$\mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}} \to \mathbf{O}_{/\mathbf{1_O}}, \ \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}} \to \mathbf{O}_{\mathbf{1_O}/}$$

and

$$\mathbf{O} \simeq \mathbf{O}_{\mathbf{1_O}//\mathbf{1_O}}, \quad V \mapsto \mathbf{1_O} \oplus V,$$

respectively.

By Sect. 3.2.3, the functor Bar_{+} is the left adjoint to the functor

$$\mathbf{O} \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}), \quad \mathbf{triv}_{\mathrm{AssocAlg}^{\mathrm{aug}}} \circ [-1].$$

I.e.,
$$\mathrm{Bar}_+ \simeq [1] \circ \mathrm{coPrim}_{\mathrm{Assoc}^{\mathrm{aug}}}.$$

3.3.2.   Similarly, we have the functors
$$\mathrm{coBar}_{/\mathbf{1}}, \ \mathrm{coBar}_{\mathbf{1}/}, \ \mathrm{coBar} \ \text{and} \ \mathrm{coBar}_+,$$
where coBar_+ is the right adjoint to the functor
$$\mathbf{O} \to \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}), \quad \mathbf{triv}_{\mathrm{CoassocCoalg}^{\mathrm{aug}}} \circ [1].$$

Hence,
$$\mathrm{coBar}_+ \simeq [-1] \circ \mathrm{Prim}_{\mathrm{Coassoc}^{\mathrm{aug}}}.$$

3.3.3.   As was mentioned in Sect. 2.3.3, we have canonical isomorphisms of operads
$$(3.5) \qquad\qquad\qquad (\mathrm{Coassoc}^{\mathrm{aug}})^\vee \simeq \mathrm{Assoc}^{\mathrm{aug}}[-1].$$

Hence, the functors $\mathrm{Bar}_{\mathbf{1}//\mathbf{1}}$ and $\mathrm{coBar}_{\mathbf{1}//\mathbf{1}}$ lift to functors
$$\mathrm{Bar}^{\mathrm{enh}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}), \quad \mathrm{Bar}_{\mathbf{1}//\mathbf{1}} \simeq \mathbf{oblv}_{\mathrm{Coassoc},\mathbf{1}//\mathbf{1}} \circ \mathrm{Bar}^{\mathrm{enh}}$$
and
$$\mathrm{coBar}^{\mathrm{enh}} : \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}), \quad \mathrm{coBar}_{\mathbf{1}//\mathbf{1}} \simeq \mathbf{oblv}_{\mathrm{Assoc},\mathbf{1}//\mathbf{1}} \circ \mathrm{coBar}^{\mathrm{enh}},$$
respectively.

3.3.4.   It is a basic feature of the isomorphism (3.5) that the above functors $\mathrm{Bar}^{\mathrm{enh}}$ and $\mathrm{coBar}^{\mathrm{enh}}$ are canonically the same as those in Sect. 3.2.5.

In particular, the functors
$$\mathrm{Bar}^{\mathrm{enh}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \rightleftarrows \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}) : \mathrm{coBar}^{\mathrm{enh}}$$
are mutually adjoint, and the functor
$$\mathrm{Bar}^{\mathrm{enh}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O})$$
is naturally symmetric monoidal.

By adjunction, we obtain that the functor $\mathrm{coBar}^{\mathrm{enh}}$ is naturally right-lax symmetric monoidal.

4. Lie algebras and co-commutative co-algebras

In this section we study of the relationship between Lie algebras and co-commutative co-algebras.

The main result of this section is Theorem 4.4.6, which says that, although the Chevalley functor from Lie algebras to co-commutative co-algebras is not fully faithful, its cousin, obtained by first looping our Lie algebra, and regarding the output as a co-commutative Hopf algebra, is fully faithful.

4.1. Koszul duality functors: commutative vs. Lie case. In this subsection we continue to suppose that $\mathbf{O}$ is a symmetric monoidal DG category. We will specialize the paradim of Koszul duality functors
$$\mathrm{coPrim}_{\mathcal{P}}^{\mathrm{enh}} : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O}) \rightleftarrows \mathcal{P}^\vee\text{-}\mathrm{Coalg}(\mathbf{O}) : \mathrm{Prim}_{\mathcal{P}^\vee}^{\mathrm{enh}}$$
to the case $\mathcal{P} = \mathrm{Lie}$.

4.1.1. First, we remark that the discussion in Sect. 3.1 renders verbatim to the situation when instead of associative algebras on $\mathbf{O}$ we talk about (co-)commutative (co-)algebras.

We note, however, the following feature of the symmetric monoidal structure on $\mathrm{ComAlg}(\mathbf{O})$: the corresponding natural transformations (3.1) are isomorphisms.

I.e., the symmetric monoidal structure on $\mathrm{ComAlg}(\mathbf{O})$, given by tensor product, equals the co-Cartesian symmetric monoidal structure, see Volume I, Chapter 1, Sect. 3.3.6. Similarly, the symmetric monoidal structure on $\mathrm{CocomCoalg}(\mathbf{O})$, given by tensor product, equals the Cartesian symmetric monoidal structure.

Note that the forgetful functors

$$\mathbf{res}^{\mathrm{Com}\to\mathrm{Assoc}} : \mathrm{ComAlg}(\mathbf{O}) \to \mathrm{AssocAlg}(\mathbf{O})$$

and

$$\mathbf{res}^{\mathrm{Cocom}\to\mathrm{Coassoc}} : \mathrm{CocomCoalg}(\mathbf{O}) \to \mathrm{CoassocCoalg}(\mathbf{O})$$

both have a natural symmetric monoidal structure.

4.1.2. We let Chev_+ denote the functor

$$\mathrm{LieAlg}(\mathbf{O}) \to \mathbf{O}, \quad [1] \circ \mathrm{coPrim}_{\mathrm{Lie}}.$$

I.e., this is the functor, left adjoint to the functor

$$\mathbf{O} \to \mathrm{LieAlg}(\mathbf{O}), \quad \mathbf{triv}_{\mathrm{Lie}} \circ [-1].$$

We let

$$\mathrm{Chev}_{\mathbf{1}//\mathbf{1}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathbf{O}_{\mathbf{1_o}//\mathbf{1_o}}$$

denote the composition of Chev_+ with the equivalence (3.3).

Composing further with the forgetful functors from $\mathbf{O}_{\mathbf{1_o}//\mathbf{1_o}}$, we obtain the corresponding functors, denoted

$$\mathrm{Chev}_{/\mathbf{1}}, \ \mathrm{Chev}_{\mathbf{1}/}, \ \mathrm{Chev},$$

from $\mathrm{LieAlg}(\mathbf{O})$ to

$$\mathbf{O}_{/\mathbf{1_o}}, \ \mathbf{O}_{\mathbf{1_o}/}, \ \mathbf{O},$$

respectively.

4.1.3. We denote by coChev the functor

$$\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}, \quad [-1] \circ \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}}.$$

I.e., this is the functor, right adjoint to the functor

$$\mathbf{O} \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}), \quad \mathbf{triv}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ [1].$$

4.1.4.   As was mentioned in Sect. 2.3.3, we have canonical isomorphisms of operads

$$(4.1) \qquad\qquad (\mathrm{Cocom}^{\mathrm{aug}})^{\vee} \simeq \mathrm{Lie}[-1].$$

Hence, the functors $\mathrm{Chev}_{\mathbf{1}//\mathbf{1}}$ and coChev lift to functors

$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}), \quad \mathrm{Chev}_{\mathbf{1}//\mathbf{1}} \simeq \mathbf{oblv}_{\mathrm{Cocom},\mathbf{1}//\mathbf{1}} \circ \mathrm{Chev}^{\mathrm{enh}}$$

and

$$\mathrm{coChev}^{\mathrm{enh}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{LieAlg}(\mathbf{O}), \quad \mathrm{coChev} \simeq \mathbf{oblv}_{\mathrm{Lie}} \circ \mathrm{coChev}^{\mathrm{enh}},$$

respectively.

Furthermore, the functors

$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathbf{O}) \rightleftarrows \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) : \mathrm{coChev}^{\mathrm{enh}}$$

are mutually adjoint.

In particular, we obtain a canonical natural transformation

$$\mathrm{Id} \to \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Chev}^{\mathrm{enh}},$$

and by applying the forgetful functor $\mathbf{oblv}_{\mathrm{Lie}}$, also the natural transformation

$$(4.2) \qquad\qquad [1] \circ \mathbf{oblv}_{\mathrm{Lie}} \to \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Chev}^{\mathrm{enh}}.$$

4.1.5.   The functor

$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$$

has a natural left-lax symmetric monoidal structure, when we consider both categories as endowed with Cartesian symmetric monoidal structure. (Recall, however, that the Cartesian symmetric monoidal structure on $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$ equals one given by the tensor product, see Sect. 4.1.1.)

In particular, we obtain that the functor

$$\mathrm{Chev} : \mathrm{LieAlg}(\mathbf{O}) \to \mathbf{O}$$

inherits a left-lax symmetric monoidal structure.

In Sect. 4.2.6 we will prove:

LEMMA 4.1.6.   *The left-lax symmetric monoidal structure on* $\mathrm{Chev}^{\mathrm{enh}}$ *is strict.*

COROLLARY 4.1.7.   *The left-lax symmetric monoidal structure on* Chev *is strict.*

4.2. The symmetric co-algebra. The symmetric (co)algebra construction

$$V \rightsquigarrow \underset{n \geq 0}{\oplus} \mathrm{Sym}^n(V)$$

is ubiquitous in algebra.

In this subsection we initiate its study in its incarnation as a *co-commutative co-algebra*.

4.2.1.   We denote by

$$\mathrm{Sym} : \mathbf{O} \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$$

the functor $\mathbf{cofree}^{\mathrm{fake}}_{\mathrm{Cocom}^{\mathrm{aug}}}$, see Sect. 2.8.5 for the notation.

4.2.2. Let us denote by $\underline{\mathrm{Sym}}$ (resp., $\underline{\mathrm{Sym}}_+$) the functor of $\mathbf{O} \to \mathbf{O}$ equal to the composition $\mathbf{oblv}_{\mathrm{Cocom}} \circ \mathrm{Sym}$ (resp., $\mathbf{oblv}_{\mathrm{Cocom},+} \circ \mathrm{Sym}$).

By definition, the endo-functor $\underline{\mathrm{Sym}}_+$ of $\mathbf{O}$ is one given by

$$V \mapsto \mathrm{Cocom}^{\mathrm{aug}} \star V.$$

Explicitly,

$$\underline{\mathrm{Sym}}(V) = \bigoplus_{n \geq 0} \mathrm{Sym}^n(V) \text{ and } \underline{\mathrm{Sym}}_+(V) = \bigoplus_{n \geq 1} \mathrm{Sym}^n(V).$$

4.2.3. By (2.11) we have

$$\tag{4.3} \mathrm{Chev}^{\mathrm{enh}} \circ \mathbf{triv}_{\mathrm{Lie}} \circ [-1] \simeq \mathrm{Sym},$$

and

$$\mathrm{Chev} \circ \mathbf{triv}_{\mathrm{Lie}} \circ [-1] \simeq \underline{\mathrm{Sym}}.$$

By adjunction from (4.3), we obtain canonical natural transformations

$$\tag{4.4} \mathbf{triv}_{\mathrm{Lie}} \circ [-1] \to \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Sym}, \quad \mathbf{O} \to \mathrm{LieAlg}(\mathbf{O})$$

and by applying $\mathbf{oblv}_{\mathrm{Lie}}$ the natural transformation

$$\tag{4.5} \mathrm{Id} \to \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Sym}.$$

By Theorem 2.9.4, we have:

THEOREM 4.2.4. *The natural transformation* (4.4) *is an isomorphism.*

COROLLARY 4.2.5. *The natural transformation* (4.5) *is an isomorphism.*

4.2.6. *Proof of Lemma 4.1.6.* We need to show that for $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathrm{LieAlg}(\mathbf{O})$, the map

$$\mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h}_1 \times \mathfrak{h}_2) \to \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h}_1) \sqcup \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h}_2) \simeq \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h}_1) \otimes \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h}_2)$$

is an isomorphism.

By Sect. 2.5.2, the above map lifts to a map

$$\mathrm{Chev}^{\mathrm{enh,Fil}}(\mathfrak{h}_1 \times \mathfrak{h}_2) \to \mathrm{Chev}^{\mathrm{enh,Fil}}(\mathfrak{h}_1) \otimes \mathrm{Chev}^{\mathrm{enh,Fil}}(\mathfrak{h}_2),$$

and it suffices to show that this map is an isomorphism in $\mathrm{CocomCoalg}(\mathbf{O}^{\mathrm{Fil}, \geq 0})$.

Since the functor ass-gr is conservative on non-negatively graded objects, in order to show that the latter map is an isomorphism, it suffices to show that the induced map

$$\text{ass-gr} \circ \mathrm{Chev}^{\mathrm{enh,Fil}}(\mathfrak{h}_1 \times \mathfrak{h}_2) \to \text{ass-gr} \circ \mathrm{Chev}^{\mathrm{enh,Fil}}(\mathfrak{h}_1) \otimes \text{ass-gr} \circ \mathrm{Chev}^{\mathrm{enh,Fil}}(\mathfrak{h}_2)$$

is an isomorphism in $\mathrm{CocomCoalg}(\mathbf{O}^{\mathrm{gr}, \geq 0})$.

By (2.4), the latter map identifies with the canonical map

$$\mathrm{Sym}^{\mathrm{gr}}(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}_1) \oplus \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}_2)) \simeq \mathrm{Sym}^{\mathrm{gr}}(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}_1 \times \mathfrak{h}_2)) \to$$
$$\to \mathrm{Sym}^{\mathrm{gr}}(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}_1)) \otimes \mathrm{Sym}^{\mathrm{gr}}(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}_2)),$$

where

$$\mathrm{Sym}^{\mathrm{gr}} : \mathbf{O} \to \mathrm{CocomCoalg}(\mathbf{O}^{\mathrm{gr}, \geq 0})$$

is the graded version of the functor Sym of Sect. 4.2.1, i.e.,

$$\mathrm{Sym}^{\mathrm{gr}} = \mathrm{Sym} \circ (\deg = 1).$$

Now, the fact that for $V_1, V_2 \in \mathbf{O}$ the map

$$\mathrm{Sym}^{\mathrm{gr}}(V_1 \oplus V_2) \to \mathrm{Sym}^{\mathrm{gr}}(V_1) \otimes \mathrm{Sym}^{\mathrm{gr}}(V_2)$$

is an isomorphism, is straightforward.

$\square$

4.3. Chevalley complex and the loop functor. The principal actor in this chapter will be the functor[6]

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomBialg}(\mathbf{O}).$$

We will see (Theorem 4.4.6) that, unlike the functor $\mathrm{Chev}^{\mathrm{enh}}$, the above functor is fully faithful (i.e., looping helps to preserve structure).

4.3.1.   Recall that by Lemma 4.1.6, the functor

$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$$

has a symmetric monoidal structure, when we consider both $\mathrm{LieAlg}(\mathbf{O})$ and $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$ as symmetric monoidal categories with respect to Cartesian product.

In particular, we obtain that $\mathrm{Chev}^{\mathrm{enh}}$ gives rise to a functor

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) : \mathrm{Grp}(\mathrm{LieAlg}(\mathbf{O})) \simeq \mathrm{Monoid}(\mathrm{LieAlg}(\mathbf{O})) \to$$
$$\to \mathrm{Monoid}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})) =: \mathrm{CocomBialg}(\mathbf{O}).$$

Moreover, its essential image automatically lies in

$$\mathrm{CocomHopf}(\mathbf{O}) := \mathrm{Grp}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})) \subset$$
$$\subset \mathrm{Monoid}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})) = \mathrm{CocomBialg}(\mathbf{O}).$$

4.3.2.   Consider now the composite functor

$$\mathbf{oblv}_{\mathrm{Monoid}} \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}).$$

We claim:

PROPOSITION 4.3.3.   *The functor* $\mathbf{oblv}_{\mathrm{Monoid}} \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{Lie}$ *identifies canonically with* $\mathrm{Sym} \circ \mathbf{oblv}_{\mathrm{Lie}}$.

PROOF.   First, we note that we have a tautological isomorphism

$$\mathbf{oblv}_{\mathrm{Monoid}} \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} \simeq \mathrm{Chev}^{\mathrm{enh}} \circ \mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathrm{Lie}}.$$

Now, by Proposition 1.7.2, we have

$$\mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathrm{Lie}} \simeq \mathbf{triv}_{\mathrm{Lie}} \circ [-1] \circ \mathbf{oblv}_{\mathrm{Lie}},$$

so

$$\mathrm{Chev}^{\mathrm{enh}} \circ \mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathrm{Lie}} \simeq \mathrm{Chev}^{\mathrm{enh}} \circ \mathbf{triv}_{\mathrm{Lie}} \circ [-1] \circ \mathbf{oblv}_{\mathrm{Lie}} \simeq \mathrm{Sym} \circ \mathbf{oblv}_{\mathrm{Lie}}.$$

$\square$

---

[6]As we will see in Sect. 6, the functor $\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}$ identifies with another familiar functor, namely, that of the universal enveloping algebra.

4.4. Primitives in bialgebras. Let $\mathfrak{h}$ be a Lie algebra. Then the universal enveloping algebra $U(\mathfrak{h})$ is naturally a cocommutative Hopf algebra. Moreover, $\mathfrak{h}$ can be recovered as the subspace of primitive elements of $U(\mathfrak{h})$.

In this subsection, we will give a higher algebra version of this statement. We show that the space of primitives of a cocommutative bi-algebra has a canonical structure of a Lie algebra and that it gives a left inverse to the functor $\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}$, (while the latter identifies with the universal enveloping algebra by Theorem 6.1.2).

The key actor in this subsection we be the functor right adjoint to

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomBialg}(\mathbf{O}).$$

We will see that this right adjoint provides a lift of the functor

$$\mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathbf{oblv}_{\mathrm{Monoid}} : \mathrm{CocomBialg}(\mathbf{O}) \to \mathbf{O}$$

to a functor

$$\mathrm{CocomBialg}(\mathbf{O}) \to \mathrm{LieAlg}(\mathbf{O}).$$

4.4.1.   Consider again the functor

$$\mathrm{coChev}^{\mathrm{enh}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{LieAlg}(\mathbf{O}).$$

Being the right adjoint of a symmetric monoidal functor (namely, $\mathrm{Chev}^{\mathrm{enh}}$), the functor $\mathrm{coChev}^{\mathrm{enh}}$ acquires a natural right-lax symmetric monoidal structure. In particular, it gives rise to a functor, denoted $\mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}})$:

$$\mathrm{CocomBialg}(\mathbf{O}) = \mathrm{Monoid}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})) \to \mathrm{Monoid}(\mathrm{LieAlg}(\mathbf{O})) \simeq \mathrm{Grp}(\mathrm{LieAlg}(\mathbf{O})).$$

By construction, the functor $\mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}})$ is the right adjoint of the functor

$$\mathrm{Grp}(\mathrm{LieAlg}(\mathbf{O})) \overset{\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}})}{\longrightarrow} \mathrm{Grp}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})) \hookrightarrow \mathrm{Monoid}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})).$$

4.4.2.   Since B_{Lie} and Ω_{Lie} are mutually inverse equivalences (see Proposition 1.6.4), the functor $\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}$ is the left adjoint of the functor

$$B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}), \quad \mathrm{CocomBialg}(\mathbf{O}) \to \mathrm{LieAlg}(\mathbf{O}).$$

4.4.3.   Note that

$$\mathbf{oblv}_{\mathrm{Lie}} \circ B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \simeq \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathbf{oblv}_{\mathrm{Monoid}}, \quad \mathrm{CocomBialg}(\mathbf{O}) \to \mathbf{O},$$

where

$$\mathbf{oblv}_{\mathrm{Monoid}} : \mathrm{CocomBialg}(\mathbf{O}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$$

is the functor of forgetting the monoid structure.

So, the functor $B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}})$ can be viewed as one upgrading the functor

$$\mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathbf{oblv}_{\mathrm{Monoid}} : \mathrm{CocomBialg}(\mathbf{O}) \to \mathbf{O}$$

to a functor $\mathrm{CocomBialg}(\mathbf{O}) \to \mathrm{LieAlg}(\mathbf{O})$.

REMARK 4.4.4. Let us repeat the last observation in words:

For a co-commutative bi-algebra A, the space of primitives of A considered just as an augmented co-commutative co-algebra, has a natural structure of Lie algebra.

This is a higher algebra version of the motto 'the tangent space of a Lie group has a structure of a Lie algebra'.

Note, however, that we defined this Lie algebra structure not be explicitly writing down the Lie bracket, but by appealing the Koszul duality of the corresponding operads:

$$(\mathrm{Cocom}^{\mathrm{aug}})^{\vee} \simeq \mathrm{Lie}[-1].$$

4.4.5. We now claim:

THEOREM 4.4.6. *The functor*

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomBialg}(\mathbf{O})$$

is fully faithful.

PROOF. We need to show that the unit of the adjunction

$$\mathrm{Id} \to \big(B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}})\big) \circ \big(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}\big)$$

is an isomorphism.

Since B_{Lie} and Ω_{Lie} are mutually inverse equivalences, it suffices to show that the natural transformation

$$\Omega_{\mathrm{Lie}} \to \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}},$$

obtained by applying the unit of the $(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}), \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}))$-adjunction to Ω_{Lie}, is an isomorphism.

For the latter, it suffices to show that the natural transformation

$$\mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathrm{Lie}} \to \mathbf{oblv}_{\mathrm{Monoid}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}$$

is an isomorphism. Note, however, that the latter natural transformation identifies with

$$\mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathrm{Lie}} \to \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Chev}^{\mathrm{enh}} \circ \mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathrm{Lie}},$$

obtained by applying the unit of the $(\mathrm{Chev}^{\mathrm{enh}}, \mathrm{coChev}^{\mathrm{enh}})$-adjunction to $\mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathrm{Lie}}$.

However, by Proposition 1.7.2, the essential image of the functor $\mathbf{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathrm{Lie}}$ belongs to the essential image of the functor $\mathbf{triv}_{\mathrm{Lie}}$, and the assertion follows from Theorem 4.2.4.

□

5. The universal enveloping algebra

In this section we recall some basic facts about the functor of universal enveloping algebra in the setting of higher algebra.

5.1. Universal enveloping algebra: definition. In this subsection we recollect the main constructions related to the functor of universal envelope of a Lie algebra.

5.1.1.   There is a canonical map of operads

(5.1) $\mathrm{Lie} \to \mathrm{Assoc}^{\mathrm{aug}}$.

From this map we obtain the restriction functor

$$\mathbf{res}^{\mathrm{Assoc}^{\mathrm{aug}} \to \mathrm{Lie}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{LieAlg}(\mathbf{O}).$$

The functor

$$U : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$$

is defined to be the left adjoint of $\mathbf{res}^{\mathrm{Assoc}^{\mathrm{aug}} \to \mathrm{Lie}}$.

5.1.2.   The map (5.1) has the following additional structure: the functor $\mathbf{res}^{\mathrm{Assoc}^{\mathrm{aug}} \to \mathrm{Lie}}$ has a natural right-lax symmetric monoidal structure, where $\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$ is a symmetric monoidal category via the tensor product, and $\mathrm{LieAlg}(\mathbf{O})$ a symmetric monoidal category via the Cartesian product.

Hence, the functor U acquires a natural left-lax symmetric monoidal structure (as we shall see shortly, this left-lax symmetric monoidal structure is actually symmetric monoidal).

Finally, we will need one more piece of structure on (5.1):

The above left-lax symmetric monoidal structure on U makes the following diagram of left-lax symmetric monoidal functors commute:

(5.2)
$$
\begin{array}{ccc}
\mathrm{LieAlg}(\mathbf{O}) & \xrightarrow{\mathrm{Chev}^{\mathrm{enh}}} & \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle \mathbf{res}^{\mathrm{Cocom}^{\mathrm{aug}} \to \mathrm{Coassoc}^{\mathrm{aug}}}} \\
\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) & \xrightarrow{\mathrm{Bar}^{\mathrm{enh}}} & \mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O}),
\end{array}
$$

such that the induced isomorphism of functors

(5.3) $\mathrm{Bar}_+ \circ U \simeq \mathbf{oblv}_{\mathrm{Coassoc},+} \circ \mathrm{Bar}^{\mathrm{enh}} \circ U \simeq$

$\simeq \mathbf{oblv}_{\mathrm{Coassoc},+} \circ \mathbf{res}^{\mathrm{Cocom}^{\mathrm{aug}} \to \mathrm{Coassoc}^{\mathrm{aug}}} \circ \mathrm{Chev}^{\mathrm{enh}} \simeq \mathbf{oblv}_{\mathrm{Cocom},+} \circ \mathrm{Chev}^{\mathrm{enh}} \simeq \mathrm{Chev}_+$

is the tautological isomorphism arising by adjunction from

$$\mathbf{res}^{\mathrm{Coassoc}^{\mathrm{aug}} \to \mathrm{Lie}} \circ \mathbf{triv}_{\mathrm{Assoc}^{\mathrm{aug}}} \simeq \mathbf{triv}_{\mathrm{Lie}}.$$

REMARK 5.1.3. In fact, one can obtain the map (5.1), along with the above properties, by defining it as corresponding to the map of co-operads

$$\mathrm{Cocom}^{\mathrm{aug}} \to \mathrm{Coassoc}^{\mathrm{aug}}$$

via the isomorphisms

$$(\mathrm{Cocom}^{\mathrm{aug}})^{\vee} \simeq \mathrm{Lie}[-1] \text{ and } (\mathrm{Coassoc}^{\mathrm{aug}})^{\vee} \simeq \mathrm{Assoc}^{\mathrm{aug}}[-1].$$

5.1.4.   Being (left-lax) monoidal, the functor U gives rise to a functor

$$\text{CocomCoalg}(\text{LieAlg}(\mathbf{O})) \to \text{CocomCoalg}(\text{AssocAlg}^{\text{aug}}(\mathbf{O})).$$

Pre-composing with the equivalence $\text{LieAlg}(\mathbf{O}) \simeq \text{CocomCoalg}(\text{LieAlg}(\mathbf{O}))$ (see Sect. 3.1.9, applied to the commutative case), we obtain a functor:

$$\text{LieAlg}(\mathbf{O}) \to \text{CocomCoalg}(\text{AssocAlg}^{\text{aug}}(\mathbf{O})).$$

Composing with the equivalence

$$\text{CocomCoalg}(\text{AssocAlg}^{\text{aug}}(\mathbf{O})) \simeq \text{CocomBialg}(\mathbf{O})$$

of Proposition C.1.3, we obtain a functor

$$U^{\text{Hopf}} : \text{LieAlg}(\mathbf{O}) \to \text{CocomBialg}(\mathbf{O}).$$

5.1.5.   By Sect. 1.5, we can upgrade the functor U^{Hopf} to a functor

$$(U^{\text{Hopf}})^{\text{Fil}} : \text{LieAlg}(\mathbf{O}) \to \text{CocomBialg}(\mathbf{O}^{\text{Fil},\geq 0}).$$

We will also consider the functor

$$U^{\text{Fil}} : \text{LieAlg}(\mathbf{O}) \to \text{AssocAlg}(\mathbf{O}^{\text{Fil},\geq 0}).$$

5.2. The PBW theorem. In this subsection we will first give a somewhat non-standard formulation of the PBW theorem, Theorem 5.2.4.

Subsequently, we will deduce from it the usual form of the PBW theorem, Corollary 5.2.6.

5.2.1.   We claim that there exists a canonically defined natural transformation

$$(5.4) \qquad\qquad U \circ \mathbf{triv}_{\text{Lie}} \to \mathbf{res}^{\text{Com}^{\text{aug}} \to \text{Assoc}^{\text{aug}}} \circ \mathbf{free}_{\text{Com}^{\text{aug}}},$$

as functors $\mathbf{O} \to \text{AssocAlg}(\mathbf{O})$.

The datum of a map (5.4) is equivalent, by adjunction, to that of a natural transformation

$$(5.5) \qquad \mathbf{triv}_{\text{Lie}} \to \mathbf{res}^{\text{Assoc}^{\text{aug}} \to \text{Lie}} \circ \mathbf{res}^{\text{Com}^{\text{aug}} \to \text{Assoc}^{\text{aug}}} \circ \mathbf{free}_{\text{Com}^{\text{aug}}}$$

as functors $\mathbf{O} \to \text{LieAlg}(\mathbf{O})$.

5.2.2.   We construct the natural transformation (5.5) as follows.

We note that map of operads

$$\text{Lie} \to \text{Assoc}^{\text{aug}} \to \text{Com}^{\text{aug}}$$

equals

$$\text{Lie} \to \mathbf{1}_{\text{Vect}^{\Sigma}} \to \text{Com}^{\text{aug}}.$$

Hence, the functor $\mathbf{res}^{\text{Assoc}^{\text{aug}} \to \text{Lie}} \circ \mathbf{res}^{\text{Com}^{\text{aug}} \to \text{Assoc}^{\text{aug}}}$ is canonically isomorphic to

$$\mathbf{triv}_{\text{Lie}} \circ \mathbf{oblv}_{\text{Com}^{\text{aug}}}.$$

Now, the datum of the natural transformation in (5.5) is obtained by applying $\mathbf{triv}_{\text{Lie}}$ to the natural transformation

$$\text{Id} \to \underline{\text{Sym}}$$

as functors $\mathbf{O} \to \mathbf{O}$.

5.2.3.   The PBW theorem says:

THEOREM 5.2.4. *The natural transformation* (5.4) *is an isomorhism.*

We will prove Theorem 5.2.4 in Sect. B. See Corollary 5.2.6 below for the relation with the more usual version of the PBW theorem.

5.2.5.   Recall the symmetric monoidal functor

$$\text{ass-gr} : \mathbf{O}^{\text{Fil}} \to \mathbf{O}^{\text{gr}},$$

and the corresponding functor

$$\text{Assoc}^{\text{aug}}(\text{ass-gr}) : \text{AssocAlg}^{\text{aug}}(\mathbf{O}^{\text{Fil}}) \to \text{AssocAlg}^{\text{aug}}(\mathbf{O}^{\text{gr}}).$$

Consider the functor

$$U^{\text{gr}} := \text{Assoc}^{\text{aug}}(\text{ass-gr}) \circ U^{\text{Fil}}, \quad \text{LieAlg}(\mathbf{O}) \to \text{AssocAlg}^{\text{aug}}(\mathbf{O}^{\text{gr}}).$$

We claim:

COROLLARY 5.2.6. *There exists a canonical isomorphism of functors*

$$\text{LieAlg}(\mathbf{O}) \to \text{AssocAlg}^{\text{aug}}(\mathbf{O}^{\text{gr}})$$

between U^{gr} *and the composition*

$$\text{LieAlg}(\mathbf{O}) \overset{\mathbf{oblv}_{\text{Lie}}}{\longrightarrow} \mathbf{O} \overset{\text{deg}=1}{\longrightarrow} \mathbf{O}^{\text{gr}} \longrightarrow \text{AssocAlg}(\mathbf{O}^{\text{gr}}),$$

where the last arrow is $\mathbf{res}^{\text{Com}^{\text{aug}} \to \text{Assoc}^{\text{aug}}} \circ \mathbf{free}_{\text{Com}^{\text{aug}}}.$

PROOF. By (1.11), the functor U^{gr} identifies canonically with

$$\text{LieAlg}(\mathbf{O}) \overset{\mathbf{oblv}_{\text{Lie}}}{\longrightarrow} \mathbf{O} \overset{\text{deg}=1}{\longrightarrow} \mathbf{O}^{\text{gr}} \overset{\mathbf{triv}_{\text{Lie}}}{\longrightarrow} \text{LieAlg}(\mathbf{O}^{\text{gr}}) \overset{U}{\longrightarrow} \text{AssocAlg}^{\text{aug}}(\mathbf{O}^{\text{gr}}).$$

Hence, the assertion of Corollary 5.2.6 follows from that of Theorem 5.2.4.

□

Corollary 5.2.6 is the usual formulation of the PBW theorem: the associated graded of the universal enveloping algebra is the symmetric algebra.

5.2.7.   From Corollary 5.2.6 we shall now deduce:

LEMMA 5.2.8. *The left-lax symmetric monoidal structure on the functor*

$$U : \text{LieAlg}(\mathbf{O}) \to \text{AssocAlg}(\mathbf{O})$$

is symmetric monoidal.

PROOF. We have to show that for $\mathfrak{h}_1, \mathfrak{h}_2 \in \text{LieAlg}(\mathbf{O})$, the morphism

$$U(\mathfrak{h}_1 \times \mathfrak{h}_2) \to U(\mathfrak{h}_1) \otimes U(\mathfrak{h}_2)$$

is an isomorphism.

It is enough to prove the corresponding fact for the functor U^{Fil}, and hence also for the functor U^{gr}. Now the assertion follows via Corollary 5.2.6 from the fact that the functor $\mathbf{free}_{\text{Com}^{\text{aug}}}$ is symmetric monoidal.   □

5.3. The Bar complex of the universal envelope. Recall the isomorphism

$$\mathrm{Bar}_+ \circ U \simeq \mathrm{Chev}_+$$

of (5.3), and the resulting isomorphism

(5.6) $$\mathrm{Bar} \circ U \simeq \mathrm{Chev}.$$

In this subsection we will upgrade the latter isomorphism to one between functors taking values in $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$.

5.3.1. Consider the functor

$$\mathrm{Bar} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$$

and note that the following diagram commutes

$$
\begin{array}{ccc}
\mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O})) & \xrightarrow{\ \mathrm{Bar}\ } & \mathrm{CocomCoalg}(\mathbf{O}) \\
{\scriptstyle \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{oblv}_{\mathrm{Cocom}})} \downarrow & & \downarrow {\scriptstyle \mathbf{oblv}_{\mathrm{Cocom}}} \\
\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) & \xrightarrow{\ \mathrm{Bar}\ } & \mathbf{O},
\end{array}
$$

since the functor $\mathbf{oblv}_{\mathrm{Cocom}} : \mathrm{CocomCoalg}(\mathbf{O}) \to \mathbf{O}$ is symmetric monoidal.

5.3.2. We claim:

PROPOSITION 5.3.3. *There exists a canonical isomorphism*

$$\mathrm{Bar} \circ U^{\mathrm{Hopf}} \simeq \mathrm{Chev}^{\mathrm{enh}}$$

as functors $\mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$, *such that the induced isomorphism*

$$\mathrm{Bar} \circ U \simeq \mathrm{Bar} \circ \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{oblv}_{\mathrm{Cocom}}) \circ U^{\mathrm{Hopf}} \simeq \mathbf{oblv}_{\mathrm{Cocom}} \circ \mathrm{Bar} \circ U^{\mathrm{Hopf}} \simeq$$

$$\simeq \mathbf{oblv}_{\mathrm{Cocom}} \circ \mathrm{Chev}^{\mathrm{enh}} \simeq \mathrm{Chev}$$

identifies with (5.6).

PROOF. Recall the commutative diagram (5.2), from which we produce the inner square in the next commutative diagram

$$
\begin{array}{ccc}
\mathrm{LieAlg}(\mathbf{O}) & \xrightarrow{\ \mathrm{Chev}^{\mathrm{enh}}\ } & \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}) \\
{\scriptstyle \sim} \downarrow & & \downarrow {\scriptstyle \sim} \\
\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{LieAlg}(\mathbf{O})) & \xrightarrow{\ \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{Chev}^{\mathrm{enh}})\ } & \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})) \\
{\scriptstyle \mathrm{CocomCoalg}^{\mathrm{aug}}(U)} \downarrow & & \downarrow {\scriptstyle \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{res}^{\mathrm{Cocom}^{\mathrm{aug}} \to \mathrm{Coassoc}^{\mathrm{aug}}})} \\
\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})) & \xrightarrow{\ \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{Bar}^{\mathrm{enh}})\ } & \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{CoassocCoalg}^{\mathrm{aug}}(\mathbf{O})) \\
{\scriptstyle \sim} \downarrow & & \downarrow {\scriptstyle \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{oblv}_{\mathrm{Coassoc}^{\mathrm{aug}}})} \\
\mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})) & \xrightarrow{\ \mathrm{Bar}\ } & \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}).
\end{array}
$$

In the above diagram, the composite left vertical arrow is, by definition, the functor U^{Hopf}, and the composite right vertical arrow is the identity functor.

$\square$

6. The universal envelope via loops

In this section we establish the main result of this chapter, Theorem 6.1.2. It says that the universal enveloping algebra of a Lie algebra can be expressed via the Chevalley functor, namely, we have a canonical isomorphism of functors

$$U^{\mathrm{Hopf}} \simeq \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}.$$

6.1. The main result. In this subsection we state the main result of this chapter, Theorem 6.1.2.

6.1.1.   Our main result is the following:

THEOREM 6.1.2. *There exists a canonical isomorphism of functors*

$$U^{\mathrm{Hopf}} \simeq \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}, \quad \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomBialg}(\mathbf{O}).$$

Several remarks are in order:

REMARK 6.1.3. The proof of Theorem 6.1.2 is such that the isomorphism stated in the theorem automatically upgrades to an isomorphism at the filtered level:

$$(U^{\mathrm{Hopf}})^{\mathrm{Fil}} \simeq \left(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}\right)^{\mathrm{Fil}}.$$

REMARK 6.1.4. One can generalize the proof of Theorem 6.1.2 to establish the isomorphims of functors

$$(6.1) \qquad\qquad U_{\mathbb{E}_n} \simeq \mathbb{E}_n\text{-}\mathrm{Alg}^{\mathrm{aug}}(\mathrm{Chev}) \circ \Omega_{\mathrm{Lie}}^{\times n},$$

where $U_{\mathbb{E}_n}$ is the left adjoint to the forgetful functor

$$\mathbf{res}^{\mathbb{E}_n^{\mathrm{aug}} \to \mathrm{Lie}} : \mathbb{E}_n\text{-}\mathrm{Alg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{LieAlg}(\mathbf{O}),$$

arising from the corresponding map of operads.

Moreover, the isomorphism (6.1) automatically upgrades to an isomorphism of the corresponding functors

$$\mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomCoalg}(\mathbb{E}_n^{\mathrm{aug}}\text{-}\mathrm{Alg}(\mathbf{O})) \simeq \mathbb{E}_n\text{-}\mathrm{Alg}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})),$$

$$(6.2) \qquad\qquad U_{\mathbb{E}_n}^{\mathrm{Hopf}} \simeq \mathbb{E}_n\text{-}\mathrm{Alg}^{\mathrm{aug}}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}^{\times n}.$$

Furthermore, the isomorphism (6.2) can be upgraded to an isomorphism of functors with values in $\mathrm{CocomCoalg}(\mathbb{E}_n\text{-}\mathrm{Alg}(\mathbf{O}^{\mathrm{Fil},\geq 0}))$.

REMARK 6.1.5. A very natural proof of the isomorphism (6.1) can be given using the language of factorization algebras. In the context of algebraic geometry, this is done in [**FraG**, Proposition 6.1.2].

6.1.6.   Note that by combining Theorem 6.1.2 with Proposition 4.3.3 we obtain:

COROLLARY 6.1.7. *There exists a canonical isomorphism of functors*

$$\mathbf{oblv}_{\mathrm{Assoc}} \circ U^{\mathrm{Hopf}} \simeq \mathrm{Sym} \circ \mathbf{oblv}_{\mathrm{Lie}}, \quad \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$$

REMARK 6.1.8. The assertion of Corollary 6.1.7 is of course well-known. The curious aspect of our proof is that it does not use the symmetrization map from the symmetric algebra to the tensor algebra, although one can show that the latter map gives the same isomorphism.

6.2. Proof of Theorem 6.1.2. The idea of the proof is the following: we consider the functor

$$\mathrm{Assoc}^{\mathrm{aug}}(U^{\mathrm{Hopf}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}\left(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O}))\right),$$

and we will compose it with two different versions of the Bar-construction: the 'inner' and the 'outer':

$$\mathrm{AssocAlg}^{\mathrm{aug}}\left(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O}))\right) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O})) \simeq$$

$$\simeq \mathrm{CocomBialg}(\mathbf{O}).$$

6.2.1.   The left-lax symmetric monoidal structure on the functor

$$U : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$$

gives rise to one on the functor

$$U^{\mathrm{Hopf}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{CocomCoalg}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})) \simeq \mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O})).$$

However, since the left-lax symmetric monoidal structure on U is strict (see Lemma 5.2.8), so is one on U^{Hopf}. Hence, the functor U^{Hopf} gives rise to a functor that we denote $\mathrm{Assoc}^{\mathrm{aug}}(U^{\mathrm{Hopf}})$:

$$\mathrm{Monoid}(\mathrm{LieAlg}(\mathbf{O})) \to \mathrm{AssocAlg}\left(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O}))\right) \simeq$$

$$\simeq \mathrm{AssocAlg}^{\mathrm{aug}}\left(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O}))\right).$$

REMARK 6.2.2.   We can think of the category $\mathrm{AssocAlg}^{\mathrm{aug}}\left(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O}))\right)$ as that of augmented $\mathbb{E}_2$-algebras in $\mathrm{CocomCoalg}(\mathbf{O})$.

6.2.3.   Consider the resulting functor
(6.3)
$$\mathrm{Assoc}^{\mathrm{aug}}(U^{\mathrm{Hopf}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}\left(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O}))\right).$$

We consider the two functors,

$$\mathrm{AssocAlg}^{\mathrm{aug}}\left(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O}))\right) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O})),$$

denoted $\mathrm{Assoc}^{\mathrm{aug}}(\mathrm{Bar})$ and Bar, corresponding to taking the Bar-complex with respect to the 'inner' and 'outer' associative algebra structure, respectively.

We claim:

(6.4) $$\mathrm{Bar} \circ \mathrm{Assoc}^{\mathrm{aug}}(U^{\mathrm{Hopf}}) \circ \Omega_{\mathrm{Lie}} \simeq U^{\mathrm{Hopf}}$$

and

(6.5) $$\mathrm{Assoc}^{\mathrm{aug}}(\mathrm{Bar}) \circ \mathrm{Assoc}^{\mathrm{aug}}(U^{\mathrm{Hopf}}) \circ \Omega_{\mathrm{Lie}} \simeq \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}$$

as functors

$$\mathrm{LieAlg}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O})).$$

6.2.4. Indeed, since the functor U^{Hopf} is symmetric monoidal, we have

$$(6.6) \qquad \mathrm{Bar} \circ \mathrm{Assoc}^{\mathrm{aug}}(U^{\mathrm{Hopf}}) \circ \Omega_{\mathrm{Lie}} \simeq U^{\mathrm{Hopf}} \circ B_{\mathrm{Lie}} \circ \Omega_{\mathrm{Lie}} \simeq U^{\mathrm{Hopf}},$$

which gives the isomorphism in (6.4).

To establish the isomorphism in (6.5), we note that the isomorphism of Proposition 5.3.3 is compatible with the symmetric monoidal structures, and, hence, gives rise to an isomorphism

$$\mathrm{Assoc}^{\mathrm{aug}}(\mathrm{Bar}) \circ \mathrm{Assoc}^{\mathrm{aug}}(U^{\mathrm{Hopf}}) \simeq \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}})$$

as functors

$$\mathrm{Grp}(\mathrm{LieAlg}(\mathbf{O})) \simeq \mathrm{Monoid}(\mathrm{LieAlg}(\mathbf{O})) \to$$
$$\to \mathrm{AssocAlg}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})) \simeq \mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O})).$$

This gives rise to the isomorphism in (6.5) by precomposing with Ω_{Lie}.

6.2.5. Recall that the symmetric monoidal structure on $\mathrm{CocomCoalg}(\mathbf{O})$ is Cartesian. In particular, we can consider the full subcategories

$$\mathrm{CocomHopf}(\mathbf{O}) := \mathrm{Grp}(\mathrm{CocomCoalg}(\mathbf{O})) \subset \mathrm{Monoid}(\mathrm{CocomCoalg}(\mathbf{O})) =$$
$$= \mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O})),$$

and

$$\mathrm{Grp}(\mathrm{Grp}(\mathrm{CocomCoalg}(\mathbf{O}))) \subset \mathrm{Monoid}(\mathrm{Monoid}(\mathrm{CocomCoalg}(\mathbf{O}))) =$$
$$= \mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathrm{CocomCoalg}(\mathbf{O}))).$$

We have the following basic fact proved below:

PROPOSITION 6.2.6. *For an ∞-category $\mathbf{C}$ endowed with the Cartesian symmetric monoidal structure, there exists a canonical isomorphism of functors*

$$\mathrm{Grp}(B) \simeq B, \quad \mathrm{Grp}(\mathrm{Grp}(\mathbf{C})) \to \mathrm{Grp}(\mathbf{C}).$$

We compose the isomorphism of Proposition 6.2.6 with the functor (6.3), and obtain an isomorphism

$$(6.7) \qquad \mathrm{Assoc}^{\mathrm{aug}}(\mathrm{Bar}) \circ \mathrm{Assoc}^{\mathrm{aug}}(U^{\mathrm{Hopf}}) \circ \Omega_{\mathrm{Lie}} \simeq \mathrm{Bar} \circ \mathrm{Assoc}^{\mathrm{aug}}(U^{\mathrm{Hopf}}) \circ \Omega_{\mathrm{Lie}}.$$

Combining the isomorphism (6.7) with the isomorphisms (6.4) and (6.5), we arrive at the conclusion of the theorem.

$\square$

6.3. Proof of Proposition 6.2.6.

6.3.1. By adjunction, the assertion of the proposition amounts to a canonical isomorphism of functors

$$(6.8) \qquad \Omega \simeq \mathrm{Grp}(\Omega) : \mathrm{Grp}(\mathbf{C}) \to \mathrm{Grp}(\mathrm{Grp}(\mathbf{C})).$$

The latter reduces the assertion to the proposition when $\mathbf{C} = \mathrm{Spc}$ is the category of spaces, by the Yoneda lemma.

6.3.2.   We start with the tautological isomorphism of functors

$$\text{(6.9)} \qquad \text{Grp}(\Omega) \circ \Omega \simeq \Omega \circ \Omega, \quad \text{Spc}_{\{*\}/} \to \text{Grp}(\text{Grp}(\text{Spc})).$$

By adjunction, we obtain a natural transformation

$$\text{(6.10)} \qquad B \circ \text{Grp}(\Omega) \to \Omega \circ B \simeq \text{Id}, \quad \text{Grp}(\text{Spc}) \to \text{Grp}(\text{Spc}).$$

Applying $\Omega : \text{Grp}(\text{Spc}) \to \text{Grp}(\text{Grp}(\text{Spc}))$ to (6.10), we obtain the desired natural transformation

$$\text{Grp}(\Omega) \simeq \Omega \circ B \circ \text{Grp}(\Omega) \to \Omega.$$

6.3.3.   To show that the resulting map $\text{Grp}(\Omega) \to \Omega$ is an isomorphism, it is enough to do so after precomposing with $\Omega : \text{Spc}_{\{*\}/} \to \text{Grp}(\text{Spc})$. However, the resulting map

$$\text{Grp}(\Omega) \circ \Omega \to \Omega \circ \Omega$$

equals that of (6.9), and hence is an isomorphism.

7. Modules

The goal of this section is to give a new perspective on the equivalence of categories

$$\mathfrak{h}\text{-mod} \simeq U(\mathfrak{h})\text{-mod}$$

for a Lie algebra $\mathfrak{h}$. We will do so using the isomorphism

$$U(\mathfrak{h})\text{-mod} \simeq \text{AssocAlg}(\mathbf{oblv}_{\text{Cocom}}) \circ \text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}),$$

given by Theorem 6.1.2.

In a sense, the upshot of this section is that one does not really need the definition of the functor

$$U : \text{LieAlg}(\mathbf{O}) \to \text{AssocAlg}^{\text{aug}}(\mathbf{O})$$

as the left adjoint of the restriction functor $\mathbf{res}^{\text{Assoc}^{\text{aug}} \to \text{Lie}}$. Namely, all the essential features of this functor are more conveniently expressed through its incarnation as $\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega$.

7.1. Left modules for associative algebras.
In this subsection we recall some basic pieces of structure pertaining to left modules over associative algebras and to the Koszul duality functor in this case.

7.1.1.   Let $\mathbf{O}$ be a monoidal category. Let A be an object of $\text{AssocAlg}(\mathbf{O})$. We let $A\text{-mod}(\mathbf{O})$ denote the category of left A-modules on $\mathbf{O}$. We have a tautological pair of adjoint functors

$$\mathbf{free}_A : \mathbf{O} \rightleftarrows A\text{-mod}(\mathbf{O}) : \mathbf{oblv}_A.$$

The monad $\mathbf{oblv}_A \circ \mathbf{free}_A$ is given by tensor product with A.

Reversing the arrows, we obtain the corresponding pieces of notations for co-modules:

$$\mathbf{oblv}_B : B\text{-comod}(\mathbf{O}) \rightleftarrows \mathbf{O} : \mathbf{cofree}_B, \quad B \in \text{CoassocCoalg}(\mathbf{O}).$$

7.1.2. Let now A be an *augmented* associative algebra. We have a canonically defined functor

$$\mathrm{Bar}^\bullet(A,-) : A\text{-mod}(\mathbf{O}) \to \mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}},$$

see [**Lu2**, Sect. 4.4.2.7].

We denote by

$$\mathrm{Bar}(A,-) : A\text{-mod}(\mathbf{O}) \to \mathbf{O}$$

the composition of $\mathrm{Bar}^\bullet(A,-)$, followed by the functor of geometric realization $\mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathbf{O}$, provided that the latter is defined.

The functor $\mathrm{Bar}(A,-)$ is the left adjoint of the functor

$$\mathbf{triv}_A : \mathbf{O} \to A\text{-mod}(\mathbf{O}),$$

given by the augmentation on A.

7.1.3. We have the following additional crucial piece of structure on the adjoint pair

$$\mathrm{Bar}(A,-) : A\text{-mod}(\mathbf{O}) \rightleftarrows \mathbf{O} : \mathbf{triv}_A.$$

Namely, the co-monad $\mathrm{Bar}(A,-) \circ \mathbf{triv}_A$ on $\mathbf{O}$ identifies canonically with one given by tensor product with the co-associative co-algebra $\mathrm{Bar}^{\mathrm{enh}}(A)$, see [**Lu2**, Sect. 5.2.2].

In particular, we have a canonically defined functor

$$\mathrm{Bar}^{\mathrm{enh}}(A,-) : A\text{-mod}(\mathbf{O}) \to \mathrm{Bar}^{\mathrm{enh}}(A)\text{-comod}(\mathbf{O}),$$

making the following diagrams commutative:

(7.1)
$$
\begin{array}{ccc}
A\text{-mod}(\mathbf{O}) & \xrightarrow{\ \mathrm{Bar}^{\mathrm{enh}}(A,-)\ } & \mathrm{Bar}^{\mathrm{enh}}(A)\text{-comod}(\mathbf{O}) \\
{\scriptstyle \mathrm{Id}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{Bar}^{\mathrm{enh}}(A)}} \\
A\text{-mod}(\mathbf{O}) & \xrightarrow{\ \mathrm{Bar}(A,-)\ } & \mathbf{O}
\end{array}
$$

and

(7.2)
$$
\begin{array}{ccc}
A\text{-mod}(\mathbf{O}) & \xrightarrow{\ \mathrm{Bar}^{\mathrm{enh}}(A,-)\ } & \mathrm{Bar}^{\mathrm{enh}}(A)\text{-comod}(\mathbf{O}) \\
{\scriptstyle \mathbf{triv}_A}\big\uparrow & & \big\uparrow{\scriptstyle \mathbf{cofree}_{\mathrm{Bar}^{\mathrm{enh}}(A)}} \\
\mathbf{O} & \xrightarrow{\ \mathrm{Id}\ } & \mathbf{O}.
\end{array}
$$

7.2. Modules over co-commutative Hopf algebras. Let $\mathbf{O}$ be a symmetric monoidal category.

The goal of this subsection is to establish the following basic fact: given a co-commutative *Hopf* algebra A, the category of modules over A as an associative algebra is equivalent to the totalization of the co-simplicial category of co-modules over $\mathrm{Bar}^\bullet(A)$, where $\mathrm{Bar}^\bullet(A)$ is considered as a simplicial co-algebra.

7.2.1. Let A be a co-commutative bi-algebra in $\mathbf{O}$. Consider the corresponding object

$$\mathrm{Bar}^\bullet(A) \in \mathrm{CocomCoalg}(\mathbf{O})^{\mathbf{\Delta}^{\mathrm{op}}}.$$

Consider the resulting simplicial category

$$\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O}),$$

i.e., the simplicial category formed by co-modules in $\mathbf{O}$ over the terms of $\mathrm{Bar}^\bullet(A)$, viewed as a simplicial co-algebra.

Passing to right adjoints, we obtain a co-simplicial category

$$\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})^R.$$

The goal of this subsection is to establish the following:

PROPOSITION-CONSTRUCTION 7.2.2. *Assume that* $A \in \mathrm{CocomHopf}(\mathbf{O})$, *and let*

$$\widetilde{A} := \mathrm{AssocAlg}(\mathbf{oblv}_{\mathrm{Cocom}})(A)$$

be the underlying associative algebra. Then there is a canonical equivalence of categories:

$$(7.3) \qquad\qquad \widetilde{A}\text{-mod} \simeq \mathrm{Tot}\left(\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})^R\right).$$

The construction of the equivalence in Proposition 7.2.2 will have the following features.

7.2.3. Recall the equivalence

$$\mathrm{Tot}\left(\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})^R\right) \simeq |\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})|$$

of Volume I, Chapter 1, Proposition 2.5.7. Thus, we obtain a functor

$$(7.4) \quad |\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})| \simeq \mathrm{Tot}\left(\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})^R\right) \simeq \widetilde{A}\text{-mod} \overset{\mathrm{Bar}(\widetilde{A},-)}{\longrightarrow} \mathbf{O}.$$

COROLLARY 7.2.4. *The functor* (7.4) *is given by the simplex-wise forgetful functors*

$$\mathbf{oblv}_{\mathrm{Bar}^m(A)} : \mathrm{Bar}^m(A)\text{-comod}(\mathbf{O}) \to \mathbf{O}.$$

PROOF. Follows by considering the corresponding right adjoints. $\square$

7.2.5. Upgrading of $\widetilde{A}$ to an object of $\mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O}))$ defines on the category $\widetilde{A}$-mod a symmetric monoidal structure. Similarly, the category $\mathrm{Tot}\left(\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})^R\right)$ is naturally symmetric monoidal.

It will follow from the construction, given below, that the equivalence (7.3) is naturally compatible with the above symmetric monoidal structures.

7.2.6.   The rest of this subsection is devoted to the proof of Proposition 7.2.2.

Using Proposition C.1.3, to A we can canonically attach an object

$$A' \in \mathrm{CocomCoalg}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})),$$

so that

$$\widetilde{A} \simeq \mathbf{oblv}_{\mathrm{Cocom}}(A').$$

Moreover, by construction, under the equivalence

$$\mathrm{CocomCoalg}(\mathbf{O})^{\mathbf{\Delta}^{\mathrm{op}}} \simeq \mathrm{CocomCoalg}(\mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}})$$

the object

$$\mathrm{Bar}^\bullet(A) \in \mathrm{CocomCoalg}(\mathbf{O})^{\mathbf{\Delta}^{\mathrm{op}}}$$

identifies with the corresponding object

$$\mathrm{Cocom}(\mathrm{Bar}^\bullet)(A') \in \mathrm{CocomCoalg}(\mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}}).$$

7.2.7.   Consider the category P that consists of pairs (B, M), where $B \in \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$ and $M \in B$-mod. This is a symmetric monoidal category under the operation of tensor product.

If B upgrades to an object $B' \in \mathrm{CocomCoalg}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}))$, then the object $(B, \mathbf{1_O}) \in \mathsf{P}$ has a natural structure of object of $\mathrm{CocomCoalg}(\mathsf{P})$, denoted $(B', \mathbf{1_O})$. Moreover, we have a naturally defined functor

$$(7.5) \qquad B\text{-mod}(\mathbf{O}) \to (B', \mathbf{1_O})\text{-comod}(\mathsf{P}), \quad M \mapsto (B, M).$$

We have a naturally defined symmetric monoidal functor

$$(7.6) \qquad \mathrm{Bar}^\bullet_{\mathrm{with\ module}} : \mathsf{P} \to \mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}}, \quad (B, M) \mapsto \mathrm{Bar}^\bullet(B, M),$$

so that for $B \in \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$, we have

$$\mathrm{Bar}^\bullet_{\mathrm{with\ module}}(B, \mathbf{1_O}) \simeq \mathrm{Bar}^\bullet(B),$$

and for $B' \in \mathrm{CocomCoalg}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}))$

$$\mathrm{Cocom}(\mathrm{Bar}^\bullet_{\mathrm{with\ module}})(B', \mathbf{1_O}) \simeq \mathrm{Cocom}(\mathrm{Bar}^\bullet)(B'),$$

as objects of $\mathrm{CocomCoalg}(\mathbf{O})^{\mathbf{\Delta}^{\mathrm{op}}}$.

7.2.8.   Combining (7.5) and (7.6) we obtain a functor

$$(7.7) \qquad \widetilde{A}\text{-mod}(\mathbf{O}) \to \mathrm{Sect}\left(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})\right),$$

where $\mathrm{Sect}(\mathbf{\Delta}^{\mathrm{op}}, -)$ denotes the category of (not necessarily co-Cartesian) sections of a given simplicial category. Specifically, this functor maps an $\widetilde{A}$-module M to the section which assigns to $[n]$, the $\mathrm{Bar}^n(A)$-comodule given by $\mathrm{Bar}^n(A, M)$ and maps given by restriction of comodules.

LEMMA 7.2.9. *If the bi-algebra A is a* Hopf *algebra, then for $M \in \widetilde{A}$-mod$(\mathbf{O})$, the section* (7.7) *defines, by passing to right adjoints, an object of*

$$\mathrm{Tot}\left(\mathrm{Bar}^\bullet(A)\text{-comod}(\mathbf{O})^R\right)$$

(i.e., the corresponding morphisms are isomorphisms for every arrow in $\mathbf{\Delta}$).

PROOF. For an $\widetilde{A}$-module M, we have the action map

$$A \otimes M \to M.$$

Applying the coinduction functor (right adjoint to restriction of comodules) to M, this gives a map

$$A \otimes M \to A \otimes M$$

of A-comodules. Unraveling the definitions, the statement of the lemma reduces to the statement that the above map is an isomorphism. This follows from the fact that A is a group object is the category of cocommutative coalgebras. □

7.2.10.   Thus, by Lemma 7.2.9, we obtain the desired functor

$$(7.8) \qquad\qquad \widetilde{A}\text{-mod}(\mathbf{O}) \to \mathrm{Tot}\left(\mathrm{Bar}^{\bullet}(A)\text{-comod}(\mathbf{O})^{R}\right).$$

Let us now show that the functor (7.8) is an equivalence.

7.2.11.   Let

$$\mathrm{ev}^{0} : \mathrm{Tot}\left(\mathrm{Bar}^{\bullet}(A)\text{-comod}(\mathbf{O})^{R}\right) \to \mathbf{O}$$

denote the functor of evaluation on 0-simplices.

It is easy to see that the co-simplicial category $\mathrm{Bar}^{\bullet}(A)\text{-comod}(\mathbf{O})^{R}$ satisfies the monadic Beck-Chevalley condition (see [**Ga3**, Defn. C.1.2] for what this means). Hence, the functor ev^{0} is monadic, and the resulting monad on $\mathbf{O}$, regarded as a plain endo-functor, is given by tensor product with

$$\mathbf{oblv}_{\mathrm{Cocom}} \circ \mathbf{oblv}_{\mathrm{Assoc}}(A) \simeq \mathbf{oblv}_{\mathrm{Assoc}}(\widetilde{A}).$$

By construction, the composite functor

$$\widetilde{A}\text{-mod}(\mathbf{O}) \to \mathrm{Tot}\left(\mathrm{Bar}^{\bullet}(A)\text{-comod}(\mathbf{O})^{R}\right) \xrightarrow{\mathrm{ev}^{0}} \mathbf{O}$$

is the tautological forgetful functor $\mathbf{oblv}_{\widetilde{A}} : \widetilde{A}\text{-mod}(\mathbf{O}) \to \mathbf{O}$. Hence, it is also monadic, and the resulting monad on $\mathbf{O}$, regarded as a plain endo-functor, is given by tensor product with $\mathbf{oblv}_{\mathrm{Assoc}}(\widetilde{A})$.

Hence, it remains to see that the homomorphism of monads on $\mathbf{O}$, induced by (7.8), is an isomorphism as plain endo-functors of $\mathbf{O}$. However, it follows from the construction that the map in question is the identity map on the functor $\mathbf{oblv}_{\mathrm{Assoc}}(\widetilde{A}) \otimes -$.

7.3. Modules for Lie algebras. Let $\mathbf{O}$ be a symmetric monoidal DG category.

In this subsection we recall some basic pieces of structure pertaining to modules over Lie algebras and the Koszul duality functor.

7.3.1.   For a Lie algebra $\mathfrak{h}$ in $\mathbf{O}$ we let $\mathfrak{h}\text{-mod}(\mathbf{O})$ the category of (operadic) $\mathfrak{h}$-modules on $\mathbf{O}$. We let

$$\mathbf{oblv}_{\mathfrak{h}} : \mathfrak{h}\text{-mod}(\mathbf{O}) \to \mathbf{O}$$

denote the tautological forgetful functor.

7.3.2.   The map from $\mathfrak{h}$ to the zero Lie algebra defines a functor

$$\mathbf{triv}_{\mathfrak{h}} : \mathbf{O} \to \mathfrak{h}\text{-mod}(\mathbf{O}).$$

This functor admits a left adjoint, denoted

$$\mathbf{coinv}(\mathfrak{h}, -) : \mathfrak{h}\text{-mod}(\mathbf{O}) \to \mathbf{O}.$$

7.3.3.   In the sequel we will need the following additional piece of structure on the adjoint pair

$$\mathbf{coinv}(\mathfrak{h}, -) : \mathfrak{h}\text{-mod}(\mathbf{O}) \rightleftarrows \mathbf{O} : \mathbf{triv}_{\mathfrak{h}}.$$

Namely, the co-monad $\mathbf{coinv}(\mathfrak{h}, -) \circ \mathbf{triv}_{\mathfrak{h}}$ on $\mathbf{O}$ identifies canonically with one given by tensor product with $\mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})$.

NB: Here we are abusing the notation slightly: we view $\mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})$ as an object of the category $\mathrm{CoassocCoalg}(\mathbf{O})$; properly, we should have written

$$\mathbf{res}^{\mathrm{Cocom} \to \mathrm{Coassoc}}(\mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})).$$

7.3.4.   In particular, we have a canonically defined functor

$$\mathbf{coinv}^{\mathrm{enh}}(\mathfrak{h}, -) : \mathfrak{h}\text{-mod}(\mathbf{O}) \to \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})\text{-comod}(\mathbf{O}),$$

making the following diagrams commutative:

$$(7.9) \qquad \begin{array}{ccc}
\mathfrak{h}\text{-mod}(\mathbf{O}) & \xrightarrow{\ \mathbf{coinv}^{\mathrm{enh}}(\mathfrak{h},-)\ } & \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})\text{-comod}(\mathbf{O}) \\
{\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})}} \\
\mathfrak{h}\text{-mod}(\mathbf{O}) & \xrightarrow{\ \mathbf{coinv}(\mathfrak{h},-)\ } & \mathbf{O}
\end{array}$$

and

$$(7.10) \qquad \begin{array}{ccc}
\mathfrak{h}\text{-mod}(\mathbf{O}) & \xrightarrow{\ \mathbf{coinv}^{\mathrm{enh}}(\mathfrak{h},-)\ } & \mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})\text{-comod}(\mathbf{O}) \\
{\scriptstyle \mathbf{triv}_{\mathfrak{h}}}\uparrow & & \uparrow{\scriptstyle \mathbf{cofree}_{\mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})}} \\
\mathbf{O} & \xrightarrow{\ \mathrm{Id}\ } & \mathbf{O}.
\end{array}$$

7.3.5.   From the commutative diagram (7.10) it follows that for $M_1, M_2 \in \mathfrak{h}\text{-mod}(\mathbf{O})$, the map
(7.11)
$$\mathrm{Maps}_{\mathfrak{h}\text{-mod}(\mathbf{O})}(M_1, M_2) \to \mathrm{Maps}_{\mathrm{Chev}^{\mathrm{enh}}(\mathfrak{h})\text{-comod}(\mathbf{O})}(\mathbf{coinv}^{\mathrm{enh}}(\mathfrak{h}, M_1), \mathbf{coinv}^{\mathrm{enh}}(\mathfrak{h}, M_2)),$$

induced by the functor $\mathbf{coinv}^{\mathrm{enh}}(\mathfrak{h}, -)$, is an isomorphism whenever M_2 lies in the essential image of the functor $\mathbf{triv}_{\mathfrak{h}}$.

7.4. Modules for a Lie algebra and its universal envelope. Let $\mathfrak{h} \in \mathrm{LieAlg}(\mathbf{O})$ be as above. In this subsection we will construct a canonical equivalence

$$(7.12) \qquad\qquad \mathfrak{h}\text{-mod}(\mathbf{O}) \simeq U(\mathfrak{h})\text{-mod}(\mathbf{O})$$

that makes the following diagrams commute:

$$\begin{array}{ccc}
\mathfrak{h}\text{-mod}(\mathbf{O}) & \longrightarrow & U(\mathfrak{h})\text{-mod}(\mathbf{O}) \\
{\scriptstyle \mathbf{oblv}_{\mathfrak{h}}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{U(\mathfrak{h})}} \\
\mathbf{O} & \xrightarrow{\ \mathrm{Id}\ } & \mathbf{O}
\end{array}$$

and

$$(7.13) \qquad \begin{array}{ccc}
\mathfrak{h}\text{-mod}(\mathbf{O}) & \longrightarrow & U(\mathfrak{h})\text{-mod}(\mathbf{O}) \\
{\scriptstyle \mathbf{coinv}(\mathfrak{h},-)}\downarrow & & \downarrow{\scriptstyle \mathrm{Bar}(U(\mathfrak{h}),-)} \\
\mathbf{O} & \xrightarrow{\ \mathrm{Id}\ } & \mathbf{O}.
\end{array}$$

In constructing (7.12) we will use the incarnation of $U(\mathfrak{h})$ as

$$\text{AssocAlg}(\mathbf{oblv}_{\text{Cocom}}) \circ \text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}),$$

given by Theorem 6.1.2.

Note that using Sect. 7.2.5, this will endow the category $\mathfrak{h}$-mod$(\mathbf{O})$ with a symmetric monoidal structure, compatible with the forgetful functor $\mathbf{oblv}_{\mathfrak{h}}$.

7.4.1.   We start with the object

$$\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}) \in \text{CocomHopf}(\mathbf{O}) \subset \text{AssocAlg}(\text{CococomCoalg}(\mathbf{O})),$$

and form the object

$$\text{Bar}^{\bullet}(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h})) \in \text{CococomCoalg}(\mathbf{O})^{\mathbf{\Delta}^{\text{op}}}.$$

Consider the resulting simplicial category

$$\text{Bar}^{\bullet}(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O}),$$

and the co-simplicial category

$$\text{Bar}^{\bullet}(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})^{R},$$

obtained by passing to right adjoints.

According to Proposition 7.2.2, the category

$$\text{AssocAlg}(\mathbf{oblv}_{\text{Cocom}}) \circ \text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O})$$

identifies with

$$\text{Tot}\left(\text{Bar}^{\bullet}(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})^{R}\right),$$

in such a way that the forgetful functor

$$\mathbf{oblv}_{\text{AssocAlg}(\mathbf{oblv}_{\text{Cocom}})\circ\text{Grp}(\text{Chev}^{\text{enh}})\circ\Omega(\mathfrak{h})} :$$
$$\text{AssocAlg}(\mathbf{oblv}_{\text{Cocom}}) \circ \text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O}) \to \mathbf{O}$$

identifies with the functor of evaluation on zero-simplices.

7.4.2.   Note that the simplicial co-algebra $\text{Bar}^{\bullet}(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}))$ identifies with

$$\text{Chev}^{\text{enh}}(\text{Bar}^{\bullet} \circ \Omega(\mathfrak{h})),$$

where

$$\text{Bar}^{\bullet} \circ \Omega(\mathfrak{h}) \in \text{LieAlg}(\mathbf{O})^{\mathbf{\Delta}^{\text{op}}}$$

is the Čech nerve in $\text{LieAlg}(\mathbf{O})$ of the map $0_{\mathbf{O}} \to \mathfrak{h}$.

Consider the co-simplicial category

(7.14) $\text{Bar}^{\bullet} \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O}).$

Since

$$\underset{\mathbf{\Delta}^{\text{op}}}{\text{colim}} \, \text{Bar}^{\bullet} \circ \Omega(\mathfrak{h}) \simeq \mathfrak{h},$$

we have

$$\mathfrak{h}\text{-mod}(\mathbf{O}) \simeq \text{Tot}\left(\text{Bar}^{\bullet} \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O})\right).$$

7.4.3. We will construct the sought-for equivalence

$$\mathfrak{h}\text{-mod}(\mathbf{O}) \simeq \text{AssocAlg}(\mathbf{oblv}_{\text{Cocom}}) \circ \text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O})$$

by constructing an equivalence

$$\text{Tot}\left(\text{Bar}^\bullet \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O})\right) \simeq \text{Tot}\left(\text{Bar}^\bullet(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})^R\right).$$

To do so, it is sufficient to show that the corresponding *semi-totalizations* are equivalent.

7.4.4. Let

$$\text{Bar}^\bullet \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O})^L$$

be the simplicial category obtained by passing to left adjoints in (7.14).

The functor $\mathbf{coinv}^{\text{enh}}(-,-)$ gives rise to a functor of simplicial categories

(7.15) $$\text{Bar}^\bullet \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O})^L \to \text{Bar}^\bullet(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O}),$$

and, in particular, a functor between the underlying semi-simplicial categories.

We have:

LEMMA 7.4.5. *For an* injective *map* $[m_1] \to [m_2]$, *the diagram of obtained by passing to right adjoints along the vertical arrows in*

$$\begin{array}{ccc}
\text{Bar}^{m_1} \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O}) & \longrightarrow & \text{Bar}^{m_1}(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O}) \\
\downarrow & & \downarrow \\
\text{Bar}^{m_2} \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O}) & \longrightarrow & \text{Bar}^{m_2}(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})
\end{array}$$

commutes.

From Lemma 7.4.5 we obtain that the term-wise application of the functor $\mathbf{coinv}^{\text{enh}}(-,-)$, gives rise to a functor from the co-semisimplicial category underlying $\text{Bar}^\bullet \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O})$ to that underlying $\text{Bar}^\bullet(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})^R$.

To prove that the resulting functor between co-semisimplicial categories induces an equivalence of semi-totalizations, it is sufficient to show that for every m, the corresponding functor

$$\mathbf{coinv}(\text{Bar}^m \circ \Omega(\mathfrak{h}), -) : \text{Bar}^m \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O}) \to \text{Bar}^m(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})$$

is fully faithful on the essential image of all the face maps $[0] \to [m]$.

However, this follows from Sect. 7.3.5.

7.4.6. It remains to establish the commutativity of the diagram (7.13).

According to Corollary 7.2.4, under the identification

$$\text{AssocAlg}(\mathbf{oblv}_{\text{Cocom}}) \circ \text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O}) \simeq$$

$$\simeq \text{Tot}\left(\text{Bar}^\bullet(\text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})^R\right)$$

of Proposition 7.2.2, the functor

$$\text{Bar}(\text{AssocAlg}(\mathbf{oblv}_{\text{Cocom}}) \circ \text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h}), -) :$$

$$\text{AssocAlg}(\mathbf{oblv}_{\text{Cocom}}) \circ \text{Grp}(\text{Chev}^{\text{enh}}) \circ \Omega(\mathfrak{h})\text{-mod}(\mathbf{O}) \to \mathbf{O}$$

corresponds to the functor

$$\operatorname{Tot}\left(\operatorname{Bar}^{\bullet}(\operatorname{Grp}(\operatorname{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})^{R}\right) \to$$
$$\to |\operatorname{Bar}^{\bullet}(\operatorname{Grp}(\operatorname{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})| \to \mathbf{O},$$

given by the forgetful functors

$$\mathbf{oblv}_{\operatorname{Bar}^{m}(\operatorname{Grp}(\operatorname{Chev}^{\mathrm{enh}})\circ\Omega(\mathfrak{h}))} : \operatorname{Bar}^{m}(\operatorname{Grp}(\operatorname{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O}) \to \mathbf{O}.$$

We have a commutative diagram

$$\begin{array}{ccc}
\operatorname{Tot}\left(\operatorname{Bar}^{\bullet}\circ\Omega(\mathfrak{h})\text{-mod}(\mathbf{O})\right) & \longrightarrow & \operatorname{Tot}\left(\operatorname{Bar}^{\bullet}(\operatorname{Grp}(\operatorname{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})^{R}\right) \\
{\scriptstyle\sim}\uparrow & & \uparrow{\scriptstyle\sim} \\
|\operatorname{Bar}^{\bullet}\circ\Omega(\mathfrak{h})\text{-mod}(\mathbf{O})^{L}| & \longrightarrow & |\operatorname{Bar}^{\bullet}(\operatorname{Grp}(\operatorname{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathbf{O})|,
\end{array}$$

where the lower horizontal arrow comes from the map of simplicial categories (7.15).

Hence, we need to show that the functor

$$|\operatorname{Bar}^{\bullet}\circ\Omega(\mathfrak{h})\text{-mod}(\mathbf{O})^{L}| \to \mathbf{O},$$

given by

$$\mathbf{coinv}(\operatorname{Bar}^{m}\circ\Omega(\mathfrak{h}), -) : \operatorname{Bar}^{m}\circ\Omega(\mathfrak{h})\text{-mod}(\mathbf{O})^{L} \to \mathbf{O},$$

corresponds under

$$(7.16) \qquad |\operatorname{Bar}^{\bullet}\circ\Omega(\mathfrak{h})\text{-mod}(\mathbf{O})^{L}| \simeq \operatorname{Tot}\left(\operatorname{Bar}^{\bullet}\circ\Omega(\mathfrak{h})\text{-mod}(\mathbf{O})\right) \simeq \mathfrak{h}\text{-mod}(\mathbf{O})$$

to the functor

$$\mathbf{coinv}(\mathfrak{h}, -) : \mathfrak{h}\text{-mod}(\mathbf{O}) \to \mathbf{O}.$$

However, this follows from the fact that the functor (7.16) is given by the functors,

$$\operatorname{Bar}^{m}\circ\Omega(\mathfrak{h})\text{-mod}(\mathbf{O}) \to \mathfrak{h}\text{-mod}(\mathbf{O})$$

left adjoint to those given by restriction.

REMARK 7.4.7. An alternate proof of the equivalence $\mathfrak{h}\text{-mod}(\mathbf{O}) \simeq U(\mathfrak{h})\text{-mod}(\mathbf{O})$ can be given as follows. Given an object $M \in \mathbf{O}$, one has the relative inner Hom

$$\underline{\operatorname{End}}_{\mathbf{O}}(M) \coloneqq \underline{\operatorname{Hom}}_{\mathbf{O}}(M, M)$$

which is an associative algebra in $\mathbf{O}$ (see Volume I, Chapter 1, Sect. 3.6.6). For any associative algebra A in $\mathbf{O}$, the structure of an A-module on M is equivalent to a map of associative algebras $A \to \underline{\operatorname{End}}_{\mathbf{O}}(M)$ [**Lu2**, Corollary 4.7.2.41]. Similarly, one can prove that for any Lie algebra $\mathfrak{h}$, the structure of an $\mathfrak{h}$-module on M is equivalent to a map of Lie algebras $\mathfrak{h} \to \underline{\operatorname{End}}_{\mathbf{O}}(M)$. The equivalence then follows from the description of $U(\mathfrak{h})$ as the algebra induced from $\mathfrak{h}$ along the map of operads $\operatorname{Lie} \to \operatorname{Assoc}^{\mathrm{aug}}$.

A. Proof of Theorem 2.9.4

Recall that Theorem 2.9.4 says that (under a certain hypothesis on the co-operad $\mathcal{Q}$) if we compute primitives in an object of $\mathcal{Q}^{\vee}\text{-Coalg}(\mathbf{O})$ of the form

$$\mathbf{cofree}_{\mathcal{Q}}^{\text{fake}}(V), \quad V \in \mathbf{O},$$

we recover V. The non-triviality here lies in the fact that $\mathbf{cofree}_{\mathcal{Q}}^{\text{fake}}$ is not the co-free $\mathcal{Q}$ co-algebra; rather, it comes from the corresponding co-free object under the functor

$$\mathcal{Q}^{\vee}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O}) \to \mathcal{Q}^{\vee}\text{-Coalg}(\mathbf{O}).$$

So, we are dealing with the difference between direct sums and direct products. At the end of the day the proof will consist of showing that a certain spectral sequence converges, and that will be achieved by taking into account t-structures (hence the assumption on $\mathcal{Q}$).

A.1. Calculation of co-primitives. Let $\mathcal{P}$ be an operad. In this subsection we will give an expression for the functor

$$\text{coPrim}_{\mathcal{P}} : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathbf{O}$$

in terms of the Koszul dual co-operad.

A.1.1. For $n \geq 1$, let

$$\iota_n : \text{Vect} \to \text{Vect}^{\Sigma}$$

be the tautological functor that produces symmetric sequences with only the n-th non-zero component.

We have the following basic fact:

LEMMA A.1.2. *For an operad $\mathcal{P}$, the object $\mathbf{1}_{\text{Vect}^{\Sigma}} \in \text{Vect}^{\Sigma}$, regarded as a right $\mathcal{P}$-module in the monoidal category Vect^{Σ}, can be canonically written as a colimit*

$$\operatorname*{colim}_{n \geq 1} \mathcal{M}_n$$

with

$$\text{coFib}(\mathcal{M}_{n-1} \to \mathcal{M}_n) \simeq \iota_n(\mathcal{P}^{\vee}(n)) \star \mathcal{P}, \quad n \geq 1.$$

A.1.3. The assertion of Lemma A.1.2 gives rise to the following more explicit way to express the functor $\text{coPrim}_{\mathcal{P}}$:

COROLLARY A.1.4. *The functor*

$$A \mapsto \text{coPrim}_{\mathcal{P}}(A), \quad \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathbf{O}$$

admits a canonical filtration by functors of the form

$$A \mapsto \mathcal{M}_n \underset{\mathcal{P}}{\star} A,$$

where $\mathcal{M}_n$ are right $\mathcal{P}$-modules, such that the associated graded of this filtration is canonically identified with

$$n \mapsto \mathcal{P}^{\vee}(n) \star \mathbf{oblv}_{\mathcal{P}}(A).$$

A.2. Computation of primitives. Our current goal is to formulate and prove an analog of Corollary A.1.4 for co-algebras over a co-operad, namely Proposition A.2.3 below.

A.2.1.   Let $\mathcal{Q}$ be a co-operad, $\mathcal{N}$ a right $\mathcal{Q}$-comodule in Vect^Σ, and $A \in \mathcal{Q}\text{-Coalg}(\mathbf{O})$.

We can form a co-simpicial object $\mathrm{coBar}_*^\bullet(\mathcal{N}, \mathcal{Q}, A)$ of $\mathbf{O}$ with the n-th term

$$\mathrm{coBar}_*^n(\mathcal{N}, \mathcal{Q}, A) := \left(\mathcal{N} \star \underbrace{\mathcal{Q} \star ... \star \mathcal{Q}}_{n} \right) * A.$$

We define

$$\mathcal{N} \overset{\mathcal{Q}}{\underset{*}{\star}} A := \mathrm{Tot}\left(\mathrm{coBar}_*^\bullet(\mathcal{N}, \mathcal{Q}, A) \right).$$

A.2.2.   We are going to prove:

PROPOSITION A.2.3.   *The functor*

$$A \mapsto \mathrm{Prim}_\mathcal{Q}(A), \quad \mathcal{Q}\text{-Coalg}(\mathbf{O}) \to \mathbf{O}$$

can be canonically written as an inverse limit of functors of the form

$$\mathcal{N}_n \overset{\mathcal{Q}}{\underset{*}{\star}} A, \quad n \geq 1,$$

where $\mathcal{N}_n$ are right $\mathcal{Q}$-comodules in Vect^Σ with

$$\mathrm{Fib}(\mathcal{N}_n \to \mathcal{N}_{n-1}) \simeq \iota_n(\mathcal{Q}^\vee(n)) \star \mathcal{Q}, \quad n > 1.$$

The rest of this subsection is devoted to the proof of this proposition.

A.2.4.   By definition, the functor $\mathrm{Prim}_\mathcal{Q}$ is calculated as

$$A \mapsto \mathbf{1}_{\mathrm{Vect}^\Sigma} \overset{\mathcal{Q}}{\underset{*}{\star}} A.$$

Now, we have the following assertion, which is an analog of Lemma A.1.2 for co-operads:

LEMMA A.2.5.   *For a co-operad $\mathcal{Q}$, the object $\mathbf{1}_{\mathrm{Vect}^\Sigma} \in \mathrm{Vect}^\Sigma$, regarded as a right $\mathcal{Q}$-comodule in the monoidal category Vect^Σ, can be canonically written as a limit*

$$\lim_{n \geq 1} \mathcal{N}_n,$$

with

$$\mathrm{Fib}(\mathcal{N}_n \to \mathcal{N}_{n-1}) \simeq \iota_n(\mathcal{Q}^\vee(n)) \star \mathcal{Q}, \quad n > 1.$$

A.2.6.   Since functor of totalization commutes with the formation of limits of terms, in order to prove Proposition A.2.3, it suffices to show that for every $m \geq 0$, the natural map

$$\mathrm{coBar}_*^m(\mathbf{1}_{\mathrm{Vect}^\Sigma}, \mathcal{Q}, A) \to \lim_n \mathrm{coBar}_*^m(\mathcal{N}_n, \mathcal{Q}, A)$$

is an isomorphism.

For the latter, by the definition of the $*$-action, it suffices to show that for any $i \geq 0$, the map

$$\left(\mathbf{1}_{\mathrm{Vect}^\Sigma} \star \underbrace{\mathcal{Q} \star ... \star \mathcal{Q}}_{m} \right)(i) \otimes A^{\otimes i} \to \lim_n \left(\mathcal{N}_n \star \underbrace{\mathcal{Q} \star ... \star \mathcal{Q}}_{m} \right)(i) \otimes A^{\otimes i}$$

is an isomorphism.

However, the required isomorphism follows from the fact that for every given i, the family

$$n \mapsto \left(\mathcal{N}_n \star \underbrace{\mathcal{Q} \star \ldots \star \mathcal{Q}}_{m} \right)(i)$$

stabilizes to

$$\left(\mathbf{1}_{\mathrm{Vect}^\Sigma} \star \underbrace{\mathcal{Q} \star \ldots \star \mathcal{Q}}_{m} \right)(i)$$

for $n \geq i$.

A.3. Proof of Theorem 2.9.4.

A.3.1. *Strategy of the proof.* We take $A := \mathbf{cofree}_{\mathcal{Q}}^{\mathrm{fake}}(V)$ for $V \in \mathbf{O}$. We need to show that the natural map

$$V \to \mathrm{Prim}_{\mathcal{Q}} \circ \mathbf{cofree}_{\mathcal{Q}}^{\mathrm{fake}}(V)$$

is an isomorphism.

We calculate the right-hand side via Proposition A.2.3. We will prove that for every $n \geq 1$, the map

$$\mathrm{coFib}\left(V \to \mathcal{N}_n \overset{\mathcal{Q}}{\underset{\star}{*}} \mathbf{cofree}_{\mathcal{Q}}^{\mathrm{fake}}(V) \right) \to \mathrm{coFib}\left(V \to \mathcal{N}_{n-1} \overset{\mathcal{Q}}{\underset{\star}{*}} \mathbf{cofree}_{\mathcal{Q}}^{\mathrm{fake}}(V) \right)$$

is zero. This will prove the required assertion.

A.3.2. *Step 0.* For a right $\mathcal{Q}$-comodule $\mathcal{N}$ in Vect^Σ, and $A \in \mathcal{Q}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O})$, consider the co-simplicial object $\mathrm{coBar}_\star^\bullet(\mathcal{N}, \mathcal{Q}, A)$ of $\mathbf{O}$ with terms

$$\mathrm{coBar}_\star^n(\mathcal{N}, \mathcal{Q}, A) := \left(\mathcal{N} \star \underbrace{\mathcal{Q} \star \ldots \star \mathcal{Q}}_{n} \right) \star A.$$

Set

$$\mathcal{N} \overset{\mathcal{Q}}{\underset{\star}{*}} A := \mathrm{Tot}\left(\mathrm{coBar}_\star^\bullet(\mathcal{N}, \mathcal{Q}, A) \right).$$

Note that for $A \in \mathcal{Q}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O})$, from (2.7) we obtain a map

(A.1) $$\mathcal{N} \overset{\mathcal{Q}}{\underset{\star}{*}} A \to \mathcal{N} \overset{\mathcal{Q}}{\underset{\star}{*}} \mathbf{res}^{\star \to *}(A).$$

We observe:

LEMMA A.3.3. *Let $\mathcal{N}$ be cofree, i.e., of the form $\mathcal{N}' \star \mathcal{Q}$ for $\mathcal{N}' \in \mathrm{Vect}^\Sigma$. Then we have a commutative diagram with vertical arrows being isomorphisms:*

$$\begin{array}{ccc}
\mathcal{N} \overset{\mathcal{Q}}{\underset{\star}{*}} A & \xrightarrow{\text{(A.1)}} & \mathcal{N} \overset{\mathcal{Q}}{\underset{\star}{*}} \mathbf{res}^{\star \to *}(A) \\
\uparrow & & \uparrow \\
\mathcal{N}' \star \mathbf{oblv}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}}(A) & \xrightarrow{\text{(2.7)}} & \mathcal{N}' \star \mathbf{oblv}_{\mathcal{Q}}^{\mathrm{ind\text{-}nilp}}(A).
\end{array}$$

COROLLARY A.3.4. *Let $\mathcal{N}$ be of the form $\iota_n(V) \star \mathcal{Q}$ for some n and $V \in \mathbf{O}$. Then the map (A.1) is an isomorphism.*

A.3.5. *Step 1.* We return to the proof of Theorem 2.9.4. We note that for any $n \geq 1$, the object $\mathcal{N}_n$ has a finite filtration by objects of the form $\iota_m(\mathcal{Q}^\vee(m)) \star \mathcal{Q}$, $m \leq n$.

By Corollary A.3.4, we obtain that for any $A \in \mathcal{Q}\text{-Coalg}^{\text{ind-nilp}}(\mathbf{O})$ the map

$$\mathcal{N}_n \overset{\mathcal{Q}}{\star} A \to \mathcal{N}_n \overset{\mathcal{Q}}{\star} \mathbf{res}^{\star \to *}(A)$$

of (A.1) is an isomorphism.

Hence, we obtain that it suffices to show that the map

$$\text{coFib}\left(V \to \mathcal{N}_n \overset{\mathcal{Q}}{\star} \mathbf{cofree}_{\mathcal{Q}^{\text{ind-nilp}}}(V)\right) \to \text{coFib}\left(V \to \mathcal{N}_{n-1} \overset{\mathcal{Q}}{\star} \mathbf{cofree}_{\mathcal{Q}^{\text{ind-nilp}}}(V)\right)$$

is zero.

A.3.6. *Step 2.* Note that each $\mathcal{N}_n \overset{\mathcal{Q}}{\star} \mathbf{cofree}_{\mathcal{Q}^{\text{ind-nilp}}}(V)$ is naturally graded by integers $d \geq 1$, such that the map

$$V \to \mathcal{N}_n \overset{\mathcal{Q}}{\star} \mathbf{cofree}_{\mathcal{Q}^{\text{ind-nilp}}}(V)$$

is an isomorphism on the degree 1 part for all n.

Hence, it remains to show that for all $d > 1$, the map

$$(A.2) \qquad \left(\mathcal{N}_n \overset{\mathcal{Q}}{\star} \mathbf{cofree}_{\mathcal{Q}^{\text{ind-nilp}}}(V)\right)^d \to \left(\mathcal{N}_{n-1} \overset{\mathcal{Q}}{\star} \mathbf{cofree}_{\mathcal{Q}^{\text{ind-nilp}}}(V)\right)^d$$

is zero, where the superscript d indicates the degree d part.

A.3.7. *Step 3.* Note now that the functor

$$V \mapsto \left(\mathcal{N}_n \overset{\mathcal{Q}}{\star} \mathbf{cofree}_{\mathcal{Q}^{\text{ind-nilp}}}(V)\right)^d$$

(resp., the natutal transformation (A.2)) is given by

$$V \mapsto (\mathcal{K}_n^d \otimes V^{\otimes d})_{\Sigma_d}$$

for some $\mathcal{K}_n^d \in \text{Rep}(\Sigma_d)$ (resp., a map $\mathcal{K}_n^d \to \mathcal{K}_{n-1}^d$).

Hence, it remains to show that for every $d > 1$ and every n, the map

$$(A.3) \qquad\qquad\qquad \mathcal{K}_n^d \to \mathcal{K}_{n-1}^d$$

is zero.

A.3.8. *Step 4.* Since the category $\text{Rep}(\Sigma_d)$ is semi-simple, the fact that (A.3) is equivalent to the map in question inducing the zero map on cohomology.

The latter reduces the assertion of the theorem to the case of $\mathbf{O} = \text{Vect}$. Namely, it suffices to show that for some/any $V \in \text{Vect}_{\text{f.d.}}^\heartsuit$ with $\dim(V) \geq d$, the map (A.2) induces the zero map on cohomology.

A.3.9. *Step 5.* Consider the operad $\mathcal{Q}^*$, and set $\mathcal{M}_n \coloneqq \mathcal{N}_n^*$. We obtain that the object

$$\left(\mathcal{N}_n \overset{\mathcal{Q}}{\star} \mathbf{cofree}_{\mathcal{Q}^{\text{ind-nilp}}}(V)\right)^d$$

is the linear dual of the object

$$(A.4) \qquad \left(\mathcal{M}_n \underset{\mathcal{Q}^*}{\star} \mathbf{free}_{\mathcal{Q}^*}(V^*)\right)^d.$$

Hence, is is enough to show that the map

$$(A.5) \qquad \left(\mathcal{M}_{n-1} \underset{\mathcal{Q}^*}{\star} \mathbf{free}_{\mathcal{Q}^*}(V^*)\right)^d \to \left(\mathcal{M}_n \underset{\mathcal{Q}^*}{\star} \mathbf{free}_{\mathcal{Q}^*}(V^*)\right)^d$$

induces a zero map on cohomology for all $n \geq 1$ and $d > 1$.

A.3.10. *Step 6.* We note that $(\mathcal{Q}^*)^\vee \simeq (\mathcal{Q}^\vee)^*$. So, by the assumption that $\mathcal{Q}^\vee[1]$ and $\mathcal{Q}^*$ are classical,

$$\mathrm{coFib}\left(\mathcal{M}_{n-1} \underset{\mathcal{Q}^*}{\star} \mathbf{free}_{\mathcal{Q}^*}(V^*) \to \mathcal{M}_n \underset{\mathcal{Q}^*}{\star} \mathbf{free}_{\mathcal{Q}^*}(V^*)\right)$$

is concentrated in cohomological degree $-n$.

Hence,

$$\mathrm{coFib}\left(\left(\mathcal{M}_{n-1} \underset{\mathcal{Q}^*}{\star} \mathbf{free}_{\mathcal{Q}^*}(V^*)\right)^d \to \left(\mathcal{M}_n \underset{\mathcal{Q}^*}{\star} \mathbf{free}_{\mathcal{Q}^*}(V^*)\right)^d\right)$$

is also concentrated in cohomological degree $-n$.

Therefore, in order to show that (A.5) induces a zero map on cohomology, it suffices to show that the colimit

$$(A.6) \qquad \underset{n}{\mathrm{colim}} \left(\mathcal{M}_n \underset{\mathcal{Q}^*}{\star} \mathbf{free}_{\mathcal{Q}^*}(V^*)\right)^d.$$

is acyclic.

A.3.11. *Step 7.* By Corollary A.1.4, the colimit (A.6) identifies with the degree d part of

$$\mathrm{coPrim}_{\mathcal{Q}^*} \circ \mathbf{free}_{\mathcal{Q}^*}(V^*).$$

However,

$$\mathrm{coPrim}_{\mathcal{Q}^*} \circ \mathbf{free}_{\mathcal{Q}^*}(V^*) \simeq V^*$$

and hence its degree d part for $d \neq 1$ vanishes.

$\square$

B. Proof of the PBW theorem

In this section we will prove the version of the PBW theorem stated in the main body of the paper as Theorem 5.2.4.

B.1. The PBW theorem at the level of operads. In this subsection we formulate a version of Theorem 5.2.4 that takes place within the category Vect^Σ.

B.1.1. We have the canonical maps

$$\phi : \mathrm{Lie} \to \mathrm{Assoc}^{\text{aug}} \quad \text{and} \quad \psi : \mathrm{Assoc}^{\text{aug}} \to \mathrm{Com}^{\text{aug}},$$

such that the composition $\psi \circ \phi$ factors through the augmentation/unit

$$\mathrm{Lie} \to \mathbf{1}_{\mathrm{Vect}^\Sigma} \to \mathrm{Com}.$$

B.1.2.   The map ϕ gives rise to the forgetful functor

$$\mathbf{res}^{\mathrm{Assoc}^{\mathrm{aug}}\to\mathrm{Lie}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{LieAlg}(\mathbf{O}),$$

and the map ψ gives rise to the forgetful functor

$$\mathbf{res}^{\mathrm{Com}^{\mathrm{aug}}\to\mathrm{Assoc}^{\mathrm{aug}}} : \mathrm{Com}^{\mathrm{aug}}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}).$$

The functor

$$U : \mathrm{LieAlg}(\mathbf{O}) \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$$

is given by

$$\mathfrak{h} \mapsto \mathrm{Assoc}^{\mathrm{aug}} \underset{\mathrm{Lie}}{\star} \mathfrak{h}.$$

B.1.3.   The functor

$$U \circ \mathbf{triv}_{\mathrm{Lie}} : \mathbf{O} \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})$$

is given by

$$V \mapsto \big(\mathrm{Assoc}^{\mathrm{aug}} \underset{\mathrm{Lie}}{\star} \mathbf{1}_{\mathrm{Vect}^{\Sigma}}\big) \star V.$$

The canonical map

$$U \circ \mathbf{triv}_{\mathrm{Lie}}(V) \to \mathbf{free}_{\mathrm{Com}^{\mathrm{aug}}}(V)$$

comes from the map in Vect^{Σ}:

(B.1) $$\mathrm{Assoc}^{\mathrm{aug}} \underset{\mathrm{Lie}}{\star} \mathbf{1}_{\mathrm{Vect}^{\Sigma}} \to \mathrm{Com}^{\mathrm{aug}},$$

which arises via the description of the map $\psi \circ \phi$ in Sect. B.1.1.

B.1.4.   The operadic PBW theorem says:

THEOREM B.1.5.   *The map* (B.1) *is an isomorphism in* Vect^{Σ}.

It is clear that Theorem B.1.5 implies Theorem 5.2.4.

B.2. Proof of Theorem B.1.5.
B.2.1.   We have the natural map in Vect^{Σ}

$$\mathrm{Com}^{\mathrm{aug}} \to \mathrm{Assoc}^{\mathrm{aug}}$$

which realizes the symmetrization map at the level of functors. This gives a map of right Lie-modules in Vect^{Σ}

$$\mathrm{Com}^{\mathrm{aug}} \star \mathrm{Lie} \to \mathrm{Assoc}^{\mathrm{aug}}.$$

It follows from the classical PBW theorem applied to a free Lie algebra on a vector space that this map is an isomorphism.

Hence, we have an isomorphism between $\mathrm{Assoc}^{\mathrm{aug}} \underset{\mathrm{Lie}}{\star} \mathbf{1}_{\mathrm{Vect}^{\Sigma}}$ and $\mathrm{Com}^{\mathrm{aug}}$. In particular, for every n, we have:

$$\Big(\mathrm{Assoc}^{\mathrm{aug}} \underset{\mathrm{Lie}}{\star} \mathbf{1}_{\mathrm{Vect}^{\Sigma}}\Big)(n) \in \mathrm{Vect}^{\heartsuit}.$$

B.2.2. It remains to show that for any $V \in \mathrm{Vect}^\heartsuit$, the map

$$H^0\left(\left(\mathrm{Assoc}^{\mathrm{aug}} \underset{\mathrm{Lie}}{\star} \mathbf{1}_{\mathrm{Vect}^\Sigma}\right) \star V\right) \to \mathbf{free}_{\mathrm{Com}^{\mathrm{aug}}}(V)$$

is an isomorphism.

Note, however, that the object $H^0\left(\left(\mathrm{Assoc} \underset{\mathrm{Lie}}{\star} \mathbf{1}_{\mathrm{Vect}^\Sigma}\right) \star V\right)$ identifies with

$$H^0\left(U \circ \mathbf{triv}_{\mathrm{Lie}}(V)\right),$$

i.e., the universal enveloping algebra of the trivial Lie algebra, taken in the world of *classical* associative algebras.

However, the latter is easily seen to map isomorphically to $\mathbf{free}_{\mathrm{Com}^{\mathrm{aug}}}(V)$.

C. Commutative co-algebras and bialgebras

Let H be a classical co-commutative bialgebra. We can regard H as either an associative algebra in the category of co-commutative co-algebras or, equivalently, a co-commutative co-algebra in the category of associative algebras.

In this section, we establish the corresponding fact in the context of higher algebra, i.e., an equivalence of $(\infty, 1)$-categories $\mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O})) \simeq \mathrm{AssocAlg}(\mathrm{CocomCoalg}(\mathbf{O}))$. The latter is not altogether obvious, as the corresponding classical assertion is proved by 'an explicit formula'.

C.1. Two incarnations of co-commutative bialgebras. Co-commutative bialgebras can be thought of in two different ways: as co-commutative co-algebras in the category of associative algebras, or as associative algebras in the category of co-commutative co-algebras. In this subsection we show that the two are equivalent.

C.1.1. In this subsection we let $\mathbf{O}$ be a symmetric monoidal category, which contains colimits, and for which the functor of tensor product preserves colimits in each variable.

The category $\mathrm{CocomBialg}(\mathbf{O})$ is defined as

$$(\mathrm{C.1}) \qquad \mathrm{AssocAlg}(\mathrm{CocomCoalg}(\mathbf{O})) \simeq \mathrm{AssocAlg}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})),$$

where the (symmetric) monoidal structure on $\mathrm{CocomCoalg}(\mathbf{O})$ is given by tensor product, which coincides with the Cartesian product in $\mathrm{CocomCoalg}(\mathbf{O})$.

Consider now the category $\mathrm{AssocAlg}(\mathbf{O})$, endowed with a symmetric monoidal structure given by tensor product. Consider the category

$$(\mathrm{C.2}) \qquad \mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O})) \simeq \mathrm{CocomCoalg}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})).$$

C.1.2.   In this section we will prove:

PROPOSITION-CONSTRUCTION C.1.3. *There exists a canonical equivalence of categories*

$$\mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O})) \simeq \mathrm{AssocAlg}(\mathrm{CocomCoalg}(\mathbf{O}))$$

that makes the diagram

$$
\begin{array}{ccc}
\mathrm{CocomCoalg}(\mathbf{O}) & \xrightarrow{\ \mathrm{Id}\ } & \mathrm{CocomCoalg}(\mathbf{O}) \\
{\scriptstyle \mathrm{CocomCoalg}(\mathbf{oblv}_{\mathrm{Assoc}})}\Big\uparrow & & \Big\uparrow {\scriptstyle \mathbf{oblv}_{\mathrm{Assoc}}} \\
\mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O})) & \longrightarrow & \mathrm{AssocAlg}(\mathrm{CocomCoalg}(\mathbf{O})) \\
{\scriptstyle \mathbf{oblv}_{\mathrm{Cocom}}}\Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{AssocAlg}(\mathbf{oblv}_{\mathrm{Cocom}})} \\
\mathrm{AssocAlg}(\mathbf{O}) & \xrightarrow{\ \mathrm{Id}\ } & \mathrm{AssocAlg}(\mathbf{O})
\end{array}
$$

commute.

C.2. Proof of Proposition C.1.3.
C.2.1. *Step 1.* We have a canonically defined symmetric monoidal functor

$$\mathrm{Bar}^{\bullet} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O}) \to \mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}}.$$

In particular, we obtain a functor

$$\mathrm{Cocom}(\mathrm{Bar}^{\bullet}) : \mathrm{CocomCoalg}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathbf{O})) \to \mathrm{CocomCoalg}(\mathbf{O}^{\mathbf{\Delta}^{\mathrm{op}}}) \simeq$$
$$\simeq \mathrm{CocomCoalg}(\mathbf{O})^{\mathbf{\Delta}^{\mathrm{op}}}.$$

Combining with (C.2), we obtain a functor

$$(\mathrm{C.3}) \qquad \mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O})) \to \mathrm{CocomCoalg}(\mathbf{O})^{\mathbf{\Delta}^{\mathrm{op}}}.$$

C.2.2. *Step 2.* Since the symmetric monoidal structure on $\mathrm{CocomCoalg}(\mathbf{O})$ is Cartesian, the functor

$$(\mathrm{C.4}) \qquad \mathrm{Bar}^{\bullet} : \mathrm{AssocAlg}(\mathrm{CocomCoalg}(\mathbf{O})) \to \mathrm{CocomCoalg}(\mathbf{O})^{\mathbf{\Delta}^{\mathrm{op}}}$$

is fully faithful.

Now, it is easy to see that the essential image of the functor (C.3) lies in that of (C.4).

This defines a functor in one direction:

$$(\mathrm{C.5}) \qquad \mathrm{CocomCoalg}(\mathrm{AssocAlg}(\mathbf{O})) \to \mathrm{AssocAlg}(\mathrm{CocomCoalg}(\mathbf{O})).$$

C.2.3. *Step 3.* Let us now prove that the functor (C.5) is an equivalence. By construction, the composite functor

$$\text{CocomCoalg}(\text{AssocAlg}(\mathbf{O})) \to \text{AssocAlg}(\text{CocomCoalg}(\mathbf{O})) \overset{\mathbf{oblv}_{\text{Assoc}}}{\longrightarrow} \text{CocomCoalg}(\mathbf{O})$$

is the tautological functor

(C.6) $\text{Cocom}(\mathbf{oblv}_{\text{Assoc}}) : \text{CocomCoalg}(\text{AssocAlg}(\mathbf{O})) \to \text{CocomCoalg}(\mathbf{O}).$

It suffices to show that the functor (C.6) and

(C.7) $\text{AssocAlg}(\text{CocomCoalg}(\mathbf{O})) \overset{\mathbf{oblv}_{\text{Assoc}}}{\longrightarrow} \text{CocomCoalg}(\mathbf{O})$

are both monadic, and that the map of monads, induced by (C.5), is an isomorphism as plain endo-functors of $\text{CocomCoalg}(\mathbf{O})$.

C.2.4. *Step 4.* The functor

$$\text{AssocAlg}(\mathbf{O}') \overset{\mathbf{oblv}_{\text{Assoc}}}{\longrightarrow} \mathbf{O}'$$

is monadic for any monoidal category $\mathbf{O}'$ (satisfying the same assumtion as $\mathbf{O}$); its left adjoint is given by

$$V \mapsto \mathbf{free}_{\text{Assoc}}(V).$$

In particular, the functor (C.7) is monadic: take $\mathbf{O}' := \text{CocomCoalg}(\mathbf{O})$.

C.2.5. *Step 5.* We have a pair of adjoint functors

$$\mathbf{free}_{\text{Assoc}} : \mathbf{O} \rightleftarrows \text{AssocAlg}(\mathbf{O}) : \mathbf{oblv}_{\text{Assoc}},$$

with the right adjoint being symmetric monoidal.

Hence, the above pair induces an adjoint pair

$$\text{CocomCoalg}(\mathbf{O}) \rightleftarrows \text{CocomCoalg}(\text{AssocAlg}(\mathbf{O})).$$

Hence, we obtain that the functor (C.6) is also monadic.

C.2.6. *Step 6.* To show that the map of monads on $\text{CocomCoalg}(\mathbf{O})$, induced by (C.5) is an isomorphism as plain endo-functors, it is enough to do so after composing with the (conservative) forgetful functor $\mathbf{oblv}_{\text{Cocom}} : \text{CocomCoalg}(\mathbf{O}) \to \mathbf{O}$.

By construction, it suffices to prove that the natural transformation

$$\mathbf{free}_{\text{Assoc}} \circ \mathbf{oblv}_{\text{Cocom}} \to \text{AssocAlg}(\mathbf{oblv}_{\text{Cocom}}) \circ \mathbf{free}_{\text{Assoc}},$$

coming by adjunction from the isomorphism

$$\mathbf{oblv}_{\text{Cocom}} \circ \mathbf{oblv}_{\text{Assoc}} \simeq \mathbf{oblv}_{\text{Assoc}} \circ \text{AssocAlg}(\mathbf{oblv}_{\text{Cocom}}),$$

is itself an isomorphism.

However, this follows from the fact that the functor

$$\mathbf{oblv}_{\text{Cocom}} : \text{CocomCoalg}(\mathbf{O}) \to \mathbf{O}$$

is symmetric monoidal and preserves coproducts.

$\square$

Formal groups and Lie algebras

Introduction

In this chapter we will use the notion of inf-scheme to give what may be regarded as the ultimate formulation of the correspondence between formal groups and Lie algebras.

0.1. Why does the tangent space of a Lie group have the structure of a Lie algebra? In classical differential geometry the process of associating a Lie algebra to a Lie group is the following:

(i) For any manifold Y, one considers the associative algebra of global differential operators, endowed with its natural filtration;

(ii) One shows that the underlying Lie algebra is compatible with the filtration, and in particular ass-gr$^1(\mathrm{Diff}(Y)) \simeq \Gamma(Y, T_Y)$ has a structure of Lie algebra;

(iii) If $Y = G$ is a Lie group, the operation of taking differential operators/vector fields, invariant with respect to left translations preserves the pieces of structure in (i) and (ii); in particular, left-invariant vector fields form a Lie algebra.

(iv) One identifies the tangent space at the identity of G with the vector space of left-invariant vector fields.

In the context of derived algebraic geometry, the process of associating a Lie algebra to a formal group is different, and we will describe it in this subsection.

0.1.1.   We will work in a relative context over a given $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ (the special case of $\mathcal{X} = \mathrm{pt}$ is still interesting and contains all the main ideas). By a *formal group* we will mean an object of the category

$$\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}),$$

where $\mathrm{FormMod}_{/\mathcal{X}}$ is the full subcategory of $(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}$ consisting of inf-schematic nil-isomorphisms $\mathcal{Y} \to \mathcal{X}$ (when $\mathcal{X} = \mathrm{pt}$, this is the category of inf-schemes $\mathcal{Y}$ with $^{\mathrm{red}}\mathcal{Y} = \mathrm{pt}$).

We will define a functor

$$\mathrm{Lie}_{\mathcal{X}} : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})),$$

and the main goal of this Chapter is to show that it is an equivalence.

When $X = \mathrm{pt}$ we obtain an equivalence between the category of group inf-schemes whose underlying reduced scheme is pt and the category of Lie algebras in Vect. Note that we impose no conditions on the cohomological degrees in which our Lie algebras are supposed to live.

0.1.2.   To explain the idea of the functor $\mathrm{Lie}_{\mathcal{X}}$, let us first carry it out for classical Lie groups; this will be a procedure of associating a Lie algebra to a Lie group different (but, of course, equivalent) to one described above.

Namely, let G be a Lie group. The space $\mathrm{Distr}(G)$ of distributions supported at the identity of G has a natural structure of a co-commutative Hopf algebra (which will ultimately be identified with the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ associated to G).

Now, the Lie algebra $\mathfrak{g}$ can then be described as the space of primitive elements of $\mathrm{Distr}(G)$.

0.1.3.   We will now describe how the functor $\mathrm{Lie}_{\mathcal{X}}$ is constructed in the context of derived algebraic geometry. The construction will be compatible with pullbacks, so we can assume that $\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$.

For any

$$(\mathcal{Y} \xrightarrow{\pi} X) \in \mathrm{FormMod}_{/X}$$

we consider

$$\mathrm{Distr}(\mathcal{Y}) := \pi_*^{\mathrm{IndCoh}}(\omega_{\mathcal{Y}}) \in \mathrm{IndCoh}(X),$$

and we observe that it has a structure of co-commutative co-algebra in $\mathrm{IndCoh}(X)$, viewed as a symmetric monoidal category with respect to the !-tensor product. (If $X = \mathrm{pt}$, the dual of $\mathrm{Distr}(\mathcal{Y})$ is the commutative algebra $\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$.)

We denote the resulting functor $\mathrm{FormMod}_{/X} \to \mathrm{CocomCoalg}(\mathrm{IndCoh}(X))$ by $\mathrm{Distr}^{\mathrm{Cocom}}$; one shows that it sends products to products. In particular, $\mathrm{Distr}^{\mathrm{Cocom}}$ gives rise to a functor

$$\mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}}) : \mathrm{Grp}(\mathrm{FormMod}_{/X}) \to \mathrm{Grp}(\mathrm{CocomCoalg}(\mathrm{IndCoh}(X))) =:$$
$$= \mathrm{CocomHopf}(\mathrm{IndCoh}(X)).$$

0.1.4.   Recall now (see Chapter 6, Sect. 4.4.2) that the category $\mathrm{CocomHopf}(\mathrm{IndCoh}(X))$ is related by a pair of adjoint functors with the category $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$:

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \rightleftarrows \mathrm{CocomHopf}(\mathrm{IndCoh}(X)) : B_{\mathrm{Lie}} \circ \mathrm{Grp}(\mathrm{coChev}^{\mathrm{enh}}),$$

with the left adjoint being fully faithful.

Finally, we set

(0.1) $\mathrm{Lie}_{\mathcal{X}} := B_{\mathrm{Lie}} \circ \mathrm{Grp}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}}).$

0.1.5.   The upshot of the above discussion is the following: the appearance of the Lie algebra structure is due to the Quillen duality at the level of operads:

$$(\mathrm{Cocom}^{\mathrm{aug}})^{\vee} \simeq \mathrm{Lie}[-1].$$

The shift $[-1]$ is compensated by delooping–this is where the group structure is used.

0.2. Formal moduli problems and Lie algebras. The equivalence

$$\mathrm{Lie}_{\mathcal{X}} : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$$

allows us to recover Lurie's equivalence ([**Lu6**, Theorem 2.0.2]) between Lie algebras and formal moduli problems, as we shall presently explain.

0.2.1. Let $\mathcal{X}$ be an object $\mathrm{PreStk}_{\mathrm{laft}}$. Consider the category $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ of pointed objects in $\mathrm{FormMod}_{/\mathcal{X}}$. I.e., this is the category of diagrams

$$(\pi : \mathcal{Y} \rightleftarrows \mathcal{X} : s), \quad \pi \circ s = \mathrm{id}$$

with the map π being an inf-schematic nil-isomorphism.

Recall also that according to Chapter 5, Theorem 1.6.4, the loop functor

$$\Omega_X : \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$$

is an equivalence, with the inverse functor denoted B_X.

0.2.2. Thus, we obtain that the composition

$$(0.2) \qquad \mathrm{Lie}_{\mathcal{X}} \circ \Omega_X : \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X))$$

is an equivalence.

0.2.3. Let us now take $\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. Let us comment on the behavior of the functor $\mathrm{Lie}_X \circ \Omega_X$ in this case.

By construction, the above functor is

$$B_{\mathrm{Lie}} \circ \mathrm{Grp}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}}) \circ \Omega_X,$$

i.e., it involves first looping our moduli problem and then delooping at the level of Lie algebras.

Note, however, that there is another functor

$$\mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X)).$$

Namely, the functor

$$\mathrm{Distr}^{\mathrm{Cocom}} : \mathrm{FormMod}_{/X} \to \mathrm{CocomCoalg}(\mathrm{IndCoh}(X))$$

gives rise to a functor

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)).$$

Composing with the functor

$$\mathrm{coChev}^{\mathrm{enh}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X)),$$

we obtain a functor

$$(0.3) \qquad \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X)).$$

0.2.4. Now, the point is that the functors $\mathrm{Lie}_X \circ \Omega_X$ and $\mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$ are *not* isomorphic.

In terms of the equivalence (0.2), the functor $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$ corresponds to

$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)).$$

Hence, the discrepancy between $\mathrm{Lie}_X \circ \Omega_X$ and $\mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$ is the endo-functor of $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$ equal to

$$\mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Chev}^{\mathrm{enh}}.$$

In particular, the unit of the adjunction defines a natural transformation

$$(0.4) \qquad \mathrm{Lie}_X \circ \Omega_X \to \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}.$$

0.3. Inf-affineness. Let $\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$. In Sect. 2 we will introduce the notion of *inf-affineness* for objects of $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$.

0.3.1.   One of the equivalent conditions for an object $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ to be inf-affine is that the map (0.4) should be an isomorphism.

Another equivalent condition for inf-affineness is that the natural map

$$T(\mathcal{Y}/X)|_X \to \mathbf{oblv}_{\mathrm{Lie}} \circ \mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})$$

should be an isomorphism.

0.3.2.   One of the ingredients in proving that Lie_X is an equivalence is the assertion that any $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$, viewed as an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$, is inf-affine.

But, in fact, a stronger assertion is true.

0.3.3.   To any $\mathcal{F} \in \mathrm{IndCoh}(X)$ we attach the *vector prestack*, denoted $\mathrm{Vect}_X(\mathcal{F})$. Namely, for

$$(f : S \rightleftarrows X : s) \in (\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ to\ } X}$$

we set

$$\mathrm{Maps}_{/X}(S, \mathrm{Vect}_X(\mathcal{F})) = \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathrm{Distr}^+(S), \mathcal{F}),$$

where

$$\mathrm{Distr}^+(S) := \mathrm{Fib}(f_*^{\mathrm{IndCoh}}(\omega_S) \to \omega_X).$$

In Corollary 2.2.3 we show that $\mathrm{Vect}_X(\mathcal{F})$ is inf-affine.

0.3.4.   It follows from Chapter 6, Corollary 1.7.3 that any $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/X})$, regarded as an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$, is canonically isomorphic to

$$\mathrm{Vect}_X(\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{Lie}(\mathcal{H}))).$$

0.4. The functor of inf-spectrum and the exponential construction. Let $\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$. Above we have introduced the functor

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)).$$

Another crucial ingredient in the proof of the fact that the functor Lie_X of (0.1) is an equivalence is the functor

$$\mathrm{Spec}^{\mathrm{inf}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{Ptd}(\mathrm{FormMod}_{/X}),$$

right adjoint to $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$.

0.4.1.   In terms of the equivalence (0.2), the functor $\mathrm{Spec}^{\mathrm{inf}}$ corresponds to the functor

$$\mathrm{coChev}^{\mathrm{enh}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X)).$$

For example, we have:

$$\mathrm{Vect}_X(\mathcal{F}) \simeq \mathrm{Spec}^{\mathrm{inf}}(\mathrm{Sym}(\mathcal{F})).$$

0.4.2.   The notions of inf-affineness and inf-spectrum are loosely analogous to those of affineness and spectrum in algebraic geometry. But the analogy is not perfect. For example, it is *not* true that the functor $\mathrm{Spec}^{\mathrm{inf}}$ is fully faithful.

Conjecturally, the functor $\mathrm{Spec}^{\mathrm{inf}}$ maps $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$ to the full subcategory of $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ consisting of inf-affine objects.

0.4.3.   We use the functor $\mathrm{Spec}^{\mathrm{inf}}$ to construct an inverse to the functor Lie_X:

$$\exp_X : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{Grp}(\mathrm{FormMod}_{/X}).$$

Namely,

$$\exp_X := \mathrm{Spec}^{\mathrm{inf}} \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}.$$

0.4.4.   The functor $\exp_{\mathcal{X}}$ (extended from the case of schemes to that of prestacks) can be used to give the following interpretation to the construction of the functor of *split square-zero extension*

$$\mathrm{IndCoh}(\mathcal{X}) \to \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}),$$

extending the functor

$$\mathrm{RealSplitSqZ} : (\mathrm{Coh}(X)^{\leq 0})^{\mathrm{op}} \to \mathrm{Ptd}((\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}})_{/X}), \quad X \in \mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}}$$

of Chapter 1, Sect. 2.1. Here we regard $(\mathrm{Coh}(X)^{\leq 0})^{\mathrm{op}}$ as a full subcategory of $\mathrm{IndCoh}(X)$ by means of

$$(\mathrm{Coh}(X)^{\leq 0})^{\mathrm{op}} \hookrightarrow \mathrm{Coh}(X)^{\mathrm{op}} \xrightarrow{\mathbb{D}_X^{\mathrm{Serre}}} \mathrm{Coh}(X) \hookrightarrow \mathrm{IndCoh}(X).$$

Namely, we have:

$$\Omega_{\mathcal{X}} \circ \mathrm{RealSplitSqZ}(\mathcal{F}) := \exp_{\mathcal{X}} \circ \mathbf{free}_{\mathrm{Lie}} \circ \Omega,$$

where Ω on $\mathrm{IndCoh}(\mathcal{X})$ is the functor of shift $[-1]$.

0.5. What else is done in this chapter? In this chapter we cover two more topics: the notion of action of objects in $\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$ on objects of $\mathrm{IndCoh}(\mathcal{X})$ and on objects of $(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}$.

0.5.1.   For $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$ we consider its Bar complex

$$B^{\bullet}(\mathcal{H}) \in (\mathrm{FormMod}_{/\mathcal{X}})^{\mathbf{\Delta}^{\mathrm{op}}}.$$

We define

$$\mathcal{H}\text{-mod}(\mathrm{IndCoh}(X)) := \mathrm{Tot}(\mathrm{IndCoh}^{!}(B^{\bullet}(\mathcal{H}))).$$

In Sect. 5 we prove that the category $\mathcal{H}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))$ identifies canonically with

$$\mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})),$$

where $\mathfrak{h} = \mathrm{Lie}_{\mathcal{X}}(\mathcal{H})$.

0.5.2. In Sect. 6 we study the (naturally defined) notion of action of $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$ on $(\mathcal{Y} \xrightarrow{\pi} \mathcal{X}) \in (\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}$.

Given an action of $\mathcal{H}$ on $\mathcal{Y}$, we construct the localization functor

$$\mathrm{Loc}_{\mathfrak{h},\mathcal{Y}/\mathcal{X}} : \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \to {}^{/\mathcal{X}}\mathrm{Crys}(\mathcal{Y}).$$

Moreover, we show that given an action of $\mathcal{H}$ on $\mathcal{Y}$, we obtain a map in $\mathrm{IndCoh}(\mathcal{Y})$

$$\pi^{!}(\mathfrak{h}) \to T(\mathcal{Y}/\mathcal{X}), \quad \mathfrak{h} = \mathrm{Lie}_{\mathcal{X}}(\mathcal{H}).$$

We also show that if $\mathfrak{h}$ is free, i.e., $\mathfrak{h} = \mathbf{free}_{\mathrm{Lie}}(\mathcal{F})$ with $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$, then the map from the space of actions of $\mathcal{H}$ on $\mathcal{Y}$ to the space of maps $\pi^{!}(\mathcal{F}) \to T(\mathcal{Y}/\mathcal{X})$ is an isomorphism.

1. Formal moduli problems and co-algebras

As was mentioned in the introduction, our goal in this chapter is to address the following old question: what is exacty the relationship between formal groups and Lie algebras. By a Lie group we understand an object of $\mathrm{Grp}(\mathrm{FormMod}_{/X})$ and by a Lie algebra an object of $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$.

In this section, we take the first step towards proving this equivalence. Namely, we establish a relationship between *pointed* formal moduli problems over X and co-commutative co-algebras in $\mathrm{IndCoh}(X)$. Specifically, we define the functor of *inf-spectrum* that assigns to a co-commutative co-algebra a pointed formal moduli problem over X.

Formal moduli problems arising in this way play a role loosely analogous to that of affine schemes in the context of usual algebraic geometry.

1.1. Co-algebras associated to formal moduli problems. To any scheme (affine or not) we can attach the commutative algebra of global sections of its structure sheaf. This functor is, obviously, contravariant.

It turns out that formal moduli problems are well-adapted for a dual operation: we send a moduli problem to the co-algebra of sections of its dualizing sheaf, which can be thought of as the co-algebra of distributions. In this subsection we describe this construction.

1.1.1. Let X be an object of ${}^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$. We regard the category $\mathrm{IndCoh}(X)$ as endowed with the symmetric monoidal structure, given by $\overset{!}{\otimes}$.

Recall the category $\mathrm{FormMod}_{/X}$. We have a canonically defined functor

$$(1.1) \qquad \mathrm{FormMod}_{/X} \to (\mathrm{DGCat}^{\mathrm{SymMon}})_{\mathrm{IndCoh}(X)/}, \quad \mathcal{Y} \mapsto \mathrm{IndCoh}(\mathcal{Y}).$$

1.1.2. Consider the following general situation. Let $\mathbf{O}$ be a fixed symmetric monoidal category, and consider the category $(\mathrm{DGCat}^{\mathrm{SymMon}})_{\mathbf{O}/}$. Let

$$'(\mathrm{DGCat}^{\mathrm{SymMon}})_{\mathbf{O}/} \subset (\mathrm{DGCat}^{\mathrm{SymMon}})_{\mathbf{O}/}$$

be the full subcategory consisting of those objects, for which the functor $\mathbf{O} \to \mathbf{O}'$ admits a left adjoint, which is compatible with the $\mathbf{O}$-module structure.

Note that for any $\phi : \mathbf{O} \to \mathbf{O}'$ as above, the object $\phi^L(\mathbf{1}_{\mathbf{O}'}) \in \mathbf{O}$ has a canonical structure of co-commutative co-algebra in $\mathbf{O}$. In particular, we obtain a canonically defined functor

$$'(\mathrm{DGCat}^{\mathrm{SymMon}})_{\mathbf{O}/} \to \mathrm{CocomCoalg}(\mathbf{O}).$$

Moreover, the functor

$$\phi^L : \mathbf{O}' \to \mathbf{O}$$

canonically factors as

$$\mathbf{O}' \to \phi^L(\mathbf{1}_{\mathbf{O}'})\text{-comod}(\mathbf{O}) \overset{\mathbf{oblv}_{\phi^L(\mathbf{1}_{\mathbf{O}'})}}{\longrightarrow} \mathbf{O},$$

in a way functorial in $\mathbf{O}' \in {}'(\mathrm{DGCat}^{\mathrm{SymMon}})_{\mathbf{O}/}$.

1.1.3. We apply the above discussion to $\mathbf{O} = \mathrm{IndCoh}(X)$. Base change (see Chapter 3, Proposition 3.1.2) implies that the functor (1.1) factors as

$$\mathrm{FormMod}_{/X} \to {}'(\mathrm{DGCat}^{\mathrm{SymMon}})_{\mathrm{IndCoh}(X)/}, \quad \mathcal{Y} \mapsto \mathrm{IndCoh}(\mathcal{Y}).$$

In particular, we obtain a functor

$$\mathrm{FormMod}_{/X} \to \mathrm{CocomCoalg}(\mathrm{IndCoh}(X)), \quad (\mathcal{Y} \overset{\pi}{\to} X) \mapsto \pi_*^{\mathrm{IndCoh}}(\omega_{\mathcal{Y}}).$$

We denote this functor by $\mathrm{Distr}^{\mathrm{Cocom}}$. We denote by

$$\mathrm{Distr} : \mathrm{FormMod}_{/X} \to \mathrm{IndCoh}(X)$$

the composition of $\mathrm{Distr}^{\mathrm{Cocom}}$ with the forgetful functor

$$\mathrm{CocomCoalg}(\mathrm{IndCoh}(X)) \overset{\mathbf{oblv}_{\mathrm{Cocom}}}{\longrightarrow} \mathrm{IndCoh}(X).$$

The functor

$$\pi_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(X)$$

canonically factors as

$$\mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{Distr}^{\mathrm{Cocom}}(\mathcal{Y})\text{-comod}(\mathrm{IndCoh}(X)) \overset{\mathbf{oblv}_{\mathrm{Distr}^{\mathrm{Cocom}}(\mathcal{Y})}}{\longrightarrow} \mathrm{IndCoh}(X)$$

in a way functorial in $\mathcal{Y}$.

1.1.4. The functor $\mathrm{Distr}^{\mathrm{Cocom}}$ defines a functor

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)),$$

and the functor Distr defines a functor

$$\mathrm{Distr}^{\mathrm{aug}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{IndCoh}(X)_{\omega_X/}.$$

We shall denote by Distr^+ the functor $\mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{IndCoh}(X)$ that sends $\mathcal{Y}$ to

$$\mathrm{coFib}(\omega_X \to \mathrm{Distr}(\mathcal{Y})) \simeq \mathrm{Fib}(\mathrm{Distr}(\mathcal{Y}) \to \omega_X).$$

1.1.5. *An example.* Note that we have a commutative diagram

$$
\begin{array}{ccc}
\left(\mathrm{Coh}(X)^{\leq 0}\right)^{\mathrm{op}} & \longrightarrow & \mathrm{Ptd}((^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\text{nil-isom to } X}) \\
\Big\downarrow{\mathbb{D}_X^{\mathrm{Serre}}} & & \Big\downarrow{\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}} \\
\mathrm{Coh}(X) & \xrightarrow{\mathbf{triv}_{\mathrm{Cocom}^{\mathrm{aug}}}} & \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)),
\end{array}
$$

(1.2)

where the top horizontal arrow is the functor of split square-zero extension.

1.1.6.   The following observation will be useful:

LEMMA 1.1.7.

(a) *The functors*
$$\mathrm{Distr} : \mathrm{FormMod}_{/X} \to \mathrm{IndCoh}(X)$$
and
$$\mathrm{Distr}^{\mathrm{Cocom}} : \mathrm{FormMod}_{/X} \to \mathrm{CocomCoalg}(\mathrm{IndCoh}(X))$$
are left Kan extensions of their respective restrictions to
$$(^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\text{nil-isom to } X} \subset \mathrm{FormMod}_{/X} .$$

(b) *The functors*
$$\mathrm{Distr}^{\mathrm{aug}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{IndCoh}(X)_{\omega_X/}$$
and
$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)),$$
are left Kan extensions of their respective restrictions to
$$\mathrm{Ptd}((^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\text{nil-isom to } X}) \subset \mathrm{Ptd}(\mathrm{FormMod}_{/X}).$$

PROOF. We prove point (a), since point (b) is similar.

Since the forgetful functor
$$\mathbf{oblv}_{\mathrm{Cocom}} : \mathrm{CocomCoalg}(\mathrm{IndCoh}(X)) \to \mathrm{IndCoh}(X)$$
commutes with colimits, it suffices to prove the assertion for the functor
$$\mathrm{Distr} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{IndCoh}(X).$$

The required assertion follows from Chapter 5, Corollary 1.5.5.

□

REMARK 1.1.8. Recall (see Chapter 6, Sect. 2.2) that for a DG category $\mathbf{O}$, in addition to the category $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O})$, one can consider the category
$$\mathrm{CocomCoalg}(\mathbf{O})^{\mathrm{aug,ind\text{-}nilp}} := \mathrm{Cocom}^{\mathrm{aug}}\text{-}\mathrm{Coalg}^{\mathrm{ind\text{-}nilp}}(\mathbf{O})$$
of ind-nilpotent co-commutatve co-algebras. This category is endowed with a forgetful functor
$$\mathbf{res}^{\star \to *} : \mathrm{CocomCoalg}^{\mathrm{aug,ind\text{-}nilp}} \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathbf{O}).$$

Using Lemma 1.1.7, one can refine the above functor
$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$$
to a functor
$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug,ind\text{-}nilp}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{CocomCoalg}^{\mathrm{aug,ind\text{-}nilp}}(\mathrm{IndCoh}(X)).$$

Namely, for $Z \in \mathrm{Ptd}((^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{nil\text{-}isom\ to}\ X}))$, the t-structure allows to naturally upgrade the object

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(Z) \in \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$$

to an object

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug},\mathrm{ind\text{-}nilp}}}(Z) \in \mathrm{CocomCoalg}^{\mathrm{aug},\mathrm{ind\text{-}nilp}}(\mathrm{IndCoh}(X)).$$

Now, we let $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug},\mathrm{ind\text{-}nilp}}}$ be the left Kan extension under

$$\mathrm{Ptd}((^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{nil\text{-}isom\ to}\ X}) \hookrightarrow \mathrm{Ptd}(\mathrm{FormMod}_{/X})$$

of the above functor $Z \mapsto \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug},\mathrm{ind\text{-}nilp}}}(Z)$.

1.2. The monoidal structure. In this subsection we will establish the compatibility of the functor $\mathrm{Distr}^{\mathrm{Cocom}}$ with symmetric monoidal structures. Namely, we show that $\mathrm{Distr}^{\mathrm{Cocom}}$ is a symmetric monoidal functor and commutes with the Bar-construction on group objects.

1.2.1. Let us consider $\mathrm{FormMod}_{/X}$ and $\mathrm{CocomCoalg}(\mathrm{IndCoh}(X))$ as symmetric monoidal categories with respect to the Cartesian structure. Tautologically, the functor $\mathrm{Distr}^{\mathrm{Cocom}}$ is left-lax symmetric monoidal. We claim:

LEMMA 1.2.2. *The left-lax symmetric monoidal structure on* $\mathrm{Distr}^{\mathrm{Cocom}}$ *is strict.*

PROOF. We need to show that for

$$\pi_1 : \mathcal{Y}_1 \to X \text{ and } \pi_2 : \mathcal{Y}_2 \to X,$$

and

$$\mathcal{Y}_1 \underset{X}{\times} \mathcal{Y}_2 =: \mathcal{Y} \xrightarrow{\pi} X,$$

the map

$$\pi^{\mathrm{IndCoh}}_*(\omega_{\mathcal{Y}}) \to (\pi_1)^{\mathrm{IndCoh}}_*(\omega_{\mathcal{Y}_1}) \overset{!}{\otimes} (\pi_2)^{\mathrm{IndCoh}}_*(\omega_{\mathcal{Y}_2})$$

is an isomorphism.

However, this follows from base change for the diagram

$$
\begin{array}{ccc}
\mathcal{Y} & \longrightarrow & \mathcal{Y}_1 \times \mathcal{Y}_2 \\
\pi \downarrow & & \downarrow \pi_1 \times \pi_2 \\
X & \longrightarrow & X \times X.
\end{array}
$$

$\square$

1.2.3. Recall the notation

$$\mathrm{CocomBialg}(\mathrm{IndCoh}(X)) := \mathrm{AssocAlg}\,(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)));$$

this is the category of associative algebras in $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$.

Recall also that

$$\mathrm{CocomHopf}(\mathrm{IndCoh}(X)) \subset \mathrm{CocomBialg}(\mathrm{IndCoh}(X))$$

denotes the full subcategory spanned by group-like objects.

1.2.4.   By Lemma 1.2.2, the functor $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$ gives rise to a functor

$$\mathrm{Grp}(\mathrm{FormMod}_{/X}) \simeq \mathrm{Monoid}(\mathrm{Ptd}(\mathrm{FormMod}_{/X})) \to$$
$$\to \mathrm{AssocAlg}\left(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))\right) = \mathrm{CocomBialg}(\mathrm{IndCoh}(X)),$$

which in fact factors through

$$\mathrm{CocomHopf}(\mathrm{IndCoh}(X)) \subset \mathrm{CocomBialg}(\mathrm{IndCoh}(X)).$$

We denote the resulting functor by $\mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}})$.

1.2.5.   The following will be useful in the sequel:

LEMMA 1.2.6.   *Let $\mathcal{H}$ be an object of $\mathrm{Grp}(\mathrm{FormMod}_{/X})$. Then the canonical map*

$$\mathrm{Bar} \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}})(\mathcal{H}) \to \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} \circ B_X(\mathcal{H})$$

is an isomorphism.

PROOF.   It is enough to establish the isomorphism in question after applying the forgetful functor $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{IndCoh}(X)$.

The left-hand side is the geometric realization of the simplicial object of $\mathrm{IndCoh}(X)$ given by

$$\mathrm{Bar}^{\bullet}(\mathrm{Distr}(\mathcal{H})) \overset{\mathrm{Lemma\ 1.2.2}}{\simeq} \mathrm{Distr}\left(B_X^{\bullet}(\mathcal{H})\right).$$

We can think of $B_X^{\bullet}(\mathcal{H})$ as the Čech nerve of the map $X \to B_X(\mathcal{H})$. Hence, by Chapter 3, Proposition 3.3.3(b), the map

$$|\mathrm{Distr}\left(B_X^{\bullet}(\mathcal{H})\right)| \to \mathrm{Distr}(B_X(\mathcal{H}))$$

is an isomorphism, as required.

$\square$

1.3. The functor of inf-spectrum.   Continuing the parallel with usual algebraic geometry, the functor Spec provides a right adjoint to the functor

$$\mathrm{Sch} \to (\mathrm{ComAlg}(\mathrm{Vect}^{\leq 0}))^{\mathrm{op}}, \quad X \mapsto \tau^{\leq 0}(\Gamma(X, \mathcal{O}_X)).$$

In this subsection we will develop its analog for formal moduli problems. This will be a functor, denoted $\mathrm{Spec}^{\mathrm{inf}}$, right adjoint to

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)).$$

1.3.1.   Starting from $\mathcal{A} \in \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$, we first define a presheaf on the category on $\mathrm{Ptd}((^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ to}\ X})$, denoted $\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})_{\mathrm{nil\text{-}isom}}$, by

$$\mathrm{Maps}(Z, \mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})_{\mathrm{nil\text{-}isom}}) := \mathrm{Maps}_{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(Z), \mathcal{A}).$$

Let $\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A}) \in (\mathrm{PreStk}_{\mathrm{laft}})_{/X}$ be the left Kan extension of $\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})_{\mathrm{nil\text{-}isom}}$ along the forgetful functor

$$(1.3) \qquad \left(\mathrm{Ptd}((^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ to}\ X})\right)^{\mathrm{op}} \to \left((^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/X}\right)^{\mathrm{op}}.$$

We claim that $\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})$ is an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$.

Indeed, this follows from Chapter 5, Corollary 1.5.2(b) and the following assertion:

LEMMA 1.3.2. *Let $Z_2' \coloneqq Z_1' \underset{Z_1}{\sqcup} Z_2$ be a push-out diagram in $\mathrm{Ptd}((^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ to}\ X})$, where the map $Z_1 \to Z_1'$ is a closed embedding. Then the canonical map*

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(Z_1') \underset{\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(Z_1)}{\sqcup} \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(Z_2) \to \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(Z_2')$$

is an isomorphism.

PROOF. Since the forgetful functor

$$\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{IndCoh}(X)_{\omega_X/}$$

commutes with colimits, it is sufficient to show that the map

$$\mathrm{Distr}^{\mathrm{aug}}(Z_1') \underset{\mathrm{Distr}^{\mathrm{aug}}(Z_1)}{\sqcup} \mathrm{Distr}^{\mathrm{aug}}(Z_2) \to \mathrm{Distr}^{\mathrm{aug}}(Z_2')$$

is an isomorphism in $\mathrm{IndCoh}(X)_{\omega_X/}$. By Serre duality, this is equivalent to showing that

$$(\pi_2')_*(\mathcal{O}_{Z_2'}) \to (\pi_1')_*(\mathcal{O}_{Z_1'}) \underset{(\pi_1)_*(\mathcal{O}_{Z_1})}{\times} (\pi_2)_*(\mathcal{O}_{Z_2})$$

is an isomorphism in $\mathrm{QCoh}(X)$, and the latter follows from the assumptions.

$\square$

1.3.3. We now claim that the assignment

$$\mathcal{A} \mapsto \mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})$$

provides a right adjoint to the functor $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$. Indeed, this follows from Chapter 5, Corollaries 1.5.2(a) and Lemma 1.1.7(b).

1.3.4. We have:

LEMMA 1.3.5. *For $\mathcal{A} \in \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$, there is a canonical isomorphism*

$$T(\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})/X)|_X \simeq \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{A}).$$

PROOF. The proof is just a repeated application of definitions. Indeed, for $\mathcal{F} \in \mathrm{Coh}(X)^{\leq 0}$ we have by definition

$$\mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}), T(\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})/X)|_X)) =$$

$$= \mathrm{Maps}_{\mathrm{Pro}(\mathrm{QCoh}(X)^-)}(T^*(\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})/X)|_X), \mathcal{F}) = \mathrm{Maps}_{X//X}(X_{\mathcal{F}}, \mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})),$$

where $X_{\mathcal{F}}$ is the split square-zero extension corresponding to $\mathcal{F}$, see Chapter 1, Sect. 2.1.1.

By definition,

$$\mathrm{Maps}_{X//X}(X_{\mathcal{F}}, \mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})) = \mathrm{Maps}_{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(S_{\mathcal{F}}), \mathcal{A}).$$

Now, by (1.2)

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(S_{\mathcal{F}}) \simeq \mathbf{triv}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F})),$$

while

$$\mathrm{Maps}_{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))}(\mathbf{triv}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F})), \mathcal{A}) = \mathrm{Maps}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}), \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{A})),$$

again by definition.

$\square$

1.3.6. Being a right adjoint to a symmetric monoidal functor, the functor $\mathrm{Spec}^{\mathrm{inf}}$ is automatically right-lax symmetric monoidal. Hence, it gives rise to a functor

$$\mathrm{CocomBialg}(\mathrm{IndCoh}(X)) \coloneqq \mathrm{AssocAlg}\left(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))\right) \to$$
$$\to \mathrm{Monoid}(\mathrm{Ptd}(\mathrm{FormMod}_{/X})) \simeq \mathrm{Grp}(\mathrm{FormMod}_{/X}).$$

We shall denote the above functor by $\mathrm{Monoid}(\mathrm{Spec}^{\mathrm{inf}})$.

REMARK 1.3.7. If instead of the category $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$ one works with the category $\mathrm{Cocom}^{\mathrm{aug,ind\text{-}nilp}}(\mathrm{IndCoh}(X))$, one obtains a functor

$$\mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}} : \mathrm{Cocom}^{\mathrm{aug,ind\text{-}nilp}}(\mathrm{IndCoh}(X)) \to \mathrm{Ptd}(\mathrm{FormMod}_{/X}).$$

However, the functors $\mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}}$ and $\mathrm{Spec}^{\mathrm{inf}}$ carry the same information: it follows formally that the functor $\mathrm{Spec}^{\mathrm{inf}}$ factors as the composition

$$\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{Cocom}^{\mathrm{aug,ind\text{-}nilp}}(\mathrm{IndCoh}(X)) \overset{\mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}}}{\longrightarrow} \mathrm{Ptd}(\mathrm{FormMod}_{/X}),$$

where the first arrow is the right adjoint to the forgetful functor

$$\mathbf{res}^{\star \to *} : \mathrm{Cocom}^{\mathrm{aug,ind\text{-}nilp}}(\mathrm{IndCoh}(X)) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)).$$

Furthermore, it follows from Chapter 6, Corollary 2.10.5(b), applied in the case of the co-operad $\mathrm{Cocom}^{\mathrm{aug}}$, and Lemma 1.3.5 that the natural map from $\mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}}$ to the composition

$$\mathrm{Cocom}^{\mathrm{aug,ind\text{-}nilp}}(\mathrm{IndCoh}(X)) \overset{\mathbf{res}^{\star \to *}}{\longrightarrow} \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \overset{\mathrm{Spec}^{\mathrm{inf}}}{\longrightarrow} \mathrm{Ptd}(\mathrm{FormMod}_{/X})$$

is an isomorphism.

1.4. An example: vector prestacks. The basic example of a scheme is the scheme attached to a finite-dimensional vector space:

$$\mathrm{Maps}(S, V) = \Gamma(S, \mathcal{O}_S) \otimes V.$$

In this subsection we describe the counterpart of this construction for formal moduli problems.

Namely, for an object $\mathcal{F} \in \mathrm{IndCoh}(X)$, we will construct a formal moduli problem $\mathrm{Vect}_X(\mathcal{F})$ over X. In the case when $\mathcal{F}$ is a coherent sheaf, $\mathrm{Vect}_X(\mathcal{F})$ will be the formal completion of the zero section of the 'vector bundle' associated to $\mathcal{F}$.

1.4.1. Let $\mathcal{F}$ be an object of $\mathrm{IndCoh}(X)$ and consider the object

$$\mathrm{Sym}(\mathcal{F}) \in \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)),$$

where, as always, the monoidal structure on $\mathrm{IndCoh}(X)$ is given by the !-tensor product. See Chapter 6, Sect. 4.2 for the notation Sym.

Consider the corresponding object

$$\mathrm{Vect}_X(\mathcal{F}) \coloneqq \mathrm{Spec}^{\mathrm{inf}}(\mathrm{Sym}(\mathcal{F})) \in \mathrm{Ptd}(\mathrm{FormMod}_{/X}).$$

1.4.2. Recall the notation Distr^+ introduced in Sect. 1.1.4. We claim:

PROPOSITION 1.4.3. *For* $Z \in \mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{nil\text{-}isom\ to\ } X}$ *the natural map*

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}(Z, \mathrm{Vect}_X(\mathcal{F})) \to \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathrm{Distr}^+(Z), \mathcal{F}),$$

given by the projection $\mathrm{Sym}(\mathcal{F}) \to \mathcal{F}$, *is an isomorphism.*

PROOF. First, we note that the presheaf on $\mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{nil\text{-}isom\ to\ } X}$, given by

$$Z \mapsto \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathrm{Distr}^+(Z), \mathcal{F}),$$

gives rise to an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$ for the same reason as $\mathrm{Spec}^{\mathrm{inf}}$ does. Denote this object by $\mathrm{Vect}'_X(\mathcal{F})$.

Hence, in order to prove that the map in question is an isomorphism, by Chapter 1, Proposition 8.3.2, its suffices to show that the map

(1.4) $$T(\mathrm{Vect}_X(\mathcal{F})/X)|_X \to T(\mathrm{Vect}'_X(\mathcal{F})/X)|_X.$$

is an isomorphism.

The commutative diagram (1.2) implies that $T(\mathrm{Vect}'_X(\mathcal{F})/X)|_X$ identifies with $\mathcal{F}$.

By Lemma 1.3.5,

$$T(\mathrm{Vect}_X(\mathcal{F})/X)|_X \simeq \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Sym}(\mathcal{F})),$$

and the map (1.4) identifies with the canonical map

$$\mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Sym}(\mathcal{F})) \to \mathcal{F}.$$

Now, the latter map is an isomorphism by Chapter 6, Corollary 4.2.5.

$\square$

Note that in the process of proof we have also shown:

COROLLARY 1.4.4. *For* $\mathcal{F} \in \mathrm{IndCoh}(X)$, *there exists a canonical isomorphism*

$$T(\mathrm{Vect}_X(\mathcal{F})/X)|_X \simeq \mathcal{F}.$$

REMARK 1.4.5. The proof of Proposition 1.4.3 used the somewhat non-trivial isomorphism of Chapter 6, Corollary 4.2.5. However, if instead of the functor $\mathrm{Spec}^{\mathrm{inf}}$, one uses the functor $\mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}}$ (see Remark 1.3.7), then the statement that

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}\left(Z, \mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}}(\mathbf{cofree}^{\mathrm{ind\text{-}nilp}}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{F}))\right) \to \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathrm{Distr}^+(Z), \mathcal{F})$$

is an isomorphism, would be tautological. Note that

$$\mathrm{Sym}(\mathcal{F}) \simeq \mathbf{res}^{\star \to *} \circ \mathbf{cofree}^{\mathrm{ind\text{-}nilp}}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{F}) =: \mathbf{cofree}^{\mathrm{fake}}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{F}).$$

Thus, we can interpret the assertion of Proposition 1.4.3 as saying that the natural map

$$\mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}}(\mathbf{cofree}^{\mathrm{ind\text{-}nilp}}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{F})) \to \mathrm{Spec}^{\mathrm{inf}}(\mathrm{Sym}(\mathcal{F})) = \mathrm{Vect}_X(\mathcal{F})$$

is an isomorphism.

Note that the latter is a particular case of the isomorphism of functors of Remark 1.3.7.

1.4.6.   We now claim:

PROPOSITION 1.4.7. *The co-unit of the adjunction*

(1.5) $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Vect}_X(\mathcal{F})) \to \mathrm{Sym}(\mathcal{F})$

is an isomorphism.

The rest of this subsection is devoted to the proof of the proposition.

1.4.8. *Step 1.* Suppose for a moment that $\mathcal{F}$ is such that $\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}) \in \mathrm{Coh}(X)^{<0}$. In this case, by Proposition 1.4.3,

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}(Z, \mathrm{Vect}_X(\mathcal{F})) \simeq \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathrm{Distr}^+(Z), \mathcal{F}) \simeq$$
$$\simeq \mathrm{Maps}_{\mathrm{QCoh}(X)}\left(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}), \mathrm{Fib}(\pi_*(\mathcal{O}_Z) \to \mathcal{O}_X)\right) \simeq$$
$$\simeq \mathrm{Maps}_{/X}\left(Z, \mathrm{Spec}_X\left(\mathbf{free}_{\mathrm{Com}}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}))\right)\right)$$

(where $\mathbf{free}_{\mathrm{Com}}$ is taken in the symmetric monoidal category $\mathrm{QCoh}(X)$), so $\mathrm{Vect}_X(\mathcal{F})$ is a scheme isomorphic to $\mathrm{Spec}_X\left(\mathbf{free}_{\mathrm{Com}}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}))\right)$, and the assertion is manifest.

1.4.9. *Step 2.* Now, we claim that both sides in (1.5), viewed as functors

$$\mathrm{IndCoh}(X) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)),$$

commute with filtered colimits in $\mathcal{F}$.

The commutation is obvious for the functor $\mathcal{F} \mapsto \mathrm{Sym}(\mathcal{F})$.

Since the functor $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$ is a left adjoint, it suffices to show that the functor

$$\mathcal{F} \mapsto \mathrm{Spec}^{\mathrm{inf}}(\mathrm{Sym}(\mathcal{F}))$$

commutes with filtered colimits.

By the construction of the functor $\mathrm{Spec}^{\mathrm{inf}}$, it suffices to show that the functor

$$\mathcal{F} \mapsto \mathrm{Spec}^{\mathrm{inf}}(\mathrm{Sym}(\mathcal{F}))_{\mathrm{nil\text{-}isom}}:$$
$$\mathrm{IndCoh}(X) \to \mathrm{Funct}\left(\left(\mathrm{Ptd}((^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ to\ }X})\right)^{\mathrm{op}}, \mathrm{Spc}\right)$$

commutes with filtered colimits.

By Proposition 1.4.3, it suffices to show that for $Z \in \mathrm{Ptd}((^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\mathrm{nil\text{-}isom\ to\ }X})$, the functor

$$\mathcal{F} \mapsto \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathrm{Distr}^+(Z), \mathcal{F})$$

commutes with filtered colimits. The latter follows from the fact that $\mathrm{Distr}^+(Z) \in \mathrm{Coh}(X) = \mathrm{IndCoh}(X)^c$.

1.4.10. *Step 3.* According to Step 2, we can assume that $\mathcal{F} \in \mathrm{Coh}(Z)$. Combining with Step 1, it remains to show that if the assertion of the proposition holds for $\mathcal{F}[-1]$, then it also holds for $\mathcal{F}$.

The description of $\mathrm{Vect}_X(\mathcal{F})$, given by Proposition 1.4.3 implies that there is a canonical isomorphism

$$\mathrm{Vect}_X(\mathcal{F}[-1]) \simeq \Omega_X(\mathrm{Vect}_X(\mathcal{F})) \in \mathrm{Grp}(\mathrm{FormMod}_{/X}),$$

and hence

$$B_X(\mathrm{Vect}_X(\mathcal{F}[-1])) \simeq \mathrm{Vect}_X(\mathcal{F}).$$

Note also that we have a canonical isomorphism in $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$:

$$\mathrm{Bar}(\mathrm{Sym}(\mathcal{F}[-1])) \simeq \mathrm{Sym}(\mathcal{F}),$$

where we regard $\mathrm{Sym}(\mathcal{F}[-1])$ as an object of

$$\mathrm{CocomBialg}(\mathrm{IndCoh}(X)) \simeq \mathrm{Assoc}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)))$$

via the structure on $\mathcal{F}[-1]$ of a group-object in $\mathrm{IndCoh}(X)$.

The following diagram commutes by adjunction

$$
\begin{array}{ccc}
\mathrm{Bar} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Vect}_X(\mathcal{F}[-1])) & \xrightarrow[\sim]{\text{Lemma 1.2.6}} & \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} \circ B_X(\mathrm{Vect}_X(\mathcal{F}[-1])) \\
\downarrow & & \downarrow \sim \\
\mathrm{Bar}(\mathrm{Sym}(\mathcal{F}[-1])) & & \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Vect}_X(\mathcal{F})) \\
\sim \downarrow & & \downarrow \\
\mathrm{Sym}(\mathcal{F}) & \xrightarrow{\mathrm{id}} & \mathrm{Sym}(\mathcal{F}).
\end{array}
$$

By assumption, the upper left vertical arrow in this diagram is an isomorphism. Hence, so is the lower right vertical arrow.

$\square$

2. Inf-affineness

In this section we study the notion of inf-affineness, which is a counterpart of the usual notion of affineness in algebraic geometry.

The naive expectation would be that an inf-affine formal moduli problem over X is one of the form $\mathrm{Spec}^{\mathrm{inf}}$ of a co-commutative co-algebra in $\mathrm{IndCoh}(X)$. However, this does not quite work as the analogy with the usual notion of affineness is not perfect: it is not true that the functor $\mathrm{Spec}^{\mathrm{inf}}$ identifies the category $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$ with that of inf-affine objects in $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$.

2.1. The notion of inf-affineness. In algebraic geometry a prestack $\mathcal{Y}$ is an affine scheme if and only if $\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is connective and for any $S \in \mathrm{Sch}^{\mathrm{aff}}$, the map

$$\mathrm{Maps}_{\mathrm{Sch}^{\mathrm{aff}}}(S, \mathcal{Y}) \to \mathrm{Maps}_{\mathrm{ComAlg}(\mathrm{Vect})}(\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), \Gamma(S, \mathcal{O}_S))$$

is an isomorphism.

In formal geometry we give a similar definition.

2.1.1.   Let as before $X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

DEFINITION 2.1.2.   *An object* $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$ *is* inf-affine, *if the functor* $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$ *induces an isomorphism*

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}(Z, \mathcal{Y}) \to$$
$$\to \mathrm{Maps}_{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(Z), \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})),$$

where $Z \in \mathrm{Ptd}(({}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{nil\text{-}isom\ to}\ X})$.

2.1.3.   Here are some basic facts related to this notion:

PROPOSITION 2.1.4.   *Any object*

$$Y \in \mathrm{Ptd}((\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{\mathrm{nil\text{-}isom\ to}\ X}) \subset \mathrm{Ptd}(\mathrm{FormMod}_{/X})$$

is inf-affine.

PROOF.   By definition, we need to show that for

$$Y_1, Y_2 \in \mathrm{Ptd}((\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{\mathrm{nil\text{-}isom\ to}\ X}),$$

with Y_1 eventually coconnective, the groupoid $\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}(Y_1, Y_2)$ maps isomorphically to

$$\mathrm{Maps}_{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))}((\pi_1)^{\mathrm{IndCoh}}_{*}(\omega_{Y_1}), (\pi_2)^{\mathrm{IndCoh}}_{*}(\omega_{Y_2})).$$

The assertion easily reduces to the case when Y_2 is eventually coconnective. In the latter case, Serre duality identifies the above groupoid with

$$\mathrm{Maps}_{\mathrm{ComAlg}^{\mathrm{aug}}(\mathrm{QCoh}(X))}((\pi_2)_{*}(\mathcal{O}_{Y_2}), (\pi_1)_{*}(\mathcal{O}_{Y_1})),$$

and the desired isomorphism is manifest.

$\square$

2.1.5.   We claim:

LEMMA 2.1.6.   *Let* $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$ *be inf-affine. Then for any* $\mathcal{Z} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$, *the map*

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}(\mathcal{Z}, \mathcal{Y}) \to$$
$$\to \mathrm{Maps}_{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Z}), \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}))$$

is an isomorphism.

PROOF.   Follows from Chapter 5, Corollary 1.5.2(a) and Lemma 1.1.7(b). $\square$

REMARK 2.1.7.   It follows from Proposition 2.3.3 below, combined with Chapter 6, Corollary 2.10.5(b) for the co-operad Cocom that if instead of $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$ one uses $\mathrm{Cocom}^{\mathrm{aug,ind\text{-}nilp}}(\mathrm{IndCoh}(X))$, one obtains the *same* notion of inf-affineness.

2.2. Inf-affineness and inf-spectrum. As was mentioned already, it is not true that the functor $\mathrm{Spec}^{\mathrm{inf}}$ identifies the category $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$ with that of inf-affine objects in $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$. The problem is that the analog of Serre's theorem fails: for a connective commutative DG algebra A, the map

$$A \to \Gamma(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$$

is an isomorphism, whereas for $\mathcal{A} \in \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$, the map

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})) \to \mathcal{A}$$

does not have to be such.

In this subsection we establish several positive facts that can be said in this direction. A more complete picture is presented in Sect. 3.3.

2.2.1.   We note:

LEMMA 2.2.2. *Let $\mathcal{A} \in \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$ be such that the co-unit of the adjunction*

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})) \to \mathcal{A}$$

is an isomorphism. Then the object $\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})$ is inf-affine.

In particular, combining with Proposition 1.4.7, we obtain:

COROLLARY 2.2.3. *For $\mathcal{F} \in \mathrm{IndCoh}(X)$, the object $\mathrm{Vect}_X(\mathcal{F}) \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$ is inf-affine.*

REMARK 2.2.4. As we shall see in Sect. 3.3.10, it is *not* true that the functor $\mathrm{Spec}^{\mathrm{inf}}$ is fully faithful. I.e., the co-unit of the adjunction

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})) \to \mathcal{A}$$

is *not* an isomorphism for all $\mathcal{A} \in \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$.

However, we will see that Chapter 6, Conjecture 2.8.9(b) for the symmetric monoidal DG category $\mathrm{IndCoh}(X)$ implies that the above map is an isomorphism for $\mathcal{A}$ lying in the essential image of the functor $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$.

We will also see that Chapter 6, Conjecture 2.8.9(a) for $\mathrm{IndCoh}(X)$ implies that the essential image of the functor $\mathrm{Spec}^{\mathrm{inf}}$ lands in the subcategory of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$ spanned by inf-affine objects.

REMARK 2.2.5. The same logic shows that Chapter 6, Conjecture 2.6.6 for the symmetric monoidal DG category $\mathrm{IndCoh}(X)$ and the Lie operad, implies that the functor $\mathrm{Spec}^{\mathrm{inf,ind\text{-}nilp}}$ is an equivalence onto the full subcategory of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$ spanned by objects that are inf-affine.

2.3. A criterion for being inf-affine. A prestack $\mathcal{Y}$ is an affine scheme if and only if $\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is connective and the canonical map

$$\mathcal{Y} \to \mathrm{Spec}(\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}))$$

is an isomorphism.

The corresponding assertion is true (but not completely tautological) also in formal geometry: an object $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$ is inf-affine if and only if the unit of the adjunction

$$\mathcal{Y} \to \mathrm{Spec}^{\mathrm{inf}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})$$

is an isomorphism.

In addition, in this subsection we will give a crucial criterion for inf-affineness in terms of the tangent space of $\mathcal{Y}$ at its distinguished point, which does not have a counterpart in usual algebraic geometry.

2.3.1.   Recall the commutative diagram (1.2).

We obtain that for $\mathcal{F} \in \mathrm{Coh}(X)^{\leq 0}$ and $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$, the functor $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$ gives rise to a canonically defined map

$$(2.1)\quad \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}), T(\mathcal{Y}/X)|_X) \to$$
$$\to \mathrm{Maps}_{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))}\Big(\mathbf{triv}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F})), \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})\Big).$$

Consider the functor

$$\mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{IndCoh}(X).$$

We can rewrite (2.1) as a map

$$(2.2)\quad \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}), T(\mathcal{Y}/X)|_X) \to$$
$$\to \mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}), \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})).$$

The map (2.2) gives rise to a well-defined map in $\mathrm{IndCoh}(X)$:

$$(2.3)\qquad\qquad T(\mathcal{Y}/X)|_X \to \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}).$$

2.3.2.   We claim:

PROPOSITION 2.3.3. *For an object $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$ the following conditions are equivalent:*
(i) *$\mathcal{Y}$ is inf-affine;*
(ii) *The unit of the adjunction $\mathcal{Y} \to \mathrm{Spec}^{\mathrm{inf}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})$ is an isomorphism;*
(iii) *The map (2.3) is an isomorphism.*

PROOF. The implication (i) $\Rightarrow$ (iii) is tautological from the definition of inf-affineness.

Suppose that $\mathcal{Y}$ satisfies (ii). Then for $Z \in \mathrm{Ptd}((^{<\infty}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{\text{nil-isom to } X})$ the map

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}(Z, \mathcal{Y}) \to \mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}\Big(Z, \mathrm{Spec}^{\mathrm{inf}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})\Big)$$

is an isomorphism, while its composition with the adjunction isomorphism

$$\mathrm{Maps}_{\mathrm{Ptd}(\mathrm{FormMod}_{/X})}\Big(Z, \mathrm{Spec}^{\mathrm{inf}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})\Big) \simeq$$
$$\simeq \mathrm{Maps}_{\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))}\Big(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(Z), \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})\Big)$$

equals the map induced by the functor $\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$. Hence, $\mathcal{Y}$ is inf-affine.

Finally, assume that $\mathcal{Y}$ satisfies (iii), and let us deduce (ii). By Chapter 1, Proposition 8.3.2, in order to show that $\mathcal{Y} \to \mathrm{Spec}^{\mathrm{inf}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})$ is an isomorphism, it suffices to show that the map

$$T(\mathcal{Y}/X)|_X \to T(\mathrm{Spec}^{\mathrm{inf}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})/X)|_X$$

is an isomorphism in $\mathrm{IndCoh}(X)$.

Recall the isomorphism $T(\mathrm{Spec}^{\mathrm{inf}}(\mathcal{A})/X)|_X \simeq \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{A})$ of Lemma 1.3.5.

Now, it is easy to see that the composed map

$$T(\mathcal{Y}/X)|_X \to T(\mathrm{Spec}^{\mathrm{inf}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})/X)|_X \simeq \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}))$$

equals the map (2.3), implying our assertion.

$\square$

3. From formal groups to Lie algebras

Let G be a Lie group. The tangent space at the identity of G has the structure of a Lie algebra. One way of describing this Lie algebra structure is the following: the Lie algebra of G is given by the space of primitive elements in the co-commutative co-algebra given by the space of distributions on G supported at the identity.

In this section, we implement this idea in the context of derived algebraic geometry and finally spell out the relationship between the categories $\mathrm{Grp}(\mathrm{FormMod}_{/X})$ and $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$, i.e., formal groups and Lie algebras:

To go from an object of $\mathrm{Grp}(\mathrm{FormMod}_{/X})$ to $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$, we first attach to it an object $\mathrm{Grp}(\mathrm{LieAlg}(\mathrm{IndCoh}(X)))$ via the functor

$$\mathrm{coChev}^{\mathrm{enh}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}$$

(i.e., we attach to an object of $\mathrm{Grp}(\mathrm{FormMod}_{/X})$ the corresponding augmented co-commutative co-algebra and use Quillen's functor $\mathrm{coChev}^{\mathrm{enh}}$ that maps $\mathrm{CocomCoalg}^{\mathrm{aug}}$ to LieAlg), and then deloop.

To go from $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$ we use the 'exponential map', incarnated by the functor

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg} \to \mathrm{CocomHopf}$$

(the latter is canonically isomorphic to the more usual construction given by the functor U^{Hopf}, the universal enveloping algebra, viewed as a co-commutatove Hopf algebra), and then apply the functor of inf-spectrum.

3.1. The exponential construction. Let as before $X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. The idea of the exponential construction is the following: for a Lie algebra $\mathfrak{h}$, the corresponding formal group $\exp_X(\mathfrak{h})$ is such that

$$\mathrm{Distr}(\exp_X(\mathfrak{h})) \simeq U(\mathfrak{h}).$$

3.1.1.   We define the functor

$$\exp_X : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{Grp}(\mathrm{FormMod}_{/X})$$

to be

$$\mathrm{Monoid}(\mathrm{Spec}^{\mathrm{inf}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}.$$

For example, for $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$, we have

$$\exp(\mathbf{triv}_{\mathrm{Lie}}(\mathcal{F})) = \mathrm{Monoid}(\mathrm{Spec}^{\mathrm{inf}})(\mathrm{Sym}(\mathcal{F})) = \mathrm{Vect}_{\mathcal{X}}(\mathcal{F}),$$

equipped with its natural group structure.

REMARK 3.1.2. To bring the above construction closer to the classical idea of the exponential map, let us recall that, according to Chapter 6, Theorem 6.1.2, we have a canonical isomorphism in $\mathrm{CocomHopf}(\mathrm{IndCoh}(X))$

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} \simeq U^{\mathrm{Hopf}}.$$

3.1.3.   In the next section we will prove:

THEOREM 3.1.4. *The functor*

$$\exp_X : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{Grp}(\mathrm{FormMod}_{/X})$$

is an equivalence.

3.2. Corollaries of Theorem 3.1.4. In this subsection we will show that the functor $\exp_X$ as defined above, has all the desired properties, i.e., that there are no unpleasant surprises.

3.2.1.   Recall (see Chapter 6, Corollary 1.7.3) that when $\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}(\mathfrak{h})$ is viewed as an object of $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$, i.e., if we forget the algebra structure, it is (canonically) isomophic to $\mathrm{Sym}(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}))$.

Hence, by Corollary 2.2.3, when we view $\exp_X(\mathfrak{h})$ as an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$, it is isomorphic to $\mathrm{Vect}_X(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}))$, and hence is inf-affine.

Therefore, as a consequence of Theorem 3.1.4 (plus Corollary 1.4.4), we obtain:

COROLLARY 3.2.2. *Every object of* $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/X})$, *when viewed by means of the forgetful functor as an object of* $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$, *is inf-affine, and we have:*

$$\mathbf{oblv}_{\mathrm{Grp}}(\mathcal{H}) \simeq \mathrm{Vect}_X(T(\mathbf{oblv}_{\mathrm{Grp}}(\mathcal{H})/X)|_X).$$

From Proposition 1.4.7 and Chapter 6, Proposition 4.3.3, we obtain:

COROLLARY 3.2.3. *The natural transformation*

(3.1) $$\mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}) \circ \exp_X \to \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}$$

is an isomorphism.

Combining the isomorphism (3.1) with the isomorphism

(3.2) $$B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} \simeq \mathrm{Id}$$

of Chapter 6, Theorem 4.4.6, we obtain:

COROLLARY 3.2.4. *There exists a canonical isomorphism of functors*

$$B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}) \circ \exp_X \simeq \mathrm{Id}.$$

3.2.5. Let us denote by

(3.3) $\mathrm{Lie}_X : \mathrm{Grp}(\mathrm{FormMod}_{/X}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X))$

the functor

$$B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}).$$

Hence:

COROLLARY 3.2.6. *The functor*

$$\mathrm{Lie}_X : \mathrm{Grp}(\mathrm{FormMod}_{/X}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X))$$

is the inverse of

$$\exp_X : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{Grp}(\mathrm{FormMod}_{/X}).$$

3.2.7. By combining Corollary 3.2.2, Proposition 2.3.3 and the tautological isomorphism

$$\mathbf{oblv}_{\mathrm{Lie}} \circ B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \simeq \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathbf{oblv}_{\mathrm{Grp}},$$

we obtain:

COROLLARY 3.2.8. *There exists a canonical isomorphism of functors*

$$\mathrm{Grp}(\mathrm{FormMod}_{/X}) \to \mathrm{IndCoh}(X), \quad \mathbf{oblv}_{\mathrm{Lie}} \circ \mathrm{Lie}_X(\mathcal{H}) \simeq T(\mathbf{oblv}_{\mathrm{Grp}}(\mathcal{H})/X)|_X.$$

In other words, this corollary says that the object of IndCoh underlying the Lie algebra corresponding to a formal group indeed identifies with the tangent space at the origin.

3.2.9. The upshot of this subsection is that in derived algebraic geometry the passage from the a formal group to its Lie algebra is given by the functor

$$\mathrm{Lie}_X := B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}).$$

3.3. Lie algebras and formal moduli problems. In this subsection we will assume Theorem 3.1.4 and deduce some further corollaries. In particular, we will show that there is an equivalence between pointed formal moduli problems over a scheme X and Lie algebras in $\mathrm{IndCoh}(X)$.

Furthermore, we will see what the functor of inf-spectrum really does, and what it means to be inf-affine. Namely, we will show that under the equivalence above, the functor $\mathrm{Spec}^{\mathrm{inf}}$ corresponds to the functor $\mathrm{coChev}^{\mathrm{enh}}$.

3.3.1. First, we claim:

COROLLARY 3.3.2. *There is the following commutative diagram of functors*

(3.4)

$$
\begin{array}{ccc}
\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) & \xleftarrow{\ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}\ } & \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \\
{\scriptstyle\mathrm{Chev}^{\mathrm{enh}}}\big\uparrow & & \sim\big\uparrow{\scriptstyle B_X} \\
\mathrm{LieAlg}(\mathrm{IndCoh}(X)) & \xleftarrow[\sim]{\ \mathrm{Lie}_X\ } & \mathrm{Grp}(\mathrm{FormMod}_{/X}).
\end{array}
$$

PROOF. Indeed, by Theorem 3.1.4, it suffices to construct a functorial isomorphism

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} \circ B_X \circ \exp_X \simeq \mathrm{Chev}^{\mathrm{enh}}.$$

However, by Lemma 1.2.6 and the isomorphism (3.1), the left-hand side identifies with

$$\mathrm{Bar} \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}) \circ \exp_X \simeq \mathrm{Bar} \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} \simeq$$

$$\simeq \mathrm{Chev}^{\mathrm{enh}} \circ B_{\mathrm{Lie}} \circ \Omega_{\mathrm{Lie}} \simeq \mathrm{Chev}^{\mathrm{enh}}.$$

$\square$

COROLLARY 3.3.3. *For* $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$ *there is a canonical isomorphism*

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}) \simeq \mathrm{Chev}^{\mathrm{enh}} \circ \mathrm{Lie}_X \circ \Omega_X(\mathcal{Y}).$$

REMARK 3.3.4. The commutative diagram (3.4) implies the following:

The functor

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$$

remembers/loses as much information as does the functor

$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)).$$

However, the functor

$$\mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}) : \mathrm{Grp}(\mathrm{FormMod}_{/X}) \to \mathrm{CocomBialg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$$

is fully faithful, as is the functor

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathrm{IndCoh}(X)) \to \mathrm{CocomBialg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)).$$

3.3.5. By passing to right adjoints in diagram (3.4) we obtain:

COROLLARY 3.3.6. *There is the following commutative diagram of functors*

$$(3.5) \qquad \begin{array}{ccc} \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) & \xrightarrow{\mathrm{Spec}^{\mathrm{inf}}} & \mathrm{Ptd}(\mathrm{FormMod}_{/X}) \\ {\scriptstyle \mathrm{coChev}^{\mathrm{enh}}} \downarrow & & \sim \downarrow {\scriptstyle \Omega_X} \\ \mathrm{LieAlg}(\mathrm{IndCoh}(X)) & \xrightarrow[\sim]{\exp_X} & \mathrm{Grp}(\mathrm{FormMod}_{/X}). \end{array}$$

REMARK 3.3.7. The commutative diagram (3.5) implies:

The functor

$$\mathrm{Spec}^{\mathrm{inf}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{Ptd}(\mathrm{FormMod}_{/X})$$

remembers/loses as much information as does the functor

$$\mathrm{coChev}^{\mathrm{enh}} : \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) \to \mathrm{LieAlg}(\mathrm{IndCoh}(X)).$$

3.3.8.   Let
$$B_X^{\text{Lie}} : \text{LieAlg}(\text{IndCoh}(X)) \to \text{Ptd}(\text{FormMod}_{/X})$$
denote the functor $B_X \circ \exp_X$.

This is the functor that associates to a Lie algebra in $\text{IndCoh}(X)$ the corresponding moduli problem. By Theorem 3.1.4, this functor is an equivalence, with the inverse being
$$\mathcal{Y} \mapsto \text{Lie}_X \circ \Omega_X(\mathcal{Y}).$$

From Proposition 2.3.3 we obtain:

COROLLARY 3.3.9. *Let $\mathcal{Y}$ be an object of* $\text{Ptd}(\text{FormMod}_{/X})$, *and let $\mathfrak{h}$ be the corresponding object of* $\text{LieAlg}(\text{IndCoh}(X))$, *i.e.,*
$$\mathfrak{h} = \text{Lie}_X \circ \Omega_X(\mathcal{Y}) \ \text{and/or} \ \mathcal{Y} := B_X^{\text{Lie}}(\mathfrak{h}).$$
Then $\mathcal{Y}$ is inf-affine if and only if unit of the adjunction
$$\mathfrak{h} \to \text{coChev}^{\text{enh}} \circ \text{Chev}^{\text{enh}}(\mathfrak{h})$$
is an isomorphism.

3.3.10.   Let $\mathcal{A}$ be an object of $\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X))$. From the diagrams (3.4) and (3.5) we obtain that the co-unit of the adjunction
$$(3.6) \qquad\qquad \text{Distr}^{\text{Cocom}^{\text{aug}}}(\text{Spec}^{\text{inf}}(\mathcal{A})) \to \mathcal{A}$$
identifies with the map
$$(3.7) \qquad\qquad \text{Chev}^{\text{enh}} \circ \text{coChev}^{\text{enh}}(\mathcal{A}) \to \mathcal{A}.$$

In partcular, we obtain that if Chapter 6, Conjecture 2.8.9(b) holds for the symmetric monoidal DG category $\text{IndCoh}(X)$ and the co-operad $\text{Cocom}^{\text{aug}}$, i.e., if the map (3.7) is an isomorphism for $\mathcal{A}$ lying in the essential image of the functor
$$\text{Chev}^{\text{enh}} : \text{LieAlg}(\text{IndCoh}(X)) \to \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)),$$
then the map (3.6) is an isomorphism for $\mathcal{A}$ lying in the essential image of the functor
$$\text{Distr}^{\text{Cocom}^{\text{aug}}} : \text{Ptd}(\text{FormMod}_{/X}) \to \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)).$$

Similarly, suppose that Chapter 6, Conjecture 2.8.9(a) holds for the symmetric monoidal DG category $\text{IndCoh}(X)$ and the co-operad $\text{Cocom}^{\text{aug}}$, i.e., if the map
$$\mathfrak{h} \to \text{coChev}^{\text{enh}} \circ \text{Chev}^{\text{enh}}(\mathfrak{h})$$
is an isomorphism for $\mathfrak{h}$ lying in the essential image of the functor
$$\text{coChev}^{\text{enh}} : \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) \to \text{LieAlg}(\text{IndCoh}(X)).$$
Then, by Corollary 3.3.9, any $\mathcal{Y} \in \text{Ptd}(\text{FormMod}_{/X})$ lying in the essential image of the functor
$$\text{Spec}^{\text{inf}} : \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) \to \text{Ptd}(\text{FormMod}_{/X}),$$
is inf-affine.

3.4. The ind-nilpotent version. For completeness, let us explain what happens
to the picture in Sect. 3.3 if we consider instead the adjoint functors

$$\text{Distr}^{\text{Cocom}^{\text{aug,ind-nil}}} : \text{Ptd}(\text{FormMod}_{/X}) \rightleftarrows \text{CocomCoalg}^{\text{aug,ind-nilp}}(\text{IndCoh}(X)) : \text{Spec}^{\text{inf,ind-nilp}}.$$

3.4.1.   First, we have the commutative diagrams

(3.8)

$$\begin{array}{ccc}
\text{CocomCoalg}^{\text{aug,ind-nilp}}(\text{IndCoh}(X)) & \xleftarrow{\text{Distr}^{\text{Cocom}^{\text{aug,ind-nilp}}}} & \text{Ptd}(\text{FormMod}_{/X}) \\
\text{Chev}^{\text{enh,ind-nilp}} \big\uparrow & & \sim \big\uparrow B_X \\
\text{LieAlg}(\text{IndCoh}(X)) & \xleftarrow[\sim]{\text{Lie}_X} & \text{Grp}(\text{FormMod}_{/X}).
\end{array}$$

and

(3.9)
$$\begin{array}{ccc}
\text{CocomCoalg}^{\text{aug,ind-nilp}}(\text{IndCoh}(X)) & \xrightarrow{\text{Spec}^{\text{inf,ind-nilp}}} & \text{Ptd}(\text{FormMod}_{/X}) \\
\text{coChev}^{\text{enh,ind-nilp}} \big\downarrow & & \sim \big\downarrow \Omega_X \\
\text{LieAlg}(\text{IndCoh}(X)) & \xrightarrow[\sim]{\exp_X} & \text{Grp}(\text{FormMod}_{/X}).
\end{array}$$

3.4.2.   Let us now assume the validity of Chapter 6, Conjecture 2.6.6 for the sym-
metric monoidal DG category $\text{IndCoh}(X)$ and the co-operad $\text{Cocom}^{\text{aug}}$.

From it we obtain:

CONJECTURE 3.4.3. *The functor*

$$\text{Spec}^{\text{inf,ind-nilp}} : \text{CocomCoalg}^{\text{aug,ind-nilp}}(\text{IndCoh}(X)) \to \text{Ptd}(\text{FormMod}_{/X})$$

is fully faithful.

3.5. Base change. As we saw in Proposition 2.3.3, the criterion of inf-affineness
involves the operation of taking primitives in an augmented co-commutative co-
algebra in $\text{IndCoh}(X)$. This operation is not guaranteed to behave well with respect
to the operation of pullback. The functor of inf-spectrum has a similar drawback,
for the same reason.

In this subsection we will establish several positive results in this direction.

3.5.1.   Let $f : X' \to X$ be a map in $^{<\infty}\text{Sch}^{\text{aff}}_{\text{ft}}$, and consider the corresponding
functor

$$f^! : \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) \to \text{Cocom}^{\text{aug}}(\text{IndCoh}(X')).$$

The following diagram commutes by construction

$$\begin{array}{ccc}
\text{Ptd}(\text{FormMod}_{/X}) & \xrightarrow{X' \underset{X}{\times} -} & \text{Ptd}(\text{FormMod}_{/X'}) \\
\text{Distr}^{\text{Cocom}^{\text{aug}}} \big\downarrow & & \big\downarrow \text{Distr}^{\text{Cocom}^{\text{aug}}} \\
\text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(X)) & \xrightarrow{f^!} & \text{Cocom}^{\text{aug}}(\text{IndCoh}(X')).
\end{array}$$

Hence, by adjunction, for $\mathcal{A} \in \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X))$ we have a canonically defined map

$$(3.10) \qquad X' \underset{X}{\times} \mathrm{Spec}^{\mathrm{inf}}(\mathcal{A}) \to \mathrm{Spec}^{\mathrm{inf}}(f^!(\mathcal{A})).$$

REMARK 3.5.2. It follows from Lemma 4.2.3 below and diagram (3.5), that the natural transformation (3.10) is an isomorphism if f is proper.

3.5.3. The following is immediate from Lemma 2.2.2 and Proposition 2.3.3:

LEMMA 3.5.4. *Assume that $\mathcal{A}$ is such that both maps*

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Spec}^{\mathrm{inf}}(\mathcal{A}) \to \mathcal{A} \ \ and \ \ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Spec}^{\mathrm{inf}}(f^!(\mathcal{A})) \to f^!(\mathcal{A})$$

are isomorphisms. Then the map (3.10) is an isomorphism for $\mathcal{A}$.

COROLLARY 3.5.5. *For $\mathcal{F} \in \mathrm{IndCoh}(X)$, the canonical map*

$$X' \underset{X}{\times} \mathrm{Vect}_X(\mathcal{F}) \to \mathrm{Vect}_{X'}(f^!(\mathcal{F}))$$

is an isomorphism.

3.5.6. By combining Corollary 3.5.5 and Chapter 6, Proposition 1.7.2, we obtain:

COROLLARY 3.5.7. *For $\mathfrak{h} \in \mathrm{LieAlg}(\mathrm{IndCoh}(X))$, the canonical map*

$$X' \underset{X}{\times} \exp_X(\mathfrak{h}) \to \exp_{X'}(f^!(\mathfrak{h}))$$

is an isomorphism.

COROLLARY 3.5.8. *For $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/X})$, the canonical map*

$$f^!(\mathrm{Lie}_X(\mathcal{H})) \to \mathrm{Lie}_{X'}(X' \underset{X}{\times} \mathcal{H})$$

is an isomorphism.

3.6. Extension to prestacks. We will now extend the equivalence $\exp_X$ to the case when the base $X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ is replaced by an arbitrary $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$.

3.6.1. Note that the discussion in Sect. 1.1 applies verbatim to the present situation (i.e., the base being an object of $\mathrm{PreStk}_{\mathrm{laft}}$). In particular, we obtain the functors

$$\mathrm{Distr} : \mathrm{FormMod}_{/\mathcal{X}} \to \mathrm{IndCoh}(\mathcal{X}),$$

$$\mathrm{Distr}^{\mathrm{Cocom}} : \mathrm{FormMod}_{/\mathcal{X}} \to \mathrm{CocomCoalg}(\mathrm{IndCoh}(\mathcal{X})),$$

$$\mathrm{Distr}^+ : \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{IndCoh}(\mathcal{X}),$$

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} : \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(\mathcal{X})),$$

and

$$\mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}) : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{CocomBial}^{\mathrm{aug}}(\mathrm{IndCoh}(\mathcal{X})).$$

We have:

THEOREM 3.6.2. *The functor*

$$\mathrm{Lie}_{\mathcal{X}} \coloneqq B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}})$$

defines an equivalence $\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$. *Furthermore, we have:*

(a) *The co-unit of the adjunction*

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} \circ \mathrm{Lie}_{\mathcal{X}} \to \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}})$$

is an isomorphism.

(b) *The composition*

$$\mathbf{oblv}_{\mathrm{Lie}} \circ \mathrm{Lie}_{\mathcal{X}} : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{IndCoh}(\mathcal{X})$$

identifies canonically with the functor

$$\mathcal{H} \mapsto T(\mathbf{oblv}_{\mathrm{Grp}}(\mathcal{H})/\mathcal{X})|_{\mathcal{X}}.$$

PROOF. Observe that for $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}}$ the functors

$$\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \to \underset{X \in ((<\infty \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}})^{\mathrm{op}}}{lim} \mathrm{LieAlg}(\mathrm{IndCoh}(X))$$

and

$$\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \underset{X \in ((<\infty \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/\mathcal{X}})^{\mathrm{op}}}{lim} \mathrm{Grp}(\mathrm{FormMod}_{/X})$$

are both equivalences, see Chapter 5, Lemma 1.1.5 for the latter statement.

Using Theorem 3.1.4, to show that the functor $\mathrm{Lie}_{\mathcal{X}}$ is an equivalence, it remains to check that the functors Lie_X, where X is a scheme, are compatible with base change. But this follows from Corollary 3.5.8.

The isomorphisms stated in (a) and (b) follow from the case of schemes.

□

As a formal consequence we obtain:

COROLLARY 3.6.3. *The category* $\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$ *contains sifted colimits, and the functor*

$$\mathcal{H} \mapsto T(\mathbf{oblv}_{\mathrm{Grp}}(\mathcal{H})/\mathcal{X})|_{\mathcal{X}} : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{IndCoh}(\mathcal{X})$$

commutes with sifted colimits.

3.6.4. Let

$$\exp_{\mathcal{X}} : \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \to \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$$

denote the equivalence, inverse to $\mathrm{Lie}_{\mathcal{X}}$.

Let

$$B^{\mathrm{Lie}}_{\mathcal{X}} : \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \to \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$$

denote the resulting equivalence

$$B_{\mathcal{X}} \circ \exp_{\mathcal{X}}, \quad \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \to \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}).$$

Note that from Corollary 3.3.2 we obtain:

COROLLARY 3.6.5. *There is a canonical isomorphism of functors*

$$\mathrm{Chev}^{\mathrm{enh}} \simeq \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}} \circ B^{\mathrm{Lie}}_{\mathcal{X}}.$$

3.6.6.   In what follows we will denote by

$$(3.11) \qquad\qquad \mathcal{F} \mapsto \mathrm{Vect}_{\mathcal{X}}(\mathcal{F})$$

the functor $\mathrm{IndCoh}(\mathcal{X}) \to \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$, given by

$$\mathcal{F} \mapsto \mathrm{Vect}_{\mathcal{X}}(\mathcal{F}) \coloneqq \mathbf{oblv}_{\mathrm{Grp}} \circ \exp_{\mathcal{X}} \circ \mathbf{triv}_{\mathrm{Lie}}(\mathcal{F}) \simeq B^{\mathrm{Lie}}_{\mathcal{X}} \circ \exp_{\mathcal{X}} \circ \mathbf{triv}_{\mathrm{Lie}}(\mathcal{F}[-1]).$$

Note that by Corollary 3.6.5 we obtain

$$\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})) \simeq \mathrm{Sym}(\mathcal{F}).$$

From Proposition 1.4.3, we obtain:

COROLLARY 3.6.7. *The functor* $\mathrm{Vect}_{\mathcal{X}}(-) : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ *is the right adjoint to the functor* Distr^+.

Note also:

COROLLARY 3.6.8. *For* $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$, *we have a canonical isomorphism*

$$\mathbf{oblv}_{\mathrm{Grp}}(\mathcal{H}) \simeq \mathrm{Vect}_{\mathcal{X}}(T(\mathbf{oblv}_{\mathrm{Grp}}(\mathcal{H})/\mathcal{X})|_{\mathcal{X}}).$$

3.6.9.   The functor (3.11) is easily seen to commute with products. Hence, it induces a functor

$$(3.12) \qquad\qquad \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{ComMonoid}(\mathrm{FormMod}_{/\mathcal{X}}),$$

see Chapter 6, Sect. 1.8 for the notation.

We claim:

COROLLARY 3.6.10. *The functor* (3.12) *is an equivalence.*

PROOF. Follows from Chapter 6, Proposition 1.8.3. □

3.7. An example: split square-zero extensions. In Chapter 1, Sect. 2.1 we discussed the functor of *split square-zero extension*

$$\mathrm{RealSplitSqZ} : (\mathrm{Coh}(X)^{\leq 0})^{\mathrm{op}} \to \mathrm{Ptd}((\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})_{/X}), \quad X \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}.$$

In this subsection we will extend this construction to the case of arbitrary objects $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$, where instead of $(\mathrm{Coh}(-)^{\leq 0})^{\mathrm{op}}$ we use all of $\mathrm{IndCoh}(\mathcal{X})$. Here for $\mathcal{X} = X \in \mathrm{Sch}_{\mathrm{aft}}$, we view $(\mathrm{Coh}(X)^{\leq 0})^{\mathrm{op}}$ as a full subcategory of $\mathrm{IndCoh}(X)$ via

$$(\mathrm{Coh}(X)^{\leq 0})^{\mathrm{op}} \hookrightarrow \mathrm{Coh}(X)^{\mathrm{op}} \overset{\mathbb{D}^{\mathrm{Serre}}_X}{\longrightarrow} \mathrm{Coh}(X) \hookrightarrow \mathrm{IndCoh}(X).$$

3.7.1. For $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$ consider the functor

$$\mathrm{RealSplitSqZ} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{Ptd}((\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{/\mathcal{X}}),$$

defined as follows:

We send $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ to

$$B_{\mathcal{X}} \circ \exp_{\mathcal{X}} \circ \mathbf{free}_{\mathrm{Lie}}(\mathcal{F}[-1]) \in \mathrm{Ptd}((\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{X}}) \in \mathrm{Ptd}((\mathrm{FormMod})_{/\mathcal{X}}) \subset \mathrm{Ptd}((\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{/\mathcal{X}}).$$

We can phrase the above construction as follows: we create the free Lie algebra on $\mathcal{F}[-1]$, then we consider the corresponding object of $\mathrm{Grp}((\mathrm{Sch}^{\mathrm{aff}})_{/X})$, and then take the formal classifying space of the latter.

By construction, we have a commutative diagram:

$$(3.13) \qquad \begin{array}{ccc} \mathrm{IndCoh}(\mathcal{X}) & \xrightarrow{\mathrm{RealSplitSqZ}} & \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \\ {\scriptstyle \mathbf{free}_{\mathrm{Lie}} \circ [-1]} \downarrow & & \downarrow {\scriptstyle \Omega_{\mathcal{X}} \, \sim} \\ \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) & \xrightarrow{\exp_{\mathcal{X}}} & \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}). \end{array}$$

3.7.2. We claim that the functor $\mathrm{RealSplitSqZ}$ can also be described as a left adjoint:

PROPOSITION 3.7.3. *The functor* $\mathrm{RealSplitSqZ}$ *is the left adjoint of the functor*

$$\mathrm{Ptd}((\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{/\mathcal{X}}) \to \mathrm{IndCoh}(\mathcal{X}), \quad \mathcal{Y} \mapsto T(\mathcal{Y}/\mathcal{X})|_{\mathcal{X}}.$$

PROOF. Given $\mathcal{Y} \in \mathrm{Ptd}((\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{/\mathcal{X}})$ and $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ we need to establish a canonical isomorphism
$$(3.14)$$
$$\mathrm{Maps}_{\mathrm{Ptd}((\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{/\mathcal{X}})}(\mathrm{RealSplitSqZ}(\mathcal{F}), \mathcal{Y}) \simeq \mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\mathcal{F}, T(\mathcal{Y}/\mathcal{X})|_{\mathcal{X}}).$$

Note that the left-hand side receives an isomorphism from $\mathrm{Maps}(\mathrm{RealSplitSqZ}(\mathcal{F}), \mathcal{Y}_{\mathcal{X}}^{\wedge})$, where $\mathcal{Y}_{\mathcal{X}}^{\wedge}$ is the formal completion of $\mathcal{Y}$ along the map $\mathcal{X} \to \mathcal{Y}$. So, with no restriction of generailty, we can assume that $\mathcal{Y} \in \mathrm{Ptd}((\mathrm{FormMod})_{/\mathcal{X}})$.

In this case, by Chapter 5, Theorem 1.6.4, we can further rewrite the left-hand side in (3.14) as

$$\mathrm{Maps}_{\mathrm{Grp}((\mathrm{FormMod})_{/\mathcal{X}})}(\exp_{\mathcal{X}} \circ \mathbf{free}_{\mathrm{Lie}}(\mathcal{F}[-1]), \Omega_{\mathcal{X}}(\mathcal{Y})),$$

and then as

$$\mathrm{Maps}_{\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))}(\mathbf{free}_{\mathrm{Lie}}(\mathcal{F}[-1]), \mathrm{Lie}_{\mathcal{X}} \circ \Omega_{\mathcal{X}}(\mathcal{Y})) \simeq$$
$$\simeq \mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\mathcal{F}[-1], \mathbf{oblv}_{\mathrm{Lie}} \circ \mathrm{Lie}_{\mathcal{X}} \circ \Omega_{\mathcal{X}}(\mathcal{Y})).$$

However, by Corollary 3.2.8, we have

$$\mathbf{oblv}_{\mathrm{Lie}} \circ \mathrm{Lie}_{\mathcal{X}} \circ \Omega_{\mathcal{X}}(\mathcal{Y}) \simeq T(\Omega_{\mathcal{X}}(\mathcal{Y})/\mathcal{X})|_{\mathcal{X}} \simeq T(\mathcal{Y}/\mathcal{X})|_{\mathcal{X}}[-1].$$

Thus, the left-hand side in (3.14) identifies with

$$\mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\mathcal{F}[-1], T(\mathcal{Y}/\mathcal{X})|_{\mathcal{X}}[-1]),$$

as required.

$\square$

REMARK 3.7.4. The above verification of the adjunction can be summarized by the commutative diagram

$$
\begin{array}{ccc}
\mathrm{IndCoh}(\mathcal{X}) & \xleftarrow{\;T(-/\mathcal{X})|_{\mathcal{X}}\;} & \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \\[2pt]
{\scriptstyle [1]\circ\mathbf{oblv}_{\mathrm{Lie}}}\Big\uparrow & & \Omega_{\mathcal{X}}\Big\downarrow{\scriptstyle\sim} \\[2pt]
\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) & \xleftarrow{\;\mathrm{Lie}_{\mathcal{X}}\;} & \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}).
\end{array}
$$

3.7.5. As a corollary of Proposition 3.7.3, we obtain:

COROLLARY 3.7.6. *The monad on* $\mathrm{IndCoh}(\mathcal{X})$, *given by the composition*

$$([-1] \circ T(-/\mathcal{X})|_{\mathcal{X}}) \circ (\mathrm{RealSplitSqZ} \circ [1])$$

is canonically isomorphic to $\mathbf{oblv}_{\mathrm{Lie}} \circ \mathbf{free}_{\mathrm{Lie}}$.

3.7.7. The next property of the functor RealSplitSqZ follows formally from Proposition 3.7.3:

COROLLARY 3.7.8. *For* $\mathcal{Y} \in (\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{\mathcal{X}/}$ *and* $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ *there is a canonical isomorphism*

$$\mathrm{Maps}_{(\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{\mathcal{X}/}}(\mathrm{RealSplitSqZ}(\mathcal{F}), \mathcal{Y}) \simeq \mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\mathcal{F}, T(\mathcal{Y})|_{\mathcal{X}}).$$

In the above corollary, by a slight abuse of notation, we view $\mathrm{RealSplitSqZ}(\mathcal{F})$ as an object of $(\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{\mathcal{X}/}$ rather than $\mathrm{Ptd}((\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{/\mathcal{X}})$.

PROOF. Set $\mathcal{Y}' := \mathcal{X} \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{Y}$, and apply the adjunction of Proposition 3.7.3. $\square$

3.7.9. Let us now compare the functor RealSplitSqZ as defined above with its version introduced in Chapter 1, Sect. 2.1:

COROLLARY 3.7.10. *For* $X \in \mathrm{Sch}_{\mathrm{aft}}$ *we have a commutative diagram*

$$
\begin{array}{ccc}
(\mathrm{Coh}(X)^{\leq 0})^{\mathrm{op}} & \xrightarrow{\;\mathrm{RealSplitSqZ}\;} & \mathrm{Ptd}((\mathrm{Sch}_{\mathrm{aft}})_{\mathrm{nil\text{-}isom\ to}\ X}) \\[2pt]
{\scriptstyle \mathbb{D}_X^{\mathrm{Serre}}}\Big\downarrow & & \Big\downarrow \\[2pt]
\mathrm{IndCoh}(X) & \xrightarrow{\;\mathrm{RealSplitSqZ}\;} & \mathrm{Ptd}((\mathrm{FormMod})_{/\mathcal{X}}).
\end{array}
$$

PROOF. Follows from Corollary 3.7.8, since the split square-zero construction of Chapter 1, Sect. 2.1 has the same universal property.

$\square$

3.7.11. It follows from the equivalence

$$B_{\mathcal{X}}^{\mathrm{Lie}} : \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \to \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$$

that the functor RealSplitSqZ takes coproducts in $\mathrm{IndCoh}(X)$ to coproducts in the category $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$. In particular, it defines a functor
(3.15)
$$\mathrm{IndCoh}(\mathcal{X})^{\mathrm{op}} \simeq \mathrm{ComMonoid}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{op}}) \to \mathrm{ComMonoid}(\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})^{\mathrm{op}}).$$

We claim:

PROPOSITION 3.7.12. *The functor* (3.15) *is an equivalence.*

PROOF. Follows from the fact that $B_{\mathcal{X}}^{\mathrm{Lie}}$ is an equivalence, combined with Chapter 6, Corollary 1.8.7. $\square$

4. Proof of Theorem 3.1.4

4.1. Step 1. In this subsection we will prove that the functor $\exp_X$ defines an equivalence from $\mathrm{LieAlg}(\mathrm{IndCoh}(X))$ to the full subcategory of $\mathrm{Grp}(\mathrm{FormMod}_{/X})$, spanned by objects that are inf-affine when viewed as objects of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$ (i.e., after forgetting the group structure).

We denote this category by $\mathrm{Grp}(\mathrm{FormMod}_{/X})'$.

4.1.1.   First, we note that by Proposition 1.4.7 and Chapter 6, Proposition 1.7.2, for any $\mathfrak{h} \in \mathrm{LieAlg}(\mathrm{IndCoh}(X))$, the object

$$\mathbf{oblv}_{\mathrm{Grp}} \circ \exp_X(\mathfrak{h}) \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$$

is inf-affine, and the canonical map

$$(4.1) \qquad \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}})(\exp_X(\mathfrak{h})) \to \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}(\mathfrak{h})$$

is an isomorphism.

4.1.2.   We claim that the functor Lie_X of Sect. 3.3, restricted to $\mathrm{Grp}(\mathrm{FormMod}_{/X})'$, provides a right adjoint to $\exp_X$. In other words, we claim that for $\mathfrak{h} \in \mathrm{LieAlg}(\mathrm{IndCoh}(X))$ and $\mathcal{H}' \in \mathrm{Grp}(\mathrm{FormMod}_{/X})'$, there is a canonical isomorphism:

$$\mathrm{Maps}_{\mathrm{Grp}(\mathrm{FormMod}_{/X})}(\exp_X(\mathfrak{h}), \mathcal{H}') \simeq \mathrm{Maps}_{\mathrm{LieAlg}(\mathrm{IndCoh}(X))}(\mathfrak{h}, \mathrm{Lie}_X(\mathcal{H}')).$$

Indeed, by Lemma 2.1.6 and (4.1), we rewrite the left-hand side as

$$\mathrm{Maps}_{\mathrm{CocomHopf}(\mathrm{IndCoh}(X))}(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}(\mathfrak{h}), \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}})(\mathcal{H}')),$$

and, further, using Chapter 6, Sect. 4.4.2 as

$$\mathrm{Maps}_{\mathrm{LieAlg}(\mathbf{O})}\left(\mathfrak{h}, B_{\mathrm{Lie}} \circ \mathrm{Monoid}(\mathrm{coChev}^{\mathrm{enh}}) \circ \mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}})(\mathcal{H}')\right),$$

as required.

4.1.3.   We claim that the unit of the adjunction

$$\mathrm{Id} \to \mathrm{Lie}_X \circ \exp_X$$

is an isomorphism.

Indeed, this follows from (4.1) and (3.2).

4.1.4.   Hence, it remains to show that the functor Lie_X, restricted to $\mathrm{Grp}(\mathrm{FormMod}_{/X})'$, is conservative. I.e., we need to show that if $\mathcal{H}_1 \to \mathcal{H}_2$ is a map in $\mathrm{Grp}(\mathrm{FormMod}_{/X})'$, such that $\mathrm{Lie}_X(\mathcal{H}_1) \to \mathrm{Lie}_X(\mathcal{H}_2)$ is an isomorphism, then the original map is also an isomorphism.

More generally, we claim that if $\mathcal{Y}_1 \to \mathcal{Y}_2$ is a map between two inf-affine objects of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$, such that the induced map

$$\mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}_1) \to \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}_2)$$

is an isomorphism in $\mathrm{IndCoh}(X)$, then the original map is also an isomorphism.

Indeed, this follows from Proposition 2.3.3 and Chapter 1, Proposition 8.3.2.

4.2. Step 2. In this subsection we will reduce the assertion of Theorem 3.1.4 to the case when X is reduced.

4.2.1.   Taking into account Step 1, the assertion of Theorem 3.1.4 is equivalent to the fact that every object $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/X})$ is inf-affine, when we consider it as an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$.

Thus, by Proposition 2.1.4, we need to show that for any $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/X})$, the canonical map

$$T(\mathcal{H}/X)|_X \to \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{H})$$

is an isomorphism.

4.2.2.   Let $f : X' \to X$ be a map in $(^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/X}$. We have the symmetric monoidal functor

$$f^! : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X'),$$

which makes the following diagram commute:

$$
\begin{array}{ccc}
\mathrm{IndCoh}(X) & \xrightarrow{\ f^!\ } & \mathrm{IndCoh}(X') \\
{\scriptstyle \mathbf{triv}_{\mathrm{Cocom}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathbf{triv}_{\mathrm{Cocom}}} \\
\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) & \xrightarrow{\ f^!\ } & \mathrm{Cocom}^{\mathrm{aug}}(\mathrm{IndCoh}(X'))
\end{array}
$$

Hence, by adjunction, we obtain a natural transformation:

(4.2) $$f^! \circ \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \to \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ f^!.$$

We claim:

LEMMA 4.2.3. *Assume that f is proper. Then the natural transformation (4.2) is an isomomorphism*

PROOF. Follows by the $(f^{\mathrm{IndCoh}}_*, f^!)$-adjunction from the commutative diagram

$$
\begin{array}{ccc}
\mathrm{IndCoh}(X) & \xleftarrow{\ f^{\mathrm{IndCoh}}_*\ } & \mathrm{IndCoh}(X') \\
{\scriptstyle \mathbf{triv}_{\mathrm{Cocom}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathbf{triv}_{\mathrm{Cocom}}} \\
\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(X)) & \xleftarrow{\ f^{\mathrm{IndCoh}}_*\ } & \mathrm{Cocom}^{\mathrm{aug}}(\mathrm{IndCoh}(X')).
\end{array}
$$

$\square$

4.2.4.   Let i denote the canonical map $X' := {}^{\mathrm{red}}X \to X$. From Lemma 4.2.3 we obtain that for $\mathcal{Y} \in \mathrm{Ptd}(\mathrm{FormMod}_{/X})$, we have a commutative diagram with vertical arrows being isomorphisms

$$
\begin{array}{ccc}
i^!(T(\mathcal{Y}/X)|_X) & \longrightarrow & i^!\Big(\mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y})\Big) \\
\Big\downarrow & & \Big\downarrow \\
T(\mathcal{Y}'/X')|_{X'} & \longrightarrow & \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}'),
\end{array}
$$

where $\mathcal{Y}' := X' \underset{X}{\times} \mathcal{Y}$.

Since the functor $i^!$ is conservative (see Volume I, Chapter 4, Corollary 6.1.5), we obtain that if

$$T(\mathcal{Y}'/X')|_{X'} \to \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}')$$

is an isomorphism, then so is

$$T(\mathcal{Y}/X)|_X) \to \mathrm{Prim}_{\mathrm{Cocom}^{\mathrm{aug}}} \circ \mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}}(\mathcal{Y}).$$

Hence, the assertion of Theorem 3.1.4 for $^{\mathrm{red}}X$ implies that for X.

4.3. Step 3. We will now show that the functor $\exp_X$ is essentially surjective onto the entire category $\mathrm{Grp}(\mathrm{FormMod}_{/X})$. By Step 2, we can assume that X is reduced.

4.3.1.   For $\mathcal{H} \in \mathrm{Grp}(\mathrm{FormMod}_{/X})$ set

$$\mathcal{Y} := B_X(\mathcal{H}) \in \mathrm{Ptd}(\mathrm{FormMod}_{/X}).$$

Using Chapter 5, Corollary 1.5.2(a), we can write

$$\mathcal{Y} \simeq \operatorname*{colim}_{\alpha \in A} Z_\alpha,$$

where the index category A is

$$\left(\mathrm{Ptd}((^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{\text{nil-isom to } X})\right)_{/\mathcal{Y}},$$

and where the colimit is taken in the category $\mathrm{PreStk}_{\mathrm{laft}}$.

We make the following observation:

LEMMA 4.3.2.   *If the scheme X is reduced, then the category A is sifted.*

PROOF.   We claim that the diagonal functor $A \to A \times A$ admits a left adjoint. Namely, it is given by sending

$$Z_1, Z_2 \to Z_1 \underset{X}{\sqcup} Z_2,$$

see Chapter 1, Proposition 7.2.2.

NB: the fact that X is reduced is used to ensure that the maps $X \to Z_i$ are closed (and hence, nilpotent embeddings).

$\square$

4.3.3.   Set

$$\mathcal{H}_\alpha := \Omega_X(Z_\alpha).$$

Since $\mathcal{H}_\alpha$ is a scheme, it is inf-affine, by Proposition 2.1.4. Hence, there exists a canonically defined functor

$$A \to \mathrm{LieAlg}(\mathrm{IndCoh}(X)), \quad \alpha \mapsto \mathfrak{h}_\alpha,$$

so that $\mathcal{H}_\alpha = \exp_X(\mathfrak{h}_\alpha)$.

Set

$$\mathfrak{h} := \operatorname*{colim}_{\alpha \in A} \mathfrak{h}_\alpha \in \mathrm{LieAlg}(\mathrm{IndCoh}(X)).$$

We are going to construct an isomorphism $\mathcal{H} \simeq \exp_X(\mathfrak{h})$.

4.3.4. By Chapter 5, Theorem 1.6.4, it suffices to construct an isomorphism

$$\mathcal{Y} \simeq B_X \circ \exp_X(\mathfrak{h})$$

in $\mathrm{Ptd}(\mathrm{FormMod}_{/X})$.

We let $\mathcal{Y} \to B_X \circ \exp_X(\mathfrak{h})$ be the map, given by the compatible system of maps

$$Z_\alpha \to B_X \circ \exp_X(\mathfrak{h})$$

that correspond under the equivalence Ω_X to the maps

$$\mathcal{H}_\alpha \simeq \exp_X(\mathfrak{h}_\alpha) \to \exp_X(\mathfrak{h}).$$

To prove that the resulting map $\mathcal{Y} \to B_X \circ \exp_X(\mathfrak{h})$ is an isomorphism, by Chapter 1, Proposition 8.3.2, it suffices to show that the induced map

$$T(\mathcal{Y}/X)|_X \to T(B_X \circ \exp_X(\mathfrak{h})/X)|_X$$

is an isomorphism in $\mathrm{IndCoh}(X)$.

4.3.5. We have a commutative diagram

$$
\begin{array}{ccc}
\underset{\alpha \in A}{\mathrm{colim}}\, T(Z_\alpha/X)|_X & \xrightarrow{\ \mathrm{id}\ } & \underset{\alpha \in A}{\mathrm{colim}}\, T(Z_\alpha/X)|_X \\
\downarrow & & \downarrow \\
T(\mathcal{Y}/X)|_X & \longrightarrow & T(B_X \circ \exp_X(\mathfrak{h})/X)|_X.
\end{array}
$$

We note that the left vertical arrow is an isomorphism by Chapter 1, Proposition 2.5.3, since the category of indices A is sifted (see Lemma 4.3.2).

Hence, it remains to show that the right vertical arrow is an isomorphism.

4.3.6. The corresponding map

$$\underset{\alpha \in A}{\mathrm{colim}}\, T(Z_\alpha/X)|_X[-1] \to T(B_X \circ \exp_X(\mathfrak{h})/X)|_X[-1]$$

identifies with

$$\underset{\alpha \in A}{\mathrm{colim}}\, T(\mathcal{H}_\alpha/X)|_X \to T(\exp_X(\mathfrak{h})/X)|_X,$$

and, further, by Proposition 2.3.3, with

$$(4.3) \qquad \underset{\alpha \in A}{\mathrm{colim}}\, \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}_\alpha) \to \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}).$$

Since the category A is sifted, in the commutative diagram

$$
\begin{array}{ccc}
\underset{\alpha \in A}{\mathrm{colim}}\, \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}_\alpha) & \longrightarrow & \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathbf{oblv}_{\mathrm{Lie}}\left(\underset{\alpha \in A}{\mathrm{colim}}\, \mathfrak{h}_\alpha\right) & \xrightarrow{\ \sim\ } & \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h})
\end{array}
$$

the vertical arrows are isomorphisms.

Hence, the map (4.3) is an isomorphism, as required. $\qquad\square$

5. Modules over formal groups and Lie algebras

In the previous sections we have constructed an equivalence between formal groups and Lie algebras. In this section we will show that under this equivalence, the datum of action of a formal group on a given object of IndCoh is equivalent to that of action of the corresponding Lie algebra.

5.1. Modules over formal groups.

5.1.1. Let $\mathcal{H}$ be an object of $\mathrm{Grp}((\mathrm{FormMod}_{\mathrm{laft}})_{/\mathcal{X}})$. We define the category $\mathcal{H}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))$ as

$$\mathrm{Tot}\left(\mathrm{IndCoh}^{!}(B^{\bullet}(\mathcal{H}))\right),$$

where $\mathrm{IndCoh}^{!}(B^{\bullet}(\mathcal{H}))$ is the co-simplicial category, obtained by applying the (contravariant) functor $\mathrm{IndCoh}^{!}_{\mathrm{PreStk}_{\mathrm{laft}}}$ to the simplicial object $B^{\bullet}(\mathcal{H})$ of $\mathrm{PreStk}_{\mathrm{laft}}$.

Denote $\mathfrak{h} := \mathrm{Lie}_{\mathcal{X}}(\mathcal{H})$. The goal of this subsection is to prove the following:

PROPOSITION-CONSTRUCTION 5.1.2. *There exists a canonical equivalence of categories*

(5.1) $\mathcal{H}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \simeq \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))$

that commutes with the forgetful functor to $\mathrm{IndCoh}(\mathcal{X})$, *and is functorial with respect to* $\mathcal{X}$.

The rest of this subsection is devoted to the proof of Proposition 5.1.2.

Without loss of generality, we can assume that $\mathcal{X} = X \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

5.1.3.   Consider the object

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}) \in \mathrm{AssocAlg}(\mathrm{CococomCoalg}(\mathrm{IndCoh}(X))).$$

Consider the corresponding simplicial object

$$\mathrm{Bar}^{\bullet}(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h})) \in \mathrm{CococomCoalg}(\mathrm{IndCoh}(X))^{\mathbf{\Delta}^{\mathrm{op}}},$$

and the simplicial category

$$\mathrm{Bar}^{\bullet}(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathrm{IndCoh}(X)).$$

According to Chapter 6, Proposition 7.2.2 and Sect. 7.4, there exist canonical equivalences

$$\mathfrak{h}\text{-mod}(\mathrm{IndCoh}(X)) \simeq \left(\mathrm{AssocAlg}(\mathbf{oblv}_{\mathrm{Cocom}}) \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h})\right)\text{-mod}(\mathrm{IndCoh}(X)) \simeq$$

$$\simeq |\mathrm{Bar}^{\bullet}(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathrm{IndCoh}(X))|.$$

5.1.4.   By Volume I, Chapter 1, Proposition 2.5.7, we have

$$\mathcal{H}\text{-mod}(\mathrm{IndCoh}(X)) = \mathrm{Tot}\left(\mathrm{IndCoh}^{!}(B^{\bullet}(\mathcal{H}))\right) \simeq |(\mathrm{IndCoh}_{*}(B^{\bullet}(\mathcal{H})))|,$$

where $\mathrm{IndCoh}_{*}(B^{\bullet}(\mathcal{H}))$ is the simplicial category, obtained by applying the functor

$$\mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}} : \mathrm{PreStk}_{\mathrm{laft}} \to \mathrm{DGCat}_{\mathrm{cont}}$$

to the simplicial object $B^{\bullet}(\mathcal{H})$ of $\mathrm{PreStk}_{\mathrm{laft}}$.

We will construct a functor between simplicial categories

(5.2) $\mathrm{IndCoh}_{*}(B^{\bullet}(\mathcal{H})) \to \mathrm{Bar}^{\bullet}(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathrm{IndCoh}(X)),$

and show that it induces an equivalence on geometric realizations.

5.1.5. Let $\pi^\bullet$ denote the augmentation $B^\bullet(\mathcal{H}) \to X$. By Sect. 1.1.3, The functor $(\pi^\bullet)_*^{\mathrm{IndCoh}}$ defines a map of simplicial categories

(5.3) $\qquad \mathrm{IndCoh}_*(B^\bullet(\mathcal{H})) \to \mathrm{Distr}^{\mathrm{Cocom}}(B^\bullet(\mathcal{H}))\text{-comod}(\mathrm{IndCoh}(X)).$

Note that by Lemma 1.2.2, we have:

$$\mathrm{Distr}^{\mathrm{Cocom}}(B^\bullet(\mathcal{H})) \simeq \mathrm{Bar}^\bullet(\mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}})(\mathcal{H})).$$

Since

$$\mathrm{Grp}(\mathrm{Distr}^{\mathrm{Cocom}^{\mathrm{aug}}})(\mathcal{H}) \simeq \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}),$$

we obtain

$$\mathrm{Distr}^{\mathrm{Cocom}}(B^\bullet(\mathcal{H})) \simeq \mathrm{Bar}^\bullet(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h})).$$

Combining with (5.3), we obtain the desired functor (5.2).

5.1.6. It remains to show that the induced functor

$$|\mathrm{IndCoh}_*(B^\bullet(\mathcal{H}))| \to |\mathrm{Bar}^\bullet(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathrm{IndCoh}(X))|$$

is an equivalence.

Consider the commutative diagram

$$
\begin{array}{ccc}
\mathrm{IndCoh}(X) & \xrightarrow{\ \mathrm{id}\ } & \mathrm{IndCoh}(X) \\
\sim\downarrow & & \downarrow\sim \\
\mathrm{IndCoh}_*(B^0(\mathcal{H})) & \longrightarrow & \mathrm{Bar}^0(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathrm{IndCoh}(X)) \\
\downarrow & & \downarrow \\
|\mathrm{IndCoh}_*(B^\bullet(\mathcal{H}))| & \longrightarrow & |\mathrm{Bar}^\bullet(\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}))\text{-comod}(\mathrm{IndCoh}(X))|.
\end{array}
$$

The functor corresponding to the composite left vertical arrow is monadic by Chapter 3, Proposition 3.3.3(a).

The functor corresponding to the composite left vertical arrow is monadic by Chapter 6, Proposition 7.2.2.

Hence, it remains to check that the resulting map of monads on $\mathrm{IndCoh}(X)$ induces an isomorphism at the level of the underlying endo-functors.

By Chapter 3, Proposition 3.3.3(a), the former endo-functor is given by !-tensor product with $\pi_*(\omega_{\mathcal{H}})$, while the latter is given by !-tensor product with

$$\mathbf{oblv}_{\mathrm{Cocom}} \circ \mathbf{oblv}_{\mathrm{Assoc}} \circ \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega(\mathfrak{h}) \simeq \mathbf{oblv}_{\mathrm{Cocom}} \circ \mathrm{Distr}^{\mathrm{Cocom}}(\mathcal{H}) \simeq \mathrm{Distr}(\mathcal{H}) \simeq \pi_*(\omega_{\mathcal{H}}).$$

Now, it is easy to see that the resulting map of endo-functors is the identity map on $\pi_*(\omega_{\mathcal{H}})$.

5.2. Relation to nil-isomorphisms. Let

$$\pi : \mathcal{Y} \leftrightarrows \mathcal{X} : s$$

be an object of $\mathrm{Ptd}((\mathrm{FormMod}_{\mathrm{laft}})_{/\mathcal{X}})$, and set $\mathcal{H} = \Omega_{\mathcal{X}}(\mathcal{Y})$.

In this subsection we will interpret various functors between the categories $\mathrm{IndCoh}(\mathcal{Y})$ and $\mathrm{IndCoh}(X)$ in terms of the equivalence of Proposition 5.1.2.

5.2.1. Set $\mathcal{H} := \Omega_{\mathcal{X}}(\mathcal{Y})$. By Chapter 3, Proposition 3.3.3(b), there is a canonical equivalence

$$(5.4) \qquad \operatorname{IndCoh}(\mathcal{Y}) \simeq \operatorname{Tot}\left(\operatorname{IndCoh}(B^{\bullet}(\mathcal{H}))\right) = \mathcal{H}\text{-mod}(\operatorname{IndCoh}(\mathcal{X})),$$

and thus

$$\operatorname{IndCoh}(\mathcal{Y}) \simeq \mathfrak{h}\text{-mod}(\operatorname{IndCoh}(\mathcal{X})).$$

Under this identification, the forgetful functor

$$\mathbf{oblv}_{\mathfrak{h}} : \mathfrak{h}\text{-mod}(\operatorname{IndCoh}(\mathcal{X})) \to \operatorname{IndCoh}(\mathcal{X})$$

corresponds to $s^{!}$, and the functor

$$\mathbf{triv}_{\mathfrak{h}} : \operatorname{IndCoh}(\mathcal{X}) \to \mathfrak{h}\text{-mod}(\operatorname{IndCoh}(\mathcal{X}))$$

corresponds to $\pi^{!}$.

5.2.2. The functor

$$\pi_{*}^{\operatorname{IndCoh}} : \operatorname{IndCoh}(\mathcal{Y}) \to \operatorname{IndCoh}(\mathcal{X}),$$

being the left adjoint of $\pi^{!}$, identifies with

$$\mathbf{coinv}(\mathfrak{h}, -) : \mathfrak{h}\text{-mod}(\operatorname{IndCoh}(\mathcal{X})) \to \operatorname{IndCoh}(\mathcal{X}).$$

The functor $\pi_{*}^{\operatorname{IndCoh}}$ naturally lifts to a functor

$$\operatorname{IndCoh}(\mathcal{Y}) \to \operatorname{Distr}^{\operatorname{Cocom}}(\mathcal{Y})\text{-comod}(\operatorname{IndCoh}(X)),$$

and the latter can be identified with

$$\mathbf{coinv}^{\operatorname{enh}}(\mathfrak{h}, -) : \mathfrak{h}\text{-mod}(\operatorname{IndCoh}(\mathcal{X})) \to \operatorname{Chev}^{\operatorname{enh}}(\mathfrak{h})\text{-comod}(\operatorname{IndCoh}(\mathcal{X})),$$

see Chapter 6, Sect. 7.3.4 for the notation.

5.2.3. The functor

$$s_{*}^{\operatorname{IndCoh}} : \operatorname{IndCoh}(\mathcal{X}) \to \operatorname{IndCoh}(\mathcal{Y}),$$

being the left adjoint of $s^{!}$, identifies with

$$\mathbf{free}_{\mathfrak{h}} : \operatorname{IndCoh}(\mathcal{X}) \to \mathfrak{h}\text{-mod}(\operatorname{IndCoh}(\mathcal{X})).$$

In particular, we obtain:

COROLLARY 5.2.4. *The monad* $s^{!} \circ s_{*}^{\operatorname{IndCoh}}$ *on* $\operatorname{IndCoh}(\mathcal{X})$ *is canonically isomorphic to the monad* $U(\mathfrak{h}) \overset{!}{\otimes} (-)$*, where* $\mathfrak{h} := \operatorname{Lie}_{\mathcal{X}}(\mathcal{H})$.

5.3. Compatibility with colimits. In this subsection we will prove the following technically important assertion: the assignment $\mathcal{Y} \rightsquigarrow \operatorname{IndCoh}(\mathcal{Y})$ commutes with sifted colimits in $\operatorname{FormMod}_{\mathcal{X}/}$. This is not tautological because the forgetful functor

$$\operatorname{FormMod}_{\mathcal{X}/} \to (\operatorname{PreStk}_{\operatorname{laft}})_{\mathcal{X}/}$$

does *not* commute with sifted colimits[1].

---

[1]Note, however, that it does commute with filtered colimits, by Chapter 1.

5.3.1. Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Let $i \mapsto \mathcal{Y}_i$ be a sifted diagram in $\mathrm{FormMod}_{\mathcal{X}/}$, and let $\mathcal{Y}$ be its colimit. Denote by f_i the canonical map $\mathcal{Y}_i \to \mathcal{Y}$.

Under the above circumstances, we have:

PROPOSITION 5.3.2. *The functor*
$$\mathrm{IndCoh}(\mathcal{Y}) \to \lim_i \mathrm{IndCoh}(\mathcal{Y}_i),$$
given by the compatible collection of functors $(f_i)^!$, *is an equivalence.*

As a formal consequence of Proposition 5.3.2

COROLLARY 5.3.3. *Under the assumptions of Proposition 5.3.2 we have:*
(a) *The functor*
$$\mathrm{colim}_i \mathrm{IndCoh}(\mathcal{Y}_i) \to \mathrm{IndCoh}(\mathcal{Y}),$$
defined by the compatible collection of functors $(f_i)_*^{\mathrm{IndCoh}}$, *is an equivalence.*
(b) *The natural map*
$$\mathrm{colim}_i (f_i)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{Y}_i}) \to \omega_{\mathcal{Y}}$$
is an isomorphism in $\mathrm{IndCoh}(\mathcal{Y})$.

The rest of this subsection is devoted to the proof of Proposition 5.3.2.

5.3.4. *Step 1.* We will first treat the case when the diagram $i \mapsto \mathcal{Y}_i$ is in $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$. In this case $\mathcal{Y}$ also naturally an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$, and identifies with the colimit of $\mathcal{Y}_i$ *in* $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$.

Let
$$i \mapsto \mathfrak{h}_i$$
be the diagram in $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$ so that $\mathcal{Y}_i = B_{\mathcal{X}}^{\mathrm{Lie}}(\mathfrak{h}_i)$. Denote
$$\mathfrak{h} := \mathrm{colim}_i \mathfrak{h}_i \in \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})),$$
so that $\mathcal{Y} \simeq B_{\mathcal{X}}^{\mathrm{Lie}}(\mathfrak{h})$.

By Proposition 5.1.2, it suffices to show that the functor
$$\mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \to \lim_i \mathfrak{h}_i\text{-mod}(\mathrm{IndCoh}(\mathcal{X})),$$
given by restriction, is an equivalence. However, this is true for any sifted diagram of Lie algebras.

5.3.5. *Step 2.* Let us now return to the general case of a sifted diagram $i \mapsto \mathcal{Y}_i$ in $\mathrm{FormMod}_{\mathcal{X}/}$. Consider the corresponding diagram
$$i \mapsto \mathcal{R}_i^{\bullet}$$
in $\mathrm{FormGrpoid}(\mathcal{X})$. Let $\mathcal{R}^{\bullet}$ be the formal groupoid corresponding to $\mathcal{Y}$.

By Chapter 5, Corollary 2.2.4, for every n, the map
$$\mathrm{colim}_i \mathcal{R}_i^n \to \mathcal{R}^n$$
is an isomorphism in $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$.

Applying Step 1, we obtain that for every n, the functor
$$\mathrm{IndCoh}(\mathcal{R}^n) \to \lim_i \mathrm{IndCoh}(\mathcal{R}_i^n)$$

is an equivalence.

Now, the equivalence

$$\mathrm{IndCoh}(\mathcal{Y}) \to \lim_i \mathrm{IndCoh}(\mathcal{Y}_i)$$

follows by descent, i.e., Chapter 5, Proposition 2.2.6. $\square$

6. Actions of formal groups on prestacks

The goal of this section is to make precise the following idea: an action of a Lie algebra on a prestack is equivalent to that of action of the corresponding formal group.

The first difficulty that we have to grapple with is to define what we mean by an action of a Lie algebra on a prestack. For now we will skirt this question by considering free Lie algebras; we will return to it in Chapter 8, Sect. 7.

6.1. Action of groups vs. Lie algebras. In this subsection we will make precise the following construction:

If a formal group $\mathcal{H}$ acts on a prestack $\mathcal{Y}$, then the Lie algebra of $\mathcal{H}$ maps to global vector fields on $\mathcal{Y}$.

6.1.1.   Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\mathrm{laft}}$. Let $\mathcal{H} \in \mathrm{Grp}((\mathrm{FormMod}_{\mathrm{laft}})_{/\mathcal{X}})$; denote $\mathfrak{h} := \mathrm{Lie}_{\mathcal{X}}(\mathcal{H})$.

Let $\pi : \mathcal{Y} \to \mathcal{X}$ be an object of $(\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{X}}$, equipped with an action of $\mathcal{H}$. Let us assume that $\mathcal{Y}$ admits deformation theory relative to $\mathcal{X}$ (see Chapter 1, Sect. 7.1.6 for what this means).

6.1.2.   We claim that the data of action gives rise to a map in $\mathrm{IndCoh}(\mathcal{Y})$;

$$(6.1) \qquad\qquad \pi^!(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h})) \to T(\mathcal{Y}/\mathcal{X}).$$

Indeed, if act denotes the action map

$$\mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{Y} \to \mathcal{Y},$$

then we have a canonically map

$$T((\mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{Y})/\mathcal{X}) \to \mathrm{act}^!(T(\mathcal{Y}/\mathcal{X})).$$

Pulling back along the unit section of $\mathcal{H}$, and composing with the canonical map

$$\pi^!(T(\mathcal{H}/X)|_X) \to T(\mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{Y}/\mathcal{X})|_{\mathcal{Y}},$$

and using the isomorphism $T(\mathcal{H}/X)|_X \simeq \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h})$ of Corollary 3.2.8, we obtain the desired map

$$\pi^!(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h})) \simeq \pi^!(T(\mathcal{H}/X)|_X) \to T(\mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{Y}/\mathcal{X})|_{\mathcal{Y}} \to \mathrm{act}^!(T(\mathcal{Y}/\mathcal{X}))|_{\mathcal{Y}} \simeq T(\mathcal{Y}/\mathcal{X}).$$

6.1.3.   Assume now that $\mathfrak{h}$ is of the form $\mathbf{free}_{\mathrm{Lie}}(\mathcal{F})$ for some $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$. Note that by adjunction we have a canonical map

$$\mathcal{F} \to \mathbf{oblv}_{\mathrm{Lie}} \circ \mathbf{free}_{\mathrm{Lie}}(\mathcal{F}).$$

Composing with (6.1), we obtain a map

$$(6.2) \qquad\qquad \pi^!(\mathcal{F}) \to T(\mathcal{Y}/\mathcal{X}).$$

6.1.4. The above construction defines a map from the groupoid of actions of $\mathcal{H}$ on $\mathcal{Y}$ to
$$\operatorname{Maps}_{\operatorname{IndCoh}(\mathcal{Y})}(\pi^!(\mathcal{F}), T(\mathcal{Y}/\mathcal{X})).$$

The goal of this section is to prove the following assertion:

THEOREM 6.1.5. *For $\mathcal{Y}$ and $\mathcal{F}$ as above, the map from groupoid of data of actions of $\mathcal{H}$ on $\mathcal{Y}$ to $\operatorname{Maps}_{\operatorname{IndCoh}(\mathcal{Y})}(\pi^!(\mathcal{F}), T(\mathcal{Y}/\mathcal{X}))$ is an isomorphism.*

6.2. Proof of Theorem 6.1.5.

6.2.1. *Idea of proof.* The statement of the theorem readily reduces to the case when $\mathcal{X} = X \in {}^{<\infty}\operatorname{Sch}_{\operatorname{ft}}^{\operatorname{aff}}$.

Let $(\pi^!)^R$ denote the (discontinuous) right adjoint of $\pi^! : \operatorname{IndCoh}(X) \to \operatorname{IndCoh}(\mathcal{Y})$, so that
$$\operatorname{Maps}_{\operatorname{IndCoh}(\mathcal{Y})}(\pi^!(\mathcal{F}), T(\mathcal{Y}/\mathcal{X})) \simeq \operatorname{Maps}_{\operatorname{IndCoh}(X)}\left(\mathcal{F}, (\pi^!)^R(T(\mathcal{Y}/\mathcal{X}))\right).$$

Starting from $\mathcal{Y}$ as above, we will construct an object
$$\operatorname{Aut}^{\operatorname{inf}}(\mathcal{Y}/X) \in \operatorname{Grp}((\operatorname{FormMod}_{\operatorname{laft}})_{/\mathcal{X}}),$$
such that for any $\mathcal{H}' \in \operatorname{Grp}((\operatorname{FormMod}_{\operatorname{laft}})_{/\mathcal{X}})$, the data of action of $\mathcal{H}'$ on $\mathcal{Y}$ is equivalent to that of a homomorphism
$$\mathcal{H}' \to \operatorname{Aut}^{\operatorname{inf}}(\mathcal{Y}/X).$$

Moreover, we will show that the map
$$(6.3) \qquad \mathbf{oblv}_{\operatorname{Lie}}\left(\operatorname{Lie}_X(\operatorname{Aut}^{\operatorname{inf}}(\mathcal{Y}/X))\right) \to (\pi^!)^R(T(\mathcal{Y}/X)),$$
arising by adjunction from (6.1), is an isomorphism.

This will prove Theorem 6.1.5, since the functor Lie_X is an equivalence.

6.2.2. By Chapter 5, Proposition 1.2.2, in order to construct $\operatorname{Aut}^{\operatorname{inf}}(\mathcal{Y}/X)$ as an object of
$$\operatorname{Monoid}((\operatorname{FormMod}_{\operatorname{laft}})_{/X}),$$
it suffices to define it as a presheaf with values in $\operatorname{Monoid}(\operatorname{Spc})$ on the category
$$({}^{<\infty}\operatorname{Sch}_{\operatorname{ft}}^{\operatorname{aff}})_{\operatorname{nil-isom\ to\ } X},$$
so that it satisfies the deformation theory conditions of Chapter 5, Proposition 1.2.2(b).

For $Z \in ({}^{<\infty}\operatorname{Sch}_{\operatorname{ft}}^{\operatorname{aff}})_{\operatorname{nil-isom\ to\ } X}$, we set
$$\operatorname{Maps}_{/X}(Z, \operatorname{Aut}^{\operatorname{inf}}(\mathcal{Y}/X)) := \operatorname{Maps}_{/X}(Z \underset{X}{\times} \mathcal{Y}, \mathcal{Y}) \underset{\operatorname{Maps}_{/X}({}^{\operatorname{red}}Z \underset{X}{\times} \mathcal{Y}, \mathcal{Y})}{\times} * \simeq$$
$$\simeq \operatorname{Maps}_{/Z}(\mathcal{Y}_Z, \mathcal{Y}_Z) \underset{\operatorname{Maps}_{/\operatorname{red}Z}(\mathcal{Y}_{\operatorname{red}Z}, \mathcal{Y}_{\operatorname{red}Z})}{\times} *,$$
(here $\mathcal{Y}_Z := Z \underset{X}{\times} \mathcal{Y}$ and $\mathcal{Y}_{\operatorname{red}Z} = {}^{\operatorname{red}}Z \underset{X}{\times} \mathcal{Y}$).

The deformation theory conditions of Chapter 5, Proposition 1.2.2(b) follow from the fact that $\mathcal{Y}$ admits deformation theory.

REMARK 6.2.3. The prestack $\operatorname{Aut}^{\operatorname{inf}}(\mathcal{Y}/X)$ constructed above is the formal completion of the full automorphism prestack $\operatorname{Aut}(\mathcal{Y}/X)$ along the identity.

6.2.4.   Thus, we have constructed $\mathrm{Aut}^{\mathrm{inf}}(\mathcal{Y}/X)$ as an object of $\mathrm{AssocAlg}((\mathrm{FormMod}_{\mathrm{laft}})_{/X})$. It belongs to $\mathrm{Grp}((\mathrm{FormMod}_{\mathrm{laft}})_{/X})$ by Chapter 5, Lemma 1.6.2.

It remains to show that the map (6.3) is an isomorphism. By construction, for $\mathcal{F} \in \mathrm{Coh}(X)$ such that $\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}) \in \mathrm{Coh}(X)^{\leq 0}$, we have

$$\mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathcal{F}, T(\mathrm{Aut}^{\mathrm{inf}}(\mathcal{Y}/X))|_X) \simeq \mathrm{Maps}_{/X}(\mathrm{RealSplitSqZ}(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F})) \underset{X}{\times} \mathcal{Y}, \mathcal{Y}).$$

By the deformation theory of $\mathcal{Y}$, the latter maps isomorphically to

$$\mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{Y})}(\pi^!(\mathcal{F}), T(\mathcal{Y}/X))),$$

and by adjunction, further (still isomorphically) to

$$\mathrm{Maps}_{\mathrm{IndCoh}(X)}\left(\mathcal{F}, (\pi^!)^R(T(\mathcal{Y}/X)))\right).$$

Furthermore, it follows from the construction that the resulting map

$$\mathrm{Maps}_{\mathrm{IndCoh}(X)}(\mathcal{F}, T(\mathrm{Aut}^{\mathrm{inf}}(\mathcal{Y}/X))|_X) \to \mathrm{Maps}_{\mathrm{IndCoh}(X)}\left(\mathcal{F}, (\pi^!)^R(T(\mathcal{Y}/X)))\right)$$

is the one induced by (6.2).

This implies the required assertion, as $\mathrm{IndCoh}(X)$ is generated by the above objects of $\mathrm{Coh}(X)$ under colimits.

$\square$

6.3. Localization of Lie algebra modules. In this subsection we show how to construct crystals on a given prestack starting from modules over a Lie algebra that acts on this prestack.

6.3.1.   Let $f : \mathcal{Y} \to \mathcal{X}$ and $\mathcal{H}$ be as in Sect. 6.1.1. Consider the prestack

$$\mathcal{Y}_{/\mathcal{X}_{\mathrm{dR}}} := \mathcal{Y}_{\mathrm{dR}} \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{X},$$

see Chapter 4, Sect. 3.3.2 for the notation.

Recall also the notation

$$^{/\mathcal{X}}\mathrm{Crys}(\mathcal{Y}) := \mathrm{IndCoh}(\mathcal{Y}_{\mathrm{dR}} \underset{\mathcal{X}_{\mathrm{dR}}}{\times} \mathcal{X}).$$

In this subsection we will construct the localization functor

$$\mathrm{Loc}_{\mathfrak{h}, \mathcal{Y}/\mathcal{X}} : \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \to {}^{/\mathcal{X}}\mathrm{Crys}(\mathcal{Y}).$$

6.3.2.   The action of $\mathcal{H}$ on $\mathcal{Y}$ defines an object

$$\mathcal{H} \underset{\mathcal{X}}{\times} \mathcal{Y} \in \mathrm{FormGrpoid}(\mathcal{Y}),$$

see Chapter 5, Sect. 2.2.1 for the notation.

By (the relative over $\mathcal{X}$ version of) Chapter 5, Theorem 2.3.2, the corresponding quotient

$$\mathcal{Y}/\mathcal{H} \in \mathrm{FormMod}_{\mathcal{Y}/}$$

is well-defined.

We have canonically defined maps of prestacks

$$\mathcal{Y}/\mathcal{H} \xrightarrow{\ f/\mathcal{H}\ } B_X(\mathcal{H})$$
$$g \downarrow$$
$$\mathcal{Y}_{/\mathcal{X}_{\mathrm{dR}}}.$$

6.3.3.   We define the sought-for functor $\mathrm{Loc}_{\mathfrak{h},\mathcal{Y}/\mathcal{X}}$ as

$$g_*^{\mathrm{IndCoh}} \circ (f/\mathcal{H})^!,$$

where we identify

$$\mathrm{IndCoh}(B_{\mathcal{X}}(\mathcal{H})) \simeq \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))$$

by means of (5.4).

6.3.4.   Note that the functor $\mathrm{Loc}_{\mathfrak{h},\mathcal{Y}/\mathcal{X}}$ is by construction the left adjoint of the (in general, discontinuous) functor

$$((f/\mathcal{H})_*^{\mathrm{IndCoh}})^R \circ g^! : {}^{/\mathcal{X}}\mathrm{Crys}(\mathcal{Y}) \to \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})).$$

We claim that the functor $((f/\mathcal{H})_*^{\mathrm{IndCoh}})^R \circ g^!$ makes the following diagram commutative:

$$
\begin{array}{ccc}
{}^{/\mathcal{X}}\mathrm{Crys}(\mathcal{Y}) & \xrightarrow{\ \mathbf{oblv}_{/\mathcal{X}_{\mathrm{dR}},\mathcal{Y}}\ } & \mathrm{IndCoh}(\mathcal{Y}) \\
{\scriptstyle ((f/\mathcal{H})_*^{\mathrm{IndCoh}})^R \circ g^!}\Big\downarrow & & \Big\downarrow{\scriptstyle (f^!)^R} \\
\mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) & \xrightarrow{\ \mathbf{oblv}_{\mathfrak{h}}\ } & \mathrm{IndCoh}(\mathcal{X}),
\end{array}
$$

where $\mathbf{oblv}_{/\mathcal{X}_{\mathrm{dR}},\mathcal{Y}}$ is by definition the !-pullback functor along

$$p_{/\mathcal{X}_{\mathrm{dR}},\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}_{/\mathcal{X}_{\mathrm{dR}}}.$$

6.3.5.   Indeed, we need to establish the commutativity of the diagram

$$
\begin{array}{ccc}
\mathrm{IndCoh}(\mathcal{Y}) & \xrightarrow{\ (f_*^{\mathrm{IndCoh}})^R\ } & \mathrm{IndCoh}(\mathcal{X}) \\
\Big\uparrow & & \Big\uparrow \\
\mathrm{IndCoh}(\mathcal{Y}/\mathcal{H}) & \xrightarrow{\ ((f/\mathcal{H})_*^{\mathrm{IndCoh}})^R\ } & \mathrm{IndCoh}(B_{\mathcal{X}}(\mathcal{H})),
\end{array}
$$

where the vertical arrows are given by !-pullback.

However, this follows by passing to right adjoints in the commutative diagram,

$$
\begin{array}{ccc}
\mathrm{IndCoh}(\mathcal{Y}) & \xleftarrow{\ f_*^{\mathrm{IndCoh}}\ } & \mathrm{IndCoh}(\mathcal{X}) \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{IndCoh}(\mathcal{Y}/\mathcal{H}) & \xleftarrow{\ (f/\mathcal{H})_*^{\mathrm{IndCoh}}\ } & \mathrm{IndCoh}(B_{\mathcal{X}}(\mathcal{H})),
\end{array}
$$

given by base-change.

CHAPTER 8

Lie algebroids

Introduction

0.1. Who are these Lie algebroids? In this chapter we initiate the study of Lie algebroids over prestacks (technically, over prestacks locally almost of finite type that admit deformation theory). The reason we decided to devote a chapter to this notion is that Lie algebroids provide a convenient language to discuss differential-geometric properties of prestacks, which will be studied in Chapter 9.

0.1.1.   In classical algebraic geometry, a Lie algebroid (over a classical scheme) X is a quasi-coherent sheaf $\mathfrak{L}$, equipped with an $\mathcal{O}_X$-linear map to the tangent sheaf and an operation of Lie bracket that satisfy some natural axioms (see Sect. 9.1).

In the setting of derived we *define* the category of Lie algebroids on $\mathcal{X}$ to be that of formal groupoids on $\mathcal{X}$. This is sensible because the category of Lie algebras in $\mathrm{IndCoh}(\mathcal{X})$ *is* equivalent to the category of formal groups over $\mathcal{X}$, due to Chapter 7, Theorem 3.6.2.

The reason we call these objects 'Lie algebroids' is that we construct various forgetful functors to more linear categories and show that Lie algebroids can be described as ind-coherent sheaves with an additional structure. However, a distinctive feature of the derived story we will explain is that the *only* description of this extra structure that we give is in terms of geometry. I.e., we could not come up with a more 'algebraic' definition.

0.1.2.   We show that with the definition of Lie algebroids as formal groupids, one can perform with them all the expected operations:

A Lie algebroid $\mathfrak{L}$ will have an associated object

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}) \in \mathrm{IndCoh}(\mathcal{X}),$$

equipped with a morphism $\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}) \to T(\mathcal{X})$, called the anchor map. The kernel of the anchor map has a structure of Lie algebra in $\mathrm{IndCoh}(\mathcal{X})$, while the space of global sections of $\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})$ has also a structure of Lie algebra (in Vect).

Thus, the category $\mathrm{LieAlgbroid}(\mathcal{X})$ is related to the category $\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$ by a pair of adjoint functors

$$\mathbf{free}_{\mathrm{LieAlgbroid}} : \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \rightleftarrows \mathrm{LieAlgbroid}(\mathcal{X}) : \mathbf{oblv}_{\mathrm{LieAlgbroid}/T},$$

and we will show that the resulting monad

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}$$

acting on $\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$ has 'the right size', see Proposition 5.3.2.

Furthermore, LieAlgbroid($\mathcal{X}$) is related to the category LieAlg(IndCoh($\mathcal{X}$)) by a pair of adjoint functors

$$\text{diag} : \text{LieAlg}(\text{IndCoh}(\mathcal{X})) \to \text{LieAlgbroid}(\mathcal{X}) : \text{ker-anch},$$

(where the meaning of diag is that an $\mathcal{O}_{\mathcal{X}}$-linear Lie algebra can be considered into a Lie algebroid with the zero anchor map, and ker-anch sends a Lie algebroid to the kernel of its anchor map[1]). The monad

$$\text{ker-anch} \circ \text{diag}_{\mathcal{X}}$$

is given by the operation of *semi-direct product* with the *inertia* Lie algebra inert$_{\mathcal{X}}$, which is again what one expects from a sensible definition of Lie algebroids.

0.1.3.   Finally, let us comment on our inability to define Lie algebroids without resorting to geometry. In fact, this is not surprising: throughout the book the only way we access Lie algebras is via the definition of the Lie operad as the Koszul dual of the commutative operad. So, it is natural that in order to define objects that generalize Lie algebras we resort to commutative objects (in our case, prestacks).

In Sect. 5.6 we present a very general categorical framework, in which one can define 'broids' as modules over a certain monad.

0.2. What is done in this chapter? We should say right away that this chapter does not contain any big theorems. Mostly, it uses the material of the previous chapters to set up the theory of Lie algebroids and also sets ground for applications in Chapter 9.

0.2.1.   In Sect. 1 we return to the study of groupoids (in spaces and then in the framework of algebraic geometry).

Given a space (resp., prestack) X, we define two functors from the category Groupoid(X) to the category of groups over X.

The first of these functors, denoted Inert, sends a groupoid to its inertia group. Applying this functor to the unit groupoid (i.e., the initial object of Groupoid(X)), we obtain the inertia group of X, denoted Inert$_X$.

The second functor, denoted Ω^{fake}, sends a groupoid R to $\Omega_X(R)$, where we view R as a pointed object over X via

$$\text{unit} : X \rightleftarrows R : p_s.$$

The above two functors are related by a fiber sequence

$$\Omega^{\text{fake}}(R) \to \text{Inert}_X \to \text{Inert}(R).$$

---

[1]Another way to look at the above adjoint pair is that the category LieAlg(IndCoh($\mathcal{X}$)) identifies with the over-category

$$(\text{LieAlgbroid}(\mathcal{X}))_{/0},$$

where 0 is the zero Lie algebroid.

0.2.2. In Sect. 2 we introduce the notion of Lie algebroid over an object $\mathcal{X} \in$ PreStk$_{\text{laft-def}}$, along with two pairs of adjoint functors

$$\mathbf{free}_{\text{LieAlgbroid}} : \text{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \rightleftarrows \text{LieAlgbroid}(\mathcal{X}) : \mathbf{oblv}_{\text{LieAlgbroid}/T},$$

and

$$\text{diag}_{\mathcal{X}} : \text{LieAlg}(\text{IndCoh}(\mathcal{X})) \to \text{LieAlgbroid}(\mathcal{X}) : \text{ker-anch}.$$

We introduce also another functor

$$\Omega^{\text{fake}} : \text{LieAlgbroid}(\mathcal{X}) \to \text{LieAlg}(\text{IndCoh}(\mathcal{X})),$$

so that for $\mathfrak{L} \in \text{LieAlgbroid}(\mathcal{X})$ we have the fiber sequence

$$\Omega^{\text{fake}}(\mathfrak{L}) \to \text{inert}_{\mathcal{X}} \to \text{ker-anch}(\mathfrak{L}),$$

where $\text{inert}_{\mathcal{X}}$ is the Lie algebra of the inertia group of $\mathcal{X}$.

We note that

$$\mathbf{oblv}_{\text{Lie}}(\text{inert}_{\mathcal{X}}) = T(X)[-1]$$

and when we apply $\mathbf{oblv}_{\text{Lie}}$ to the map $\Omega^{\text{fake}}(\mathfrak{L}) \to \text{inert}_{\mathcal{X}}$, we recover the shift by $[-1]$ of the anchor map, i.e., of the object

$$\mathbf{oblv}_{\text{LieAlgbroid}/T} \in \text{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}.$$

0.2.3. In Sect. 3 we consider the basic examples of Lie algebroids: the tangent algebroid, the zero algebroid, the Lie algebroid attached to a map of prestacks, and the Atiyah algebroid attached to an object of $\text{QCoh}(\mathcal{X})^{\text{perf}}$.

0.2.4. In Sect. 4 we introduce the notion of module over a Lie algebroid, and define the universal enveloping algebra of a Lie algebroid.

0.2.5. In Sect. 5 we study the relationship between square-zero extensions and Lie algebroids. Recall that according to Chapter 5, Theorem 2.3.2, for a given $\mathcal{X} \in \text{PreStk}_{\text{laft-def}}$, the category of formal moduli problems *under* $\mathcal{X}$ is equivalent to that of formal groupoids over $\mathcal{X}$, and thus to the category of Lie algebroids.

Using this equivalence, we construct functor

$$\text{RealSqZExt} : \text{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \to \text{FormMod}_{\mathcal{X}/}$$

to correspond to the functor

$$\mathbf{free}_{\text{LieAlgbroid}} : \text{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \to \text{LieAlgbroid}(\mathcal{X}).$$

We show that the functor is the left adjoint to the functor that sends $\mathcal{X} \to \mathcal{Y}$ to $T(\mathcal{X}/\mathcal{Y}) \in \text{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$, i.e., it really behaves like a square-zero extension.

We also show that the notion of square-zero extension developed in the present section using Lie algebroids is equivalent to one developed in Chapter 1, Sect. 10, which was bootstrapped from the case of schemes.

0.2.6. In Sect. 6 we introduce the Atiyah class, which is a functorial assignment for any $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ of a map

$$T(\mathcal{X})[-1] \overset{!}{\otimes} \mathcal{F} \overset{\alpha_{\mathcal{F}}}{\to} \mathcal{F}.$$

We show that if $i : \mathcal{X} \to \mathcal{X}'$ is a square-zero extension of $\mathcal{X}$, given by

$$\mathcal{F}' \overset{\gamma}{\to} T(\mathcal{X}),$$

then the category $\mathrm{IndCoh}(\mathcal{X}')$ can be described as the category consisting of $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$, equipped with a null-homotopy of the composite map

$$\mathcal{F}' \overset{!}{\otimes} \mathcal{F} \overset{\gamma \overset{!}{\otimes} \mathrm{id}}{\longrightarrow} T(\mathcal{X})[-1] \overset{!}{\otimes} \mathcal{F} \overset{\alpha_{\mathcal{F}}}{\to} \mathcal{F}.$$

We deduce that the dualizing object $\omega_{\mathcal{X}'} \in \mathrm{IndCoh}(\mathcal{X}')$ fits into the exact triangle

$$i_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}'} \to i_*^{\mathrm{IndCoh}}(\mathcal{F}'),$$

further justifying the terminology 'square-zero extension'.

0.2.7. In Sect. 7 we show that the space of global sections of a Lie algebroid carries a canonical structure of Lie algebra. (In particular, global vector fields carry a structure of Lie algebra.)

We also show that $\mathfrak{h}$ is a Lie algebra object in $\mathrm{IndCoh}(\mathcal{X})$ obtained as $\Omega^{\mathrm{fake}}(\mathfrak{L})$ for a Lie algebroid $\mathfrak{L}$, then the Lie algebra structure on the space of global sections of $\mathfrak{h}$ is the trivial one.

0.2.8. In Sect. 8 we present another point of view on the category $\mathrm{LieAlgbroid}(\mathcal{X})$. Namely, we show that the functor

$$\mathrm{ker\text{-}anch} : \mathrm{LieAlgbroid}(\mathcal{X}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$$

is monadic.

I.e., the category $\mathrm{LieAlgbroid}(\mathcal{X})$ can be realized as the category of modules for the monad

$$\mathsf{M}_{\mathrm{Inert}_{\mathcal{X}}^{\mathrm{inf}}} := \mathrm{ker\text{-}anch} \circ \mathrm{diag}$$

acting on the category $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$.

The monad $\mathsf{M}_{\mathrm{Inert}_{\mathcal{X}}^{\mathrm{inf}}}$ is given by the operation of 'semi-direct product' with the inertia Lie algebra $\mathrm{inert}_{\mathcal{X}}$. So in a sense, this gives a very manageable presentation of the category $\mathrm{LieAlgbroid}(\mathcal{X})$. We learned this idea from J. Francis.

Thus, there are (at least) two ways to exhibit $\mathrm{LieAlgbroid}(\mathcal{X})$ as modules over a monad acting on some category: one is what we just said above, and another via the adjunction

$$\mathbf{free}_{\mathrm{LieAlgbroid}} : \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \rightleftarrows \mathrm{LieAlgbroid}(\mathcal{X}) : \mathbf{oblv}_{\mathrm{LieAlgbroid}/T}.$$

0.2.9.   Finally, in Sect. 9 we compare our definition of Lie algebroids with the usual (i.e., classical) one, when our prestack $\mathcal{X}$ is a classical scheme X.

We show (see Theorem 9.1.5) that the subcategory consisting of Lie algebroids $\mathfrak{L}$, for which the object $\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}) \in \mathrm{IndCoh}(X)$ lies in the essential image of the functor

$$\mathrm{QCoh}(X)^\heartsuit \hookrightarrow \mathrm{QCoh}(X) \overset{\Upsilon_X}{\hookrightarrow} \mathrm{IndCoh}(X)$$

is canonically equivalent to that of classical Lie algebroids.

Further, we show that if $\mathfrak{L} \in \mathrm{LieAlgbroid}(\mathcal{X})$ is such that $\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}) \in \mathrm{IndCoh}(X)$ lies in the essential image of the functor

$$\mathrm{QCoh}(X)^{\heartsuit,\mathrm{flat}} \hookrightarrow \mathrm{QCoh}(X)^\heartsuit \hookrightarrow \mathrm{QCoh}(X) \overset{\Upsilon_X}{\hookrightarrow} \mathrm{IndCoh}(X),$$

then the category $\mathfrak{L}\text{-mod}(\mathrm{IndCoh}(X))$ agrees with the classical definition of the category of modules over a Lie algebroid.

1. The inertia group

In this section we return to the discussion of groupoids, first in the category Spc and then in formal geometry.

We show that there are two forgetful functors from the category of groupoids (on a given space or prestack) $\mathcal{X}$ to that of groups over $\mathcal{X}$. The first functor is given by the inertia group, i.e. the morphisms with the same source and target. The second is given by taking the relative loop space of the groupoid. We also establish a relationship between these two functors: namely, they fit into a fiber sequence with the inertia group of the identity groupoid in the middle.

1.1. Inertia group of a groupoid. In this subsection we work in the category of spaces. We introduce the notion of inertia group of a groupoid.

1.1.1.   Recall the setting of Chapter 5, Sect. 2.1. For $X \in \mathrm{Spc}$, note that we have a tautological forgetful functor

$$\mathrm{diag} : \mathrm{Grp}(\mathrm{Spc}_{/X}) \to \mathrm{Grpoid}(X).$$

In fact,

$$\mathrm{Grp}(\mathrm{Spc}_{/X}) \simeq \mathrm{Grpoid}(X)_{/\mathrm{diag}_X}.$$

Hence, the functor diag admits a right adjoint, denoted Inert, given by Cartesian product (inside $\mathrm{Grpoid}(X)$) with diag_X.

Concretely, as a space

$$\mathrm{Inert}(R) := X \underset{X \times X}{\times} R,$$

(we recall that $X \times X$ is the final object in $\mathrm{Grpoid}(X)$).

1.1.2. For $R = \mathrm{diag}_X$ being the identity groupoid, we thus obtain an object of $\mathrm{Grp}(\mathrm{Spc}_{/X})$, denoted Inert_X.

I.e., as an object of $\mathrm{Grp}(\mathrm{Spc}_{/X})$, we have:

$$\mathrm{Inert}_X = X \underset{X \times X}{\times} X = \Omega_X(X \times X),$$

where $X \times X$ is regarded as an object of $\mathrm{Ptd}(\mathrm{Spc}_{/X})$ via the maps

$$\Delta_X : X \rightleftarrows X \times X : p_s.$$

The object $\mathrm{Inert}_X \in \mathrm{Grp}(\mathrm{Spc}_{/X})$ is called the *inertia group*[2] of X.

1.1.3. For $R = X \underset{Y}{\times} X$, we have:

$$\mathrm{Inert}(R) = X \underset{Y}{\times} \mathrm{Inert}_Y .$$

1.1.4. There is another functor

$$\Omega^{\mathrm{fake}} : \mathrm{Grpoid}(X) \to \mathrm{Grp}(\mathrm{Spc}_{/X}).$$

Namely,

$$\Omega^{\mathrm{fake}}(R) := \Omega_X(R),$$

where in the left-hand side Ω_X is the loop functor $\mathrm{Ptd}(\mathrm{Spc}_{/X}) \to \mathrm{Grp}(\mathrm{Spc}_{/X})$, where we view R as as an object of $\mathrm{Ptd}(\mathrm{Spc}_{/X})$ via

$$\mathrm{unit} : X \rightleftarrows R : p_s.$$

For example,

$$\mathrm{Inert}_X = \Omega^{\mathrm{fake}}(X \times X).$$

1.1.5. Since $X \times X$ is the final object in $\mathrm{Grpoid}(X)$, for any groupoid R we have a tautological map $R \to X \times X$, which gives rise to a map

$$\Omega^{\mathrm{fake}}(R) \to \mathrm{Inert}_X .$$

In addition, the unit map $\mathrm{diag}_X \to R$ gives rise to a map in $\mathrm{Grp}(\mathrm{Spc}_{/X})$

$$\mathrm{Inert}_X \to \mathrm{Inert}(R).$$

It is easy to see that

$$\Omega^{\mathrm{fake}}(R) \to \mathrm{Inert}_X \to \mathrm{Inert}(R)$$

is a fiber sequence in $\mathrm{Grp}(\mathrm{Spc}_{/X})$.

1.1.6. Note also that the composed endo-functor of $\mathrm{Groupoid}(X)$

$$\mathrm{diag} \circ \Omega^{\mathrm{fake}}$$

identifies with

$$R \mapsto \mathrm{diag}_X \underset{R}{\times} \mathrm{diag}_X,$$

where the fiber product is taken in $\mathrm{Groupoid}(X)$.

―――――――――

[2]Note that we can also think of Inert_X as X^{S^1}, i.e., the free loop space of X.

1.2. Infinitesimal inertia. In this subsection we translate the material from Sect. 1.1 to the context of infinitesimal algebraic geometry. I.e., instead of Spc, we will work with the category $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$, and instead of groupoids we will consider objects of $\mathrm{FormGrpoid}(\mathcal{X})$ over a given prestack $\mathcal{X}$.

1.2.1. Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Consider the category $\mathrm{FormGrpoid}(\mathcal{X})$.

Note that $\mathrm{FormGrpoid}(\mathcal{X})$ admits a final object equal to $(\mathcal{X} \times \mathcal{X})^{\wedge}$, the formal completion of the diagonal in $\mathcal{X} \times \mathcal{X}$.

The initial object in $\mathrm{FormGrpoid}(\mathcal{X})$ is $\mathrm{diag}_{\mathcal{X}}$, and we have a canonical identification
$$\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \simeq \mathrm{FormGrpoid}(\mathcal{X})_{/\mathrm{diag}_{\mathcal{X}}}.$$

1.2.2. Consider the forgetful functor
$$\mathrm{diag} : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{FormGrpoid}(\mathcal{X}).$$

It admits a right adjoint, denoted $\mathrm{Inert}^{\mathrm{inf}}$, and given by Cartesian product (inside the category $\mathrm{FormGrpoid}(\mathcal{X})$) with the unit groupoid $\mathrm{diag}_{\mathcal{X}}$. Explicitly,
$$\mathrm{Inert}^{\mathrm{inf}}(\mathcal{R}) = \mathcal{X} \underset{(\mathcal{X} \times \mathcal{X})^{\wedge}}{\times} \mathcal{R}.$$

1.2.3. For $\mathcal{R} = \mathrm{diag}_{\mathcal{X}}$ being the identity groupoid, we thus obtain an object of $\mathrm{Grp}(\mathrm{Spc}_{/X})$, denoted $\mathrm{Inert}^{\mathrm{inf}}_{\mathcal{X}}$. We call it the *infinitesimal inertial group* of $\mathcal{X}$.

I.e., as an object of PreStk, we have:
$$\mathrm{Inert}^{\mathrm{inf}}_{\mathcal{X}} = \mathcal{X} \underset{(\mathcal{X} \times \mathcal{X})^{\wedge}}{\times} \mathcal{X}.$$

1.2.4. We reserve the notation $\mathrm{Inert}_{\mathcal{X}}$ for the object
$$\mathcal{X} \underset{\mathcal{X} \times \mathcal{X}}{\times} \mathcal{X} \in \mathrm{Grp}(\mathrm{PreStk}_{/\mathcal{X}}),$$
i.e., the *usual* (=non-infinitesimal) inertia group of $\mathcal{X}$.

It is easy to see that $\mathrm{Inert}^{\mathrm{inf}}_{\mathcal{X}}$ is obtained from $\mathrm{Inert}_{\mathcal{X}}$ by completion along the unit section.

1.2.5. There is another functor
$$\Omega^{\mathrm{fake}} : \mathrm{FormGrpoid}(\mathcal{X}) \to \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}).$$

Namely,
$$\Omega^{\mathrm{fake}}(\mathcal{R}) := \Omega_{\mathcal{X}}(\mathcal{R}),$$
where in the left-hand side $\Omega_{\mathcal{X}}$ is the loop functor $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$, where we view $\mathcal{R}$ as as an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ via
$$\mathrm{unit} : \mathcal{X} \rightleftarrows \mathcal{R} : p_s.$$

For example,
$$\mathrm{Inert}^{\mathrm{inf}}_{\mathcal{X}} = \Omega^{\mathrm{fake}}((\mathcal{X} \times \mathcal{X})^{\wedge}).$$

1.2.6.   Since $(\mathcal{X} \times \mathcal{X})^\wedge$ is the final object in $\mathrm{FormGrpoid}(\mathcal{X})$, for any groupoid $\mathcal{R}$ we have a tautological map $\mathcal{R} \to (\mathcal{X} \times \mathcal{X})^\wedge$, which gives rise to a map

$$\Omega^{\mathrm{fake}}(\mathcal{R}) \to \mathrm{Inert}_{\mathcal{X}}^{\mathrm{inf}} .$$

In addition, the unit map $\mathcal{X} \to \mathcal{R}$ gives rise to a map in $\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$

$$\mathrm{Inert}_{\mathcal{X}}^{\mathrm{inf}} \to \mathrm{Inert}^{\mathrm{inf}}(\mathcal{R}).$$

It is easy to see that

(1.1) $\Omega^{\mathrm{fake}}(\mathcal{R}) \to \mathrm{Inert}_{\mathcal{X}}^{\mathrm{inf}} \to \mathrm{Inert}^{\mathrm{inf}}(\mathcal{R})$

is a fiber sequence.

1.3. Inertia Lie algebras. In this subsection we will introduce Lie algebra counterparts of the constructions in Sect. 1.2.

1.3.1.   In what follows we denote

$$\mathrm{inert}_{\mathcal{X}} := \mathrm{Lie}_{\mathcal{X}}(\mathrm{Inert}_{\mathcal{X}}^{\mathrm{inf}}) \in \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})).$$

Note that

$$\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{inert}_{\mathcal{X}}) \simeq T(\mathcal{X})[-1].$$

1.3.2.   For $\mathcal{R} \in \mathrm{FormGrpoid}(\mathcal{X})$, denote

$$\mathrm{inert}(\mathcal{R}) := \mathrm{Lie}(\mathrm{Inert}^{\mathrm{inf}}(\mathcal{R})).$$

From the fiber sequence (1.1) we obtain a fiber sequence in $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$:

(1.2) $\mathrm{Lie}(\Omega^{\mathrm{fake}}(\mathcal{R})) \to \mathrm{inert}_{\mathcal{X}} \to \mathrm{inert}(\mathcal{R}).$

1.3.3.   If $\mathcal{R}$ is the groupoid corresponding to a formal moduli problem $\mathcal{X} \to \mathcal{Y}$ (i.e., $\mathcal{R} = \mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X}$), then

$$\mathrm{inert}(\mathcal{R}) \simeq \mathrm{inert}_{\mathcal{Y}}|_{\mathcal{X}}.$$

In particular,

$$\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{inert}(\mathcal{R})) \simeq T(\mathcal{Y})|_{\mathcal{X}}[-1].$$

The canonical map

$$\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{inert}_{\mathcal{X}}) \to \mathbf{oblv}_{\mathrm{Lie}}(\mathrm{inert}(\mathcal{R})),$$

induced by $\mathrm{Inert}_{\mathcal{X}}^{\mathrm{inf}} \to \mathrm{Inert}^{\mathrm{inf}}(\mathcal{R})$, is the shft by $[-1]$ of the differential $T(\mathcal{X}) \to T(\mathcal{Y})|_{\mathcal{X}}$.

Note also that

$$\mathbf{oblv}_{\mathrm{Lie}} \circ \mathrm{Lie}(\Omega^{\mathrm{fake}}(\mathcal{R})) \simeq T(\mathcal{X}/\mathcal{Y})[-1].$$

Applying $\mathbf{oblv}_{\mathrm{Lie}}$ to (1.2), we obtain the shift by $[-1]$ of the tautological exact triangle

$$T(\mathcal{X}/\mathcal{Y}) \to T(\mathcal{X}) \to T(\mathcal{Y})|_{\mathcal{X}}.$$

2. Lie algebroids: definition and basic pieces of structure

In this section we introduce the category $\mathrm{LieAlgbroid}(\mathcal{X})$ of Lie algebroids on $\mathcal{X}$ as the category of formal groupoids on $\mathcal{X}$ and study several forgetful functors to the categories $\mathrm{IndCoh}(\mathcal{X})$ and $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$, including those induced by the functors from Sect. 1.

2.1. Lie algebroids and the main forgetful functor.

We *define* the category $\mathrm{LieAlgbroid}(\mathcal{X})$ to be the same as $\mathrm{FormGrpoid}(\mathcal{X})$. The difference will only express itself in our point of view: we will (try to) view Lie algebroids as objects of a linear category (namely, $\mathrm{IndCoh}(X)$), equipped with an extra structure.

According to Chapter 5, Theorem 2.3.2, we can also identify

$$\mathrm{LieAlgbroid}(\mathcal{X}) \simeq \mathrm{FormMod}_{\mathcal{X}/}.$$

2.1.1.   Our 'main' forgetful functor is denoted

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} : \mathrm{LieAlgbroid}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X}))_{/T(\mathcal{X})},$$

and is constructed as follows:

It associates to a formal moduli problem $\mathcal{X} \to \mathcal{Y}$ the object of $\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$ equal to

$$T(\mathcal{X}/\mathcal{Y}) \to T(\mathcal{X}).$$

The functor $\mathbf{oblv}_{\mathrm{LieAlgbroid}/T}$ is conservative by Chapter 1, Proposition 8.3.2.

2.1.2.   We will think of a Lie algebroid $\mathfrak{L}$ as the corresponding object $\mathbf{oblv}_{\mathrm{LieAlgbroid}/T}(\mathfrak{L})$ of $\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$, abusively denoted by the same character $\mathfrak{L}$, equipped with an extra structure.

We shall denote by $\mathbf{oblv}_{\mathrm{LieAlgbroid}}$ the composition of $\mathbf{oblv}_{\mathrm{LieAlgbroid}/T}$ and the forgetful functor

$$\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \to \mathrm{IndCoh}(\mathcal{X}).$$

The corresponding map

$$(2.1) \qquad\qquad \mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}) \overset{\mathrm{anch}}{\to} T(\mathcal{X})$$

is usually referred to as the *anchor map.*

PROPOSITION 2.1.3.

(a) *The category* $\mathrm{LieAlgbroid}(\mathcal{X})$ *admits sifted colimits, and the functor* $\mathbf{oblv}_{\mathrm{LieAlgbroid}/T}$ *commutes with sifted colimits.*

(b) *The functor* $\mathbf{oblv}_{\mathrm{LieAlgbroid}/T}$ *admits a left adjoint.*

PROOF. Point (a) of the proposition follows from Chapter 5, Corollary 2.2.4. To prove point (b), by the Adjoint Functor Theorem, it is enough to show that the functor $\mathbf{oblv}_{\mathrm{LieAlgbroid}/T}$ commutes with limits, while the latter is obvious from the definitions.

$$\square$$

We will denote the functor

$$\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \to \mathrm{LieAlgbroid}(\mathcal{X}),$$

left adjoint to $\mathbf{oblv}_{\mathrm{LieAlgbroid}/T}$, by $\mathbf{free}_{\mathrm{LieAlgbroid}}$. In Sect. 5 we will clarify the geometric meaning of this functor.

2.1.4.   Note that Corollary 2.1.3 implies:

COROLLARY 2.1.5. *The functor*

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} : \mathrm{LieAlgbroid}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$$

is monadic.

2.1.6.   The above discussion can be rendered into the relative setting, where instead of the category $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$, we consider the category $(\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{/\mathcal{Z}}$ over a fixed $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$.

For $\mathcal{X} \in (\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{/\mathcal{Z}}$, we denote the resulting category of relative Lie algebroids by

$$\mathrm{LieAlgbroid}(\mathcal{X}/\mathcal{Z}).$$

Its natural forgetful functor, denoted by the same symbol $\mathbf{oblv}_{\mathrm{LieAlgbroid}/T}$ takes values in the category $\mathrm{IndCoh}(\mathcal{X})_{T(/\mathcal{X}/\mathcal{Z})}$. I.e., we now take tangent spaces *relative* to $\mathcal{Z}$.

2.2. From Lie algebroids to Lie algebras. It turns out that there are two forgetful functors from $\mathrm{LieAlgbroid}(\mathcal{X})$ to $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$, induced by the two functors from groupoids to groups in Sect. 1. We will explore these two functors in the present subsection.

2.2.1.   We define the functor

$$\mathrm{ker\text{-}anch} : \mathrm{LieAlgbroid}(\mathcal{X}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$$

so that the diagram

$$\begin{array}{ccc}
\mathrm{FormGrpoid}(\mathcal{X}) & \xrightarrow{\ \sim\ } & \mathrm{LieAlgbroid}(\mathcal{X}) \\
{\scriptstyle \mathrm{Inert}^{\mathrm{inf}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{ker\text{-}anch}} \\
\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) & \xrightarrow[\sim]{\ \mathrm{Lie}\ } & \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))
\end{array}$$

is commutative.

I.e., if $\mathfrak{L}$ is the algebroid corresponding to the groupoid $\mathcal{R}$, we have

$$\mathrm{ker\text{-}anch}(\mathfrak{L}) := \mathrm{inert}(\mathcal{R}),$$

in the notation of Sect. 1.3.2.

Note that by construction, for $\mathfrak{L} \in \mathrm{LieAlgbroid}(\mathcal{X})$, we have:

$$\mathbf{oblv}_{\mathrm{Lie}} \circ \mathrm{ker\text{-}anch}(\mathfrak{L}) \simeq \mathrm{Fib}\Big(\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}) \overset{\mathrm{anch}}{\to} T(\mathcal{X})\Big),$$

functorially in $\mathfrak{L}$.

In particular, the functor $\mathrm{ker\text{-}anch}$ is conservative.

2.2.2.   Another forgetful functor, denoted $\Omega^{\mathrm{fake}} : \mathrm{LieAlgbroid}(\mathcal{X}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$, is defined so that the diagram

$$
\begin{array}{ccc}
\mathrm{FormGrpoid}(\mathcal{X}) & \xrightarrow{\ \sim\ } & \mathrm{LieAlgbroid}(\mathcal{X}) \\
{\scriptstyle \Omega^{\mathrm{fake}}} \downarrow & & \downarrow {\scriptstyle \Omega^{\mathrm{fake}}} \\
\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) & \xrightarrow[\sim]{\ \mathrm{Lie}\ } & \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))
\end{array}
$$

commutes.

In particular, the fiber sequence (1.2) translates as

(2.2) $\Omega^{\mathrm{fake}}(\mathfrak{L}) \to \mathrm{inert}_{\mathcal{X}} \to \mathrm{ker\text{-}anch}(\mathfrak{L})$.

Note that by construction

$$
\mathbf{oblv}_{\mathrm{Lie}} \circ \Omega^{\mathrm{fake}}(\mathfrak{L}) \simeq \mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[-1].
$$

I.e., the object of $\mathrm{IndCoh}(\mathcal{X})$, underlying the shift by $[-1]$ of a Lie algebroid, carries a natural structure of Lie algebra in $\mathrm{IndCoh}(\mathcal{X})$.

The functor Ω^{fake} is also conservative.

REMARK 2.2.3.   In Sect. 7, we shall see that the object of Vect equal to global sections of $\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})$ for a Lie algebroid $\mathfrak{L}$ itself carries a structure of Lie algebra.

2.2.4.   We will refer to the canonical map

(2.3) $\Omega^{\mathrm{fake}}(\mathfrak{L}) \to \mathrm{inert}_{\mathcal{X}}$

as the *shifted anchor map*. After applying $\mathbf{oblv}_{\mathrm{Lie}}$, the map (2.3) becomes the shift by $[-1]$ of the map (2.1).

Applying $\mathbf{oblv}_{\mathrm{Lie}}$ to (2.2), we obtain a fiber sequence in $\mathrm{IndCoh}(\mathcal{X})$ that is equal to the shift by $[-1]$ of the tautological sequence

$$
\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{ker\text{-}anch}(\mathfrak{L})) \to \mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}) \to T(\mathcal{X}).
$$

2.2.5.   The functor ker-anch admits a left adjoint, denoted

$$
\mathrm{diag} : \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \to \mathrm{LieAlgbroid}(\mathcal{X}).
$$

Tautologically, it makes the following diagram commute

$$
\begin{array}{ccc}
\mathrm{FormGrpoid}(\mathcal{X}) & \xrightarrow{\ \sim\ } & \mathrm{LieAlgbroid}(\mathcal{X}) \\
{\scriptstyle \mathrm{diag}} \uparrow & & \uparrow {\scriptstyle \mathrm{diag}} \\
\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) & \xrightarrow[\sim]{\ \mathrm{Lie}\ } & \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})).
\end{array}
$$

We note:

LEMMA 2.2.6. *The following diagram of functors commutes:*

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{X}) & \xrightarrow{\mathbf{free}_{\mathrm{Lie}}} & \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \\ \downarrow & & \downarrow{\scriptstyle\mathrm{diag}} \\ \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} & \xrightarrow{\mathbf{free}_{\mathrm{LieAlgbroid}}} & \mathrm{LieAlgbroid}(\mathcal{X}), \end{array}$$

where the left vertical arrow sends $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ *to* $(\mathcal{F} \overset{0}{\to} T(\mathcal{X}))$.

PROOF. Follows by adjunction from the commutativity of the corresponding diagram of right adjoints

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{X}) & \xleftarrow{\mathbf{oblv}_{\mathrm{Lie}}} & \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \\ \uparrow & & \uparrow{\scriptstyle\mathrm{ker\text{-}anch}} \\ \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} & \xleftarrow{\mathbf{oblv}_{\mathrm{LieAlgbroid}/T}} & \mathrm{LieAlgbroid}(\mathcal{X}), \end{array}$$

where the left vertical arrow sends

$$(\mathcal{F} \overset{\gamma}{\to} T(\mathcal{X})) \mapsto \mathrm{Fib}(\gamma).$$

$\square$

3. Examples of Lie algebroids

In this section we discuss four main examples of Lie algebroids: the tangent algebroid, the zero algebroid, the Lie algebroid attached to a map, and the Atiyah algebroid attached to a perfect complex.

3.1. The tangent and zero Lie algebroids. In this subsection we introduce two most basic Lie algebroids.

3.1.1. The most basic example of a Lie algebroid is the final object of $\mathrm{LieAlgbroid}(\mathcal{X})$, denoted $\mathcal{T}(\mathcal{X})$. It is called the *tangent Lie algebroid*.

It corresponds to the formal moduli problem $\mathcal{X} \overset{p_{\mathrm{dR},\mathcal{X}}}{\longrightarrow} \mathcal{X}_{\mathrm{dR}}$. The corresponding groupoid is $(\mathcal{X} \times \mathcal{X})^{\wedge}$.

We have

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}/T}(\mathcal{T}(\mathcal{X})) = (T(\mathcal{X}) \overset{\mathrm{id}}{\to} T(\mathcal{X})).$$

We also have:

$$\mathrm{ker\text{-}anch}(\mathcal{T}(\mathcal{X})) = 0 \text{ and } \Omega^{\mathrm{fake}}(\mathcal{T}(\mathcal{X})) \simeq \mathrm{inert}_{\mathcal{X}}.$$

3.1.2. For a Lie algebroid $\mathfrak{L}$, we define the notion of *splitting* to be the right inverse of the canonical map $\mathfrak{L} \to \mathcal{T}(\mathcal{X})$.

3.1.3.   The initial object in $\mathrm{LieAlgbroid}(\mathcal{X})$ is the 'zero' Lie algebroid, denoted

$$0 \in \mathrm{LieAlgbroid}(\mathcal{X}).$$

It equals $\mathrm{diag}(0)$, and corresponds to the groupoid $\mathrm{diag}_{\mathcal{X}}$. The corresponding formal moduli problem is

$$\mathcal{X} \overset{\mathrm{id}}{\to} \mathcal{X}.$$

We have:

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}/T}(0) = (0 \to T(\mathcal{X})).$$

We also have

$$\mathrm{ker\text{-}anch}(0) = \mathrm{inert}_{\mathcal{X}} \ \text{ and } \ \Omega^{\mathrm{fake}}(0) = 0.$$

3.1.4.   Note that the composite endo-functor of $\mathrm{LieAlgbroid}(\mathcal{X})$

$$\mathrm{diag} \circ \Omega^{\mathrm{fake}}$$

identifies with

$$\mathfrak{L} \mapsto 0 \underset{\mathfrak{L}}{\times} 0,$$

where the fiber product is taken in $\mathrm{LieAlgbroid}(X)$.

3.2. The Lie algebroid attached to a map. In this subsection we discuss the Lie algebroid attached to a map of prestacks.

3.2.1.   Let $\mathcal{X} \to \mathcal{Y}$ be a map in $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Consider the corresponding map

$$\mathcal{X} \to \mathcal{Y}^{\wedge}_{\mathcal{X}},$$

where

$$\mathcal{Y}^{\wedge}_{\mathcal{X}} := \mathcal{X}_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y},$$

and the corresponding formal groupoid

$$(\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X})^{\wedge},$$

see Chapter 5, Sect. 2.3.3.

We denote the corresponding algebroid by $\mathcal{T}(\mathcal{X}/\mathcal{Y})$. We have

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}/T}(\mathcal{T}(\mathcal{X}/\mathcal{Y})) = (T(\mathcal{X}/\mathcal{Y}) \to T(\mathcal{X})).$$

We also have:

$$\mathrm{ker\text{-}anch}(\mathcal{T}(\mathcal{X}/\mathcal{Y})) \simeq \mathrm{inert}_{\mathcal{Y}}|_{\mathcal{X}},$$

and therefore

$$\mathbf{oblv}_{\mathrm{LieAlg}}(\mathrm{ker\text{-}anch}(\mathcal{T}(\mathcal{X}/\mathcal{Y}))) \simeq f^{!}(T(\mathcal{Y}))[-1].$$

3.2.2.   Note that we recover $\mathcal{T}(\mathcal{X})$ as $\mathcal{T}(\mathcal{X}/\mathrm{pt})$.

Note also that the zero Lie algebroid can be recovered as $\mathcal{T}(\mathcal{X}/\mathcal{X})$.

3.2.3.   By definition, the datum of *splitting* of the Lie algebroid $\mathcal{T}(\mathcal{X}/\mathcal{Y})$ is equivalent to that of factoring the map $\mathcal{X} \to \mathcal{Y}$ as

$$\mathcal{X} \overset{p_{\mathrm{dR},\mathcal{X}}}{\longrightarrow} \mathcal{X}_{\mathrm{dR}} \to \mathcal{Y}.$$

3.3. Digression: the universal classifying space. In this subsection we introduce the prestack responsible for the functor that sends an affine scheme to the (space underlying) the category of perfect complexes on this scheme. We will use this prestack in the next subsection in order to construct the Atiyah algebroid of a perfect complex.

3.3.1.   We define the prestack Perf by setting

$$\mathrm{Maps}(S, \mathrm{Perf}) = (\mathrm{QCoh}(S)^{\mathrm{perf}})^{\mathrm{Spc}}, \quad S \in \mathrm{Sch}^{\mathrm{aff}},$$

where we recall that the superscript 'Spc' stands for taking the space obtained from a given $(\infty, 1)$-category by discarding non-invertible morphisms.

PROPOSITION 3.3.2. *The prestack* Perf *belongs to* $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$.

PROOF. First, we note that Perf is convergent (see Volume I, Chapter 3, Proposition 3.6.10). In order to prove that Perf belongs to $\mathrm{PreStk}_{\mathrm{laft}}$, it is sufficient to show that the functor

$$S \mapsto \mathrm{QCoh}(S)^{\mathrm{perf}}$$

takes filtered limits (on all of $\mathrm{Sch}^{\mathrm{aff}}$) to colimits. However, this follows from [**DrGa2**, Lemma 1.9.5].

Thus, it remains to show that Perf admits deformation theory. This will be done in Sect. A.2.

$\square$

3.3.3.   We will now describe the Lie algebra $\mathrm{inert}_{\mathrm{Perf}}$.

Let $\mathcal{E}_{\mathrm{univ}}$ be the tautological object of $\mathrm{QCoh}(\mathrm{Perf})^{\mathrm{perf}}$. Consider the object

$$\mathrm{End}(\mathcal{E}_{\mathrm{univ}}) \in \mathrm{AssocAlg}(\mathrm{QCoh}(\mathrm{Perf})).$$

Applying the symmetric monoidal functor

$$\Upsilon : \mathrm{QCoh}(-) \to \mathrm{IndCoh}(-)$$

(see Volume I, Chapter 6, Sect. 3.3), we obtain an object

$$\Upsilon_{\mathrm{Perf}}(\mathrm{End}(\mathcal{E}_{\mathrm{univ}})) \in \mathrm{AssocAlg}(\mathrm{IndCoh}(\mathrm{Perf})).$$

We claim:

PROPOSITION 3.3.4. *The object* $\mathrm{inert}_{\mathrm{Perf}} \in \mathrm{LieAlg}(\mathrm{IndCoh}(\mathrm{Perf}))$ *identifies canonically with the Lie algebra obtained from* $\Upsilon_{\mathrm{Perf}}(\mathrm{End}(\mathcal{E}_{\mathrm{univ}}))$ *by applying the forgetful functor*

$$\mathbf{res}^{\mathrm{Assoc} \to \mathrm{Lie}} : \mathrm{AssocAlg}(\mathrm{IndCoh}(\mathrm{Perf})) \to \mathrm{LieAlg}(\mathrm{IndCoh}(\mathrm{Perf})).$$

PROOF. The rest of this subsection is devoted to the proof of this proposition.

3.3.5.   Consider first the object

$$\mathrm{Inert}_{\mathrm{Perf}} \in \mathrm{Grp}(\mathrm{PreStk}_{/\,\mathrm{Perf}}).$$

By definition, for $S \in \mathrm{Sch}^{\mathrm{aff}}$, the groupoid $\mathrm{Maps}(S, \mathrm{Inert}_{\mathrm{Perf}})$ consists of the data $(\mathcal{E}, g)$, where $\mathcal{E} \in \mathrm{QCoh}(S)^{\mathrm{perf}}$ and g is an automorphism of $\mathcal{E}$.

We need to show that the Lie algebra of the completion $\mathrm{Inert}_{\mathrm{Perf}}^{\mathrm{inf}}$ of $\mathrm{Inert}_{\mathrm{Perf}}$ along the unit section (obtained by the functor $\mathrm{Lie}_{\mathrm{Perf}}$ of Chapter 7, Theorem 3.6.2) identifies canonically with

$$\mathbf{res}^{\mathrm{Assoc}\to\mathrm{Lie}}\left(\Upsilon_{\mathrm{Perf}}(\mathrm{End}(\mathcal{E}_{\mathrm{univ}}))\right).$$

3.3.6.   Consider

$$\Upsilon_{\mathrm{Perf}}(\mathcal{E}_{\mathrm{univ}}) \in \mathrm{IndCoh}(\mathrm{Perf}).$$

The above description of $\mathrm{Inert}_{\mathrm{Perf}}$ implies that $\Upsilon_{\mathrm{Perf}}(\mathcal{E}_{\mathrm{univ}})$ naturally lifts to an object of

$$\mathrm{Inert}_{\mathrm{Perf}}\text{-mod}(\mathrm{IndCoh}(\mathrm{Perf}));$$

see Chapter 7, Sect. 5.1.1 for the notation.

In particular, by restriction, we can view $\Upsilon_{\mathrm{Perf}}(\mathcal{E}_{\mathrm{univ}})$ as an object of

$$\mathrm{Inert}_{\mathrm{Perf}}^{\mathrm{inf}}\text{-mod}(\mathrm{IndCoh}(\mathrm{Perf})).$$

By Chapter 7, Proposition 5.1.2, we can view $\Upsilon_{\mathrm{Perf}}(\mathcal{E}_{\mathrm{univ}})$ as an object of

$$\mathrm{inert}_{\mathrm{Perf}}\text{-mod}(\mathrm{IndCoh}(\mathrm{Perf})),$$

and by Chapter 6, Sect. 7.4 as an object of

$$U(\mathrm{inert}_{\mathrm{Perf}})\text{-mod}(\mathrm{IndCoh}(\mathrm{Perf})).$$

Hence, we obtain a map of associative algebras

$$U(\mathrm{inert}_{\mathrm{Perf}}) \to \mathrm{End}(\Upsilon_{\mathrm{Perf}}(\mathcal{E}_{\mathrm{univ}})) \simeq \Upsilon_{\mathrm{Perf}}(\mathrm{End}(\mathcal{E}_{\mathrm{univ}})).$$

By adjunction, we obtain a map of Lie algebras

(3.1) $$\mathrm{inert}_{\mathrm{Perf}} \to \mathbf{res}^{\mathrm{Assoc}\to\mathrm{Lie}}\left(\Upsilon_{\mathrm{Perf}}(\mathrm{End}(\mathcal{E}_{\mathrm{univ}}))\right).$$

It remains to see that the latter map is an isomorphism.

3.3.7.   By definition,

$$\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{inert}_{\mathrm{Perf}}) = T(\mathrm{Inert}_{\mathrm{Perf}}/\mathrm{Perf})|_{\mathrm{Perf}},$$

and deformation theory identifies the latter with $\mathbf{oblv}_{\mathrm{Assoc}}\left(\Upsilon_{\mathrm{Perf}}(\mathrm{End}(\mathcal{E}_{\mathrm{univ}}))\right)$.

Morover, by unwinding the constructions, we obtain that the resulting map

$$\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{inert}_{\mathrm{Perf}}) \to \mathbf{oblv}_{\mathrm{Assoc}}\left(\Upsilon_{\mathrm{Perf}}(\mathrm{End}(\mathcal{E}_{\mathrm{univ}}))\right)$$

equals the map obtained from (3.1) by applying the functor $\mathbf{oblv}_{\mathrm{Lie}}$.

Hence, we obtain that the map (3.1) induces an isomorphism of the underlying objects of $\mathrm{IndCoh}(\mathrm{Perf})$, as required.

$\square$

3.4. The Atiyah algebroid. In this subsection we introduce the Atiyah algebroid corresponding to an object of $\mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}}$ for $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Furthermore, we show, that as in the classical case, the Atiyah algebroid controls the obstruction to giving such an object a structure of crystal on $\mathcal{X}$.

3.4.1.   Recall that for $\mathcal{X} \in \mathrm{PreStk}$ the category

$$\mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}} \subset \mathrm{QCoh}(\mathcal{X})$$

is defined as

$$\lim_{S \in (\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{X}})^{\mathrm{op}}} \mathrm{QCoh}(S)^{\mathrm{perf}}.$$

Therefore,

$$\mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}} \simeq \mathrm{Maps}(\mathcal{X}, \mathrm{Perf}),$$

where Perf is as in Sect. 3.3.

3.4.2.   For $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$, and given an object $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}}$, and thus a map

$$\mathcal{X} \to \mathrm{Perf},$$

we define the Atiyah algebroid of $\mathcal{E}$, denoted $\mathrm{At}(\mathcal{E})$, to be $\mathcal{T}(\mathcal{X}/\mathrm{Perf})$.

Note that

$$\mathrm{ker\text{-}anch}(\mathrm{At}(\mathcal{E})) \simeq \mathrm{inert}(\mathrm{Perf})|_{\mathcal{X}},$$

and the latter identifies with $\Upsilon_{\mathcal{X}}(\mathrm{End}(\mathcal{E}))$ by Proposition 3.3.4.

3.4.3.   By Sect. 3.2.3, the datum of splitting of $\mathrm{At}(\mathcal{E})$ is equivalent to that of factoring the map $\mathcal{X} \to \mathrm{Perf}$, corresponding to $\mathcal{E}$, as

$$\mathcal{X} \overset{p_{\mathrm{dR},\mathcal{X}}}{\longrightarrow} \mathcal{X}_{\mathrm{dR}} \to \mathrm{Perf}.$$

I.e., this is equivalent to a structure of *left crystal* on $\mathcal{E}$, see [**GaRo2**, Sect. 2.1] for what this means.

According to [**GaRo2**, Proposition 2.4.4], this is equivalent to a structure of crystal on $\Upsilon_{\mathcal{X}}(\mathcal{E})$.

4. Modules over Lie algebroids and the universal enveloping algebra

4.1. Modules over Lie algebroids. In this subsection we introduce the notion of module over a Lie algebroid.

In particular, we show that for $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}}$, the ind-coherent sheaf $\Upsilon_{\mathcal{X}}(\mathcal{E}) \in \mathrm{IndCoh}(\mathcal{X})$ has a canonical structure of a module over the Atiyah algebroid $\mathrm{At}(\mathcal{E})$; Moreover, the Atiyah algebroid is the universal Lie algebroid that acts on $\Upsilon_{\mathcal{X}}(\mathcal{E})$; i.e. an action of an algebroid $\mathfrak{L}$ on $\Upsilon_{\mathcal{X}}(\mathcal{E})$ is equivalent to a map of Lie algebroids $\mathfrak{L} \to \mathrm{At}(\mathcal{E})$.

4.1.1.   Let $\mathfrak{L}$ be a Lie algebroid on $\mathcal{X}$, corresponding to a groupoid $\mathcal{R}$. We *define* the category $\mathfrak{L}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))$ to be

$$\mathrm{IndCoh}(\mathcal{X})^{\mathcal{R}},$$

see Chapter 5, Sect. 2.2.5 for the notation.

We let

$$\mathbf{ind}_{\mathfrak{L}} : \mathrm{IndCoh}(\mathcal{X}) \rightleftarrows \mathfrak{L}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) : \mathbf{oblv}_{\mathfrak{L}}$$

denote the corresponding adjoint pair of functors.

4.1.2. Let $(\mathcal{X} \xrightarrow{\pi} \mathcal{Y}) \in \mathrm{FormMod}_{\mathcal{X}/}$ be the object corresponding to $\mathfrak{L}$. By Chapter 5, Proposition 2.2.6, we have a canonical equivalence

$$\mathrm{IndCoh}(\mathcal{Y}) \simeq \mathfrak{L}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})).$$

Under this equivalence, the functor $\mathbf{oblv}_{\mathfrak{L}}$ corresponds to $\pi^!$, and the functor $\mathbf{ind}_{\mathfrak{L}}$ corresponds to π_*^{IndCoh}.

4.1.3. Assume for a moment that $\mathfrak{L}$ is of the form $\mathrm{diag}(\mathfrak{h})$ for $\mathfrak{h} \in \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})$. In this case, by Chapter 7, Sect. 5.2.1, we have a canonical identification

$$\mathfrak{L}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \simeq \mathfrak{h}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})).$$

Under this equivalence, the functor $\mathbf{oblv}_{\mathfrak{L}}$ goes over to $\mathbf{oblv}_{\mathfrak{h}}$, and the functor $\mathbf{ind}_{\mathfrak{L}}$ corresponds to $\mathbf{ind}_{\mathfrak{h}}$.

4.1.4. *Examples.* For $\mathfrak{L} = \mathcal{T}(\mathcal{X})$ we obtain:

$$\mathcal{T}(X)\text{-mod}(\mathrm{IndCoh}(X)) = \mathrm{IndCoh}(\mathcal{X}_{\mathrm{dR}}) =: \mathrm{Crys}(\mathcal{X}).$$

For $\mathfrak{L} = 0$, we have

$$\mathcal{T}(X)\text{-mod}(\mathrm{IndCoh}(X)) = \mathrm{IndCoh}(\mathcal{X}).$$

4.1.5. Let now $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}}$. By construction, $\Upsilon_{\mathcal{X}}(\mathcal{E})$ has a canonical structure of module over $\mathrm{At}(\mathcal{E})$.

Hence, for a Lie algebroid $\mathfrak{L}$, a homomorphism $\mathfrak{L} \to \mathrm{At}(\mathcal{E})$ defines on $\Upsilon_{\mathcal{X}}(\mathcal{E})$ a structure of $\mathfrak{L}$-module.

PROPOSITION 4.1.6. *The above map from the space of homomorphisms $\mathfrak{L} \to \mathrm{At}(\mathcal{E})$ to that of structures of $\mathfrak{L}$-module on $\Upsilon_{\mathcal{X}}(\mathcal{E})$ is an isomorphism.*

PROOF. Let $\mathcal{X} \xrightarrow{\pi} \mathcal{Y}$ be the object of $\mathrm{FormMod}_{\mathcal{X}/}$ corresponding to $\mathfrak{L}$. The space of homomorphisms $\mathfrak{L} \to \mathrm{At}(\mathcal{E})$ is isomorphic to the space of factorizations of the map $\mathcal{X} \to \mathrm{Perf}$, corresponding to $\mathcal{E}$ as

$$\mathcal{X} \xrightarrow{\pi} \mathcal{Y} \to \mathrm{Perf}.$$

I.e., this is the space of ways to write $\mathcal{E}$ as $\pi^*(\mathcal{E}')$ for $\mathcal{E}' \in \mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}}$.

The space of structures of $\mathfrak{L}$-module on $\Upsilon_{\mathcal{X}}(\mathcal{E})$ is isomorphic to the space of ways to write $\Upsilon_{\mathcal{X}}(\mathcal{E})$ as $\pi^!(\Upsilon_{\mathcal{Y}}(\mathcal{E}'))$. I.e., we have to show that the diagram of categories

$$
\begin{array}{ccc}
\mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}} & \xrightarrow{\Upsilon_{\mathcal{Y}}} & \mathrm{IndCoh}(\mathcal{Y}) \\
{\scriptstyle \pi^*}\big\downarrow & & \big\downarrow{\scriptstyle \pi^!} \\
\mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}} & \xrightarrow{\Upsilon_{\mathcal{X}}} & \mathrm{IndCoh}(\mathcal{X})
\end{array}
$$

is a pullback square. However, this follows by descent from Volume I, Chapter 6, Lemma 3.3.7.

$\square$

4.2. The universal enveloping algebra. In this subsection we associate to a Lie algebroid $\mathfrak{L}$ its universal enveloping algebra, viewed as an algebra object in the category of endo-functors of $\mathrm{IndCoh}(-)$.

4.2.1.   Let $\mathfrak{L}$ be a Lie algebroid on $\mathcal{X}$. Consider the monad on $\mathrm{IndCoh}(\mathcal{X})$ corresponding to the adjunction

$$\mathbf{ind}_{\mathfrak{L}} : \mathrm{IndCoh}(\mathcal{X}) \rightleftarrows \mathfrak{L}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) : \mathbf{oblv}_{\mathfrak{L}}.$$

We denote by $U(\mathfrak{L})$ the corresponding algebra object in the monoidal DG category

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})).$$

Tautologically,

$$\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L})) = \mathbf{oblv}_{\mathfrak{L}} \circ \mathbf{ind}_{\mathfrak{L}}.$$

4.2.2.   Assume for a moment that $\mathfrak{L}$ is of the form $\mathrm{diag}(\mathfrak{h})$ for $\mathfrak{h} \in \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$.

In this case, by Chapter 7, Proposition 5.1.2, $U(\mathfrak{L})$ is given by tensor product with $U(\mathfrak{h})$.

REMARK 4.2.3. In Chapter 9 we will see that $U(\mathfrak{L})$ possesses an extra structure: namely a filtration. This extra structure will allow us to develop *infinitesimal differential geometry* on prestacks.

4.3. The co-algebra structure. In the classical situation, the unversal enveloping algebra of a Lie algebroid, when considered as a left $\mathcal{O}_X$-module, has a natural structure of co-commutative co-algebra. In this subsection we will establish the corresponding property in the derived setting.

4.3.1.   Consider the functor

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})) \rightarrow \mathrm{IndCoh}(\mathcal{X}),$$

given by precomposition with

$$p_{\mathcal{X}}^! : \mathrm{Vect} \rightarrow \mathrm{IndCoh}(\mathcal{X}).$$

Let $U(\mathfrak{L})^L \in \mathrm{IndCoh}(\mathcal{X})$ denote the image of

$$\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L})) \in \mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X}))$$

under this functor.

The object $U(\mathfrak{L})^L$ corresponds to the functor

$$\mathbf{oblv}_{\mathfrak{L}} \circ \mathbf{ind}_{\mathfrak{L}} \circ p_{\mathcal{X}}^! : \mathrm{Vect} \rightarrow \mathrm{IndCoh}(\mathcal{X}).$$

4.3.2.   Note that the category $\mathfrak{L}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \simeq \mathrm{IndCoh}(\mathcal{Y})$ carries a natural symmetric monoidal structure, and the functor $\mathbf{oblv}_{\mathfrak{L}}$ is symmetric monoidal. Hence, the functor $\mathbf{ind}_{\mathfrak{L}}$ has a natural left-lax symmetric monoidal structure.

Hence, the functor $\mathbf{oblv}_{\mathfrak{L}} \circ \mathbf{ind}_{\mathfrak{L}} \circ p_{\mathcal{X}}^!$ also has a left-lax symmetric monoidal structure. This defines on $U(\mathfrak{L})^L \in \mathrm{IndCoh}(\mathcal{X})$ a structure of co-commutative co-algebra in the symmetric monoidal category $\mathrm{IndCoh}(\mathcal{X})$, and the map $0 \rightarrow \mathfrak{L}$ defines an augmentation.

4.3.3. Thus, we can view $U(\mathfrak{L})^L$ as an object of

$$\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(\mathcal{X})).$$

We are going to prove:

PROPOSITION 4.3.4. *There exists a canonical isomorphism in* $\mathrm{CocomCoalg}(\mathrm{IndCoh}(\mathcal{X}))$*:*

$$U(\mathfrak{L})^L \simeq \mathrm{Chev}^{\mathrm{enh}}(\Omega^{\mathrm{fake}}(\mathfrak{L})).$$

PROOF. Let $p_s, p_t : \mathcal{R} \rightrightarrows \mathcal{X}$ be the formal groupoid corresponding to $\mathfrak{L}$. We can rewrite the functor $\mathbf{oblv}_{\mathfrak{L}} \circ \mathbf{ind}_{\mathfrak{L}} \circ p_{\mathcal{X}}^!$ as

$$(p_s)_*^{\mathrm{IndCoh}} \circ p_{\mathcal{R}}^!$$

(here $p_{\mathcal{R}}$ is the projection $\mathcal{R} \to \mathrm{pt}$), where the left-lax symmetric monoidal structure comes from the symmetric monoidal structure on $p_{\mathcal{R}}^!$ and the left-lax symmetric monoidal structure on $(p_s)_*^{\mathrm{IndCoh}}$, the latter obtained by adjunction from the symmetric monoidal structure on $p_s^!$.

Let us regard $\mathcal{R}$ as an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ via the maps $\Delta_{\mathcal{X}}$ and p_s. Now, the statement of the proposition follows from Chapter 7, Sect. 5.2.2.

$\square$

5. Square-zero extensions and Lie algebroids

In this section, we will show that under the equivalence $\mathrm{LieAlgbroid}(\mathcal{X}) \simeq \mathrm{FormMod}_{\mathcal{X}/}$, free Lie algebroids on $\mathcal{X}$ correspond to square-zero extensions.

This is parallel to Chapter 7, Corollary 3.7.8, which says that *split* square zero extensions correspond to free Lie algebras.

5.1. Square-zero extensions of prestacks. Let X be a scheme. Consider the full subcategory

$$\mathrm{Sch}_{X/,\mathrm{inf}\text{-}\mathrm{closed}} \subset \mathrm{Sch}_{X/},$$

see Chapter 1, Sect. 5.1.2.

Recall that we have a pair of mutually adjoint functors

$$\mathrm{RealSqZ} : ((\mathrm{QCoh}(X)^{\leq -1})_{T^*(X)/})^{\mathrm{op}} \rightleftarrows \mathrm{Sch}_{X/,\mathrm{inf}\text{-}\mathrm{closed}},$$

where the right adjoint sends $(X \to Y) \mapsto T^*(X/Y)$.

We will now carry out parallel constructions in the setting of formal moduli problems under an arbitrary object of $\mathrm{PreStk}_{\mathrm{laft}\text{-}\mathrm{def}}$.

5.1.1. For $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft}\text{-}\mathrm{def}}$ consider the category $\mathrm{FormMod}_{\mathcal{X}/}$.

Consider the functor

$$(5.1) \qquad \mathrm{FormMod}_{\mathcal{X}/} \to \mathrm{IndCoh}(X)_{/T(\mathcal{X})}, \quad (\mathcal{X} \to \mathcal{Y}) \mapsto (T(\mathcal{X}/\mathcal{Y}) \to T(\mathcal{X})).$$

Note that under the equivalence

$$\mathrm{FormMod}_{\mathcal{X}/} \simeq \mathrm{LieAlgbroid}(\mathcal{X}),$$

the functor (5.1) corresponds to $\mathbf{oblv}_{\mathrm{LieAlgbroid}/T}$.

Hence, by Proposition 2.1.3(b), the functor (5.1) admits a left adjoint. In what follows, we shall denote the left adjoint to (5.1) by

$$\mathrm{RealSqZ} : \mathrm{IndCoh}(X)_{/T(\mathcal{X})} \to \mathrm{FormMod}_{\mathcal{X}/}.$$

The following diagram commutes by definition:

(5.2)
$$
\begin{array}{ccc}
\mathrm{FormMod}_{/\mathcal{X}} & \xrightarrow{\ \sim\ } & \mathrm{FormGrpoid}(\mathcal{X}) \\
\scriptstyle{\mathrm{RealSqZ}}\big\uparrow & & \sim\big\uparrow \\
\mathrm{IndCoh}(X)_{/T(\mathcal{X})} & \xrightarrow{\ \mathbf{free}_{\mathrm{LieAlgbroid}}\ } & \mathrm{LieAlgbroid}(\mathcal{X}).
\end{array}
$$

5.1.2.   We have:

LEMMA 5.1.3. *For any $(\mathcal{X} \overset{f}{\to} \mathcal{Z}) \in (\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{\mathcal{X}/}$ and $(\mathcal{F} \overset{\gamma}{\to} T(\mathcal{X})) \in$* $\mathrm{IndCoh}(X)_{/T(\mathcal{X})}$, *the space of extensions of f to a map*

$$
\mathrm{RealSqZ}(\mathcal{F} \overset{\gamma}{\to} T(\mathcal{X})) \to \mathcal{Z}
$$

is canonically isomorphic to that of nul-homotopies of the composed map

$$
\mathcal{F} \overset{\gamma}{\to} T(\mathcal{X}) \to T(\mathcal{Z})|_{\mathcal{X}}.
$$

PROOF. We can replace $\mathcal{Z}$ by

$$
\mathcal{Z}' := \mathcal{Z} \underset{\mathcal{Z}_{\mathrm{dR}}}{\times} \mathcal{X}_{\mathrm{dR}},
$$

so that $\mathcal{Z}' \in \mathrm{FormMod}_{\mathcal{X}/}$, and then the assertion follows from the definition. $\square$

5.1.4.   Recall the functor RealSplitSqZ of Chapter 7, Sect. 3.7. By Chapter 7, Proposition 3.7.3, it is the left adjoint to

$$
\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \to \mathrm{IndCoh}(\mathcal{X}), \quad (\mathcal{X} \to \mathcal{Y} \to \mathcal{X}) \mapsto T(\mathcal{Y}/\mathcal{X})|_{\mathcal{X}},
$$

and by construction corresponds under the equivalence

$$
\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \overset{\Omega_{\mathcal{X}}}{\longrightarrow} \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \overset{\mathrm{Lie}}{\longrightarrow} \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))
$$

to the functor

$$
\mathrm{IndCoh}(\mathcal{X}) \overset{[-1]}{\longrightarrow} \mathrm{IndCoh}(\mathcal{X}) \overset{\mathbf{free}_{\mathrm{Lie}}}{\longrightarrow} \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})).
$$

The commutative diagram of Lemma 2.2.6 translates into the commutative diagram

(5.3)
$$
\begin{array}{ccc}
\mathrm{IndCoh}(\mathcal{X}) & \xrightarrow{\ \mathrm{RealSplitSqZ}\,\circ\,[1]\ } & \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}) \\
\big\downarrow & & \big\downarrow \\
\mathrm{IndCoh}(X)_{/T(\mathcal{X})} & \xrightarrow{\ \mathrm{RealSqZ}\ } & \mathrm{FormMod}_{\mathcal{X}/},
\end{array}
$$

where the left vertical arrow sends $\mathcal{F} \mapsto (\mathcal{F} \overset{0}{\to} T(\mathcal{X}))$, and the right vertical arrow is the tautological forgetful functor.

5.2. Tangent complex of a square-zero extension. In this subsection we approach the following question: how to describe the relative tangent complex of a square-zero extension?

This question makes sense even for schemes, however, it turns out that it is more convenient to answer in the framework of arbitrary objects of $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$ and formal moduli problems.

By answering this question we will also arrive to an alternative definition of Lie algebroids as modules over a certain monad.

5.2.1.   From the commutative diagram (5.2) we obtain (compare with Chapter 7, Corollary 3.7.6):

COROLLARY 5.2.2. *The monad on* $\mathrm{IndCoh}(X)_{/T(\mathcal{X})}$ *given by the composition*

$$T(\mathcal{X}/-) \circ \mathrm{RealSqZ}$$

is canonically isomorphic to

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}.$$

In other words, Corollary 5.2.2 gives a description of the relative tangent complex of a square-zero extension in terms of the 'more linear' functor $\mathbf{free}_{\mathrm{LieAlgbroid}}$.

REMARK 5.2.3. In Sect. 5.3 we will give an 'estimate' of what the monad

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}$$

looks like when viewed as a plain endo-functor.

5.2.4.   From Corollary 2.1.5, we obtain:

COROLLARY 5.2.5. *There exists a canonical equivalence of categories*

$$(5.4) \qquad \mathrm{LieAlgbroid}(\mathcal{X}) \simeq (T(\mathcal{X}/-) \circ \mathrm{RealSqZ})\text{-}\mathrm{mod}(\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}).$$

Note that Corollary 5.2.5 implies that we can use the right-hand side of (5.4) as an alternative definition of the category $\mathrm{LieAlgbroid}(\mathcal{X})$.

5.3. Filtration on the free algebroid. The main result of this subsection, Proposition 5.3.2 gives an estimate of what the monad

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}} : \mathrm{IndCoh}(X)_{/T(\mathcal{X})} \to \mathrm{IndCoh}(X)_{/T(\mathcal{X})}$$

looks like as a plain endo-functor, see Proposition 5.3.2 below.

5.3.1.   The goal of this subsection is to prove:

PROPOSITION 5.3.2. *For* $(\mathcal{F} \xrightarrow{\gamma} T(\mathcal{X})) \in \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$, *the object*

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}\,/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}(\mathcal{F} \to T(\mathcal{X})) \in \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$$

can be naturally lifted to

$$(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0})_{/T(\mathcal{X})}$$

(where $T(\mathcal{X})$ *is regarded as a filtered object placed in degree* 0*), such that its associated graded identifies with*

$$\mathbf{oblv}_{\mathrm{Lie}} \circ \mathbf{free}_{\mathrm{Lie}}(\mathcal{F}) \xrightarrow{0} T(\mathcal{X})$$

with its natural grading.

The rest of this subsection is devoted to the proof of Proposition 5.3.2. In the proof we will appeal to the material from Chapter 9, Sect. 1. Let us explain the idea:

Given an object $(\mathcal{F} \xrightarrow{\gamma} T(\mathcal{X})) \in \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$, scaling γ to zero gives (by applying Chapter 9, Sect. 1) a filtration on $(\mathcal{F} \xrightarrow{\gamma} T(\mathcal{X}))$, such that the associated graded is $\mathcal{F} \xrightarrow{0} T(\mathcal{X})$.

The result then follows by applying $\mathbf{free}_{\mathrm{LieAlgbroid}}$ to this filtered object, because

$$\mathbf{free}_{\mathrm{LieAlgbroid}}((\mathcal{F} \xrightarrow{0} T(\mathcal{X}))$$

is the free Lie algebra generated by $\mathcal{F}$.

5.3.3.   Consider the following presheaves of categories

$$\mathcal{P}_1 \text{ and } \mathcal{P}_2, \quad (\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}})^{\mathrm{op}} \to 1\text{-Cat}.$$

The functor $\mathcal{P}_1$ sends an affine scheme S to

$$\mathrm{IndCoh}(\mathcal{X} \times S)_{/T(\mathcal{X})|_{\mathcal{X} \times S}}.$$

The functor $\mathcal{P}_2$ sends an affine scheme S to

$$\mathrm{FormMod}_{\mathcal{X} \times S //S}.$$

Here $\mathrm{FormMod}_{\mathcal{X} \times S //S}$ stands for formal moduli problems under $X \times S$, equipped with a map of prestacks to S.

The functors

$$\mathrm{RealSqZ}_{/S} : \mathrm{IndCoh}(\mathcal{X} \times S)_{/T(\mathcal{X})|_{\mathcal{X} \times S}} \rightleftarrows \mathrm{FormMod}_{\mathcal{X} \times S //S} : T(\mathcal{X} \times S/-)$$

give rise to a pair of natural transformations

$$(5.5) \qquad\qquad\qquad\qquad \mathcal{P}_1 \rightleftarrows \mathcal{P}_2,$$

see Sect. 5.4.1 below for the notation.

5.3.4.   We regard $\mathcal{P}_1$ and $\mathcal{P}_2$ as endowed with the *trivial* action of the monoid $\mathbb{A}^1$ (we refer the reader to Chapter 9, Sect. 1.2 for the formalism of actions of monoids on presheaves of categories). The functors in (5.5) are (obviously) $\mathbb{A}^1$-equivariant.

5.3.5.   We now consider the presheaf of categories $\mathcal{P}_0$, represented by the monoid $\mathbb{A}^1$, equipped with an action on itself by *multiplication*.

The object $(\mathcal{F} \xrightarrow{\gamma} T(\mathcal{X})) \in \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})})$ gives rise to a natural transformation

$$(5.6) \qquad\qquad\qquad\qquad \mathcal{P}_0 \to \mathcal{P}_1$$

defined as follows: the corresponding object of $\mathcal{P}_1(\mathbb{A}^1)$ is

$$\mathcal{F}|_{\mathcal{X} \times \mathbb{A}^1} \xrightarrow{\gamma_{\mathrm{scaled}}} T(\mathcal{X})|_{\mathcal{X} \times \mathbb{A}^1},$$

where the value of γ_{scaled} over $\lambda \in \mathbb{A}^1$ is $\lambda \cdot \gamma$.

It is easy to see that the above natural transformation $\mathcal{P}_0 \to \mathcal{P}_1$ has a canonical structure of *left-lax* equivariance with respect to $\mathbb{A}^1$.

Note that by Chapter 9, Lemma 1.5.5(a), the category of left-lax equivariant functors $\mathbb{A}^1 \to \mathcal{P}^1$ identifies with

$$(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil}, \geq 0})_{/T(\mathcal{X})}.$$

Under this identification, the above functor (5.6) is given by

$$\mathcal{F} \overset{\gamma}{\to} T(\mathcal{X}),$$

where $\mathcal{F}$ (resp., $T(\mathcal{X})$) is regarded as a filtered object placed in degree 1 (resp., 0).

5.3.6. Thus, we obtain that the composite functor

$$\mathcal{P}_0 \to \mathcal{P}_1 \to \mathcal{P}_2 \to \mathcal{P}_1$$

has a structure of left-lax equivariance with respect to $\mathbb{A}^1$.

The corresponding object of $(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil}, \geq 0})_{/T(\mathcal{X})}$ is the desired lift of

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}(\mathcal{F} \to T(\mathcal{X})).$$

$\square$

5.4. Pullbacks of square-zero extensions. In this subsection we will show that the functor

$$\mathrm{RealSqZ} : \mathrm{IndCoh}(X)_{/T(\mathcal{X})} \to \mathrm{FormMod}_{\mathcal{X}/}$$

introduced above, is compatible with base change.

This will allow us, in the next subsection, to compare RealSqZ with another notion of square-zero extension of a prestack, namely, the one from Chapter 1, Sect. 10.1.

5.4.1. Let $\mathcal{X}_0$ be an object of $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$, and let $\mathcal{X} \in (\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{/\mathcal{X}_0}$. The functor RealSqZ defines a functor

$$\mathrm{RealSqZ}_{/\mathcal{X}_0} : \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X}/\mathcal{X}_0)} \to (\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{\mathcal{X}//\mathcal{X}_0}.$$

5.4.2. Let $f_0 : \mathcal{Y}_0 \to \mathcal{X}_0$ be a map in $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$, and set $\mathcal{Y} := \mathcal{Y}_0 \underset{\mathcal{X}_0}{\times} \mathcal{X}$. Let f denote the resulting map $\mathcal{X} \to \mathcal{Y}$. Tautologically,

$$f^!(T(\mathcal{X}/\mathcal{X}_0)) \simeq T(\mathcal{Y}/\mathcal{Y}_0).$$

By adjunction, for

$$(\mathcal{F}_{\mathcal{X}} \xrightarrow{\gamma_{\mathcal{X}}} T(\mathcal{X}/\mathcal{X}_0)) \in \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X}/\mathcal{X}_0)}$$

and its pullback by means of $f^!$

$$(\mathcal{F}_{\mathcal{Y}} \xrightarrow{\gamma_{\mathcal{Y}}} T(\mathcal{Y}/\mathcal{Y}_0)) \in \mathrm{IndCoh}(\mathcal{Y})_{/T(\mathcal{Y}/\mathcal{Y}_0)},$$

we have a canonical map in $(\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{\mathcal{Y}//\mathcal{Y}_0}$

$$(5.7) \qquad \mathrm{RealSqZ}_{/\mathcal{Y}_0}(\gamma_{\mathcal{Y}}) \to \mathcal{Y}_0 \underset{\mathcal{X}_0}{\times} \mathrm{RealSqZ}_{/\mathcal{X}_0}(\gamma_{\mathcal{X}}).$$

We claim:

PROPOSITION 5.4.3. *The map* (5.7) *is an isomorphism.*

We can depict the assertion of Proposition 5.4.3 by the commutative diagram

$$(5.8)$$

$$
\begin{array}{ccc}
\mathrm{IndCoh}(\mathcal{Y})_{/T(\mathcal{Y}/\mathcal{Y}_0)} & \xleftarrow{\ f^!\ } & \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X}/\mathcal{X}_0)} \\
{\scriptstyle \mathrm{RealSqZ}_{/\mathcal{Y}_0}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{RealSqZ}_{/\mathcal{X}_0}} \\
(\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{\mathcal{Y}//\mathcal{Y}_0} & \xleftarrow{\ \mathcal{Y}_0 \underset{\mathcal{X}_0}{\times} -\ } & (\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{\mathcal{X}//\mathcal{X}_0}.
\end{array}
$$

5.4.4. *Proof of Proposition 5.4.3.* We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{IndCoh}(\mathcal{Y}) & \xleftarrow{\ f^!\ } & \mathrm{IndCoh}(\mathcal{X}) \\
\big\downarrow & & \big\downarrow \\
\mathrm{IndCoh}(\mathcal{Y})_{/T(\mathcal{Y}/\mathcal{Y}_0)} & \xleftarrow{\ f^!\ } & \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X}/\mathcal{X}_0)},
\end{array}
$$

where the vertical arrows are as in Lemma 2.2.6. Since the essential image of

$$\mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X}/\mathcal{X}_0)}$$

generates the target category under sifted colimits, and since the horizontal arrows in (5.8) commute with colimits, it suffices to show that the outer diagram in

$$
\begin{array}{ccc}
\mathrm{IndCoh}(\mathcal{Y}) & \xleftarrow{\ f^!\ } & \mathrm{IndCoh}(\mathcal{X}) \\
\big\downarrow & & \big\downarrow \\
\mathrm{IndCoh}(\mathcal{Y})_{/T(\mathcal{Y}/\mathcal{Y}_0)} & \xleftarrow{\ f^!\ } & \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X}/\mathcal{X}_0)} \\
{\scriptstyle \mathrm{RealSqZ}_{/\mathcal{Y}_0}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{RealSqZ}_{/\mathcal{X}_0}} \\
(\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{\mathcal{Y}//\mathcal{Y}_0} & \xleftarrow{\ \mathcal{Y}_0 \underset{\mathcal{X}_0}{\times} -\ } & (\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{\mathcal{X}//\mathcal{X}_0}
\end{array}
$$

commutes.

However, by Lemma 2.2.6, the outer diagram identifies with

$$
\begin{array}{ccc}
\mathrm{IndCoh}(\mathcal{Y}) & \xleftarrow{\ f^!\ } & \mathrm{IndCoh}(\mathcal{X}) \\
{\scriptstyle \mathbf{free}_{\mathrm{Lie}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{free}_{\mathrm{Lie}}} \\
\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{Y})) & \xleftarrow{\ f^!\ } & \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \\
{\scriptstyle B_{\mathcal{Y}}\circ\exp}\big\downarrow & & \big\downarrow{\scriptstyle B_{\mathcal{X}}\circ\exp} \\
\mathrm{Ptd}((\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{/\mathcal{Y}}) & \xleftarrow{\ \mathcal{Y}_0 \underset{\mathcal{X}_0}{\times} -\ } & \mathrm{Ptd}((\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{/\mathcal{X}}) \\
\big\downarrow & & \big\downarrow \\
(\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{\mathcal{Y}//\mathcal{Y}_0} & \xleftarrow{\ \mathcal{Y}_0 \underset{\mathcal{X}_0}{\times} -\ } & (\mathrm{PreStk}_{\mathrm{laft\text{-}def}})_{\mathcal{X}//\mathcal{X}_0}.
\end{array}
$$

Now, the commutativity of the latter diagram is manifest, since the middle vertical arrows are equivalences.

$$\square$$

5.5. Relation to another notion of square-zero extension. In this subsection, we will relate the category $\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$ and the functor RealSqZ to the construction considered in Chapter 1, Sect. 10.1.

5.5.1.   Assume for a moment that $\mathcal{X} = X \in \mathrm{Sch}_{\mathrm{aft}}$ and let us start with a map

$$T^*(X) \to \mathcal{I}, \quad \mathcal{I} \in \mathrm{Coh}(X)^{\leq -1}.$$

On the one hand, the construction of Chapter 1, Sect. 5.1, produces from $T^*(X) \to \mathcal{I}$ an object

$$\mathrm{RealSqZ}(T^*(X) \to \mathcal{I}) \in (\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-isom from } X} \subset (\mathrm{Sch}_{\mathrm{aft}})_{X/}.$$

On the other hand, setting $\mathcal{F} = \mathbb{D}_X^{\mathrm{Serre}}(\mathcal{I})$, we obtain an object

$$(\mathcal{F} \to T(X)) \in \mathrm{IndCoh}(X)_{/T(\mathcal{X})}.$$

It follows from that under the embedding

$$(\mathrm{Sch}_{\mathrm{aft}})_{\text{nil-isom from } X} \hookrightarrow \mathrm{FormMod}_{X/},$$

we have an isomorphism

$$\mathrm{RealSqZ}(T^*(X) \to \mathcal{I}) \simeq \mathrm{RealSqZ}(\mathcal{F} \to T(X)),$$

functorially in

$$(T^*(X) \to \mathcal{I}) \in ((\mathrm{Coh}(X)^{\leq -1})_{T^*(X)/})^{\mathrm{op}}.$$

Indeed, both objects satisfy the same universal property on the category $\mathrm{FormMod}_{X/}$.

5.5.2.   Let $\mathcal{X}$ be an object of PreStk, and let $\mathcal{I} \in \mathrm{QCoh}(\mathcal{X})^{\leq 0}$. In this case, the construction of Chapter 1, Sect. 10.1.1 produces a category (in fact, a space) $\mathrm{SqZ}(\mathcal{X}, \mathcal{I})$, equipped with a forgetful functor

(5.9) $$\mathrm{SqZ}(\mathcal{X}, \mathcal{I}) \to \mathrm{PreStk}_{\mathcal{X}/}.$$

5.5.3.   Assume now that $\mathcal{X} \in \mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Assume, moreover, that $\mathcal{I}$, regarded as an object of

$$\mathrm{QCoh}(\mathcal{X})^{\leq 0} \subset \mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}},$$

belongs to

$$\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)_{\mathrm{laft}}^{\mathrm{fake}} \subset \mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}},$$

see Chapter 1, Sect. 4.3.6 for what this means.

This condition can be rewritten as follows: for any $S \in (\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{aff}})_{/\mathcal{X}}$, the pullback $\mathcal{I}|_S \in \mathrm{QCoh}(S)^{\leq 0}$ has *coherent* cohomologies.

5.5.4.   Set

$$\mathcal{F} := \mathbb{D}_{\mathcal{X}}^{\mathrm{Serre}}(\mathcal{I}[1]) \in \mathrm{IndCoh}(\mathcal{X}).$$

We claim:

PROPOSITION 5.5.5.   *There exists a canonically defined isomorphism of spaces*

$$\mathrm{SqZ}(\mathcal{X}, \mathcal{I}) \simeq \mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\mathcal{F}, T(\mathcal{X}))$$

that makes the diagram

$$
\begin{array}{ccc}
\mathrm{SqZ}(\mathcal{X}, \mathcal{I}) & \longrightarrow & \mathrm{PreStk}_{\mathcal{X}/} \\
\sim \uparrow & & \uparrow \\
\mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\mathcal{F}, T(\mathcal{X})) & \xrightarrow{\mathrm{RealSqZ}} & \mathrm{FormMod}_{\mathcal{X}/}
\end{array}
$$

commute.

The rest of this subsection is devoted to the proof of Proposition 5.5.5.

5.5.6.   Note that

$$\mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\mathcal{F}, T(\mathcal{X})) \simeq \mathrm{Maps}_{\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)^{\mathrm{fake}}_{\mathrm{laft}}}(T^*(\mathcal{X}), \mathcal{I}[1]).$$

Hence, we have a map

$$\mathrm{SqZ}(\mathcal{X}, \mathcal{I}) \to \mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\mathcal{F}, T(\mathcal{X})),$$

given by the construction in Chapter 1, Sect. 10.2.

We will now construct the inverse map.

5.5.7.   For $(\mathcal{F}_{\mathcal{X}} \overset{\gamma_{\mathcal{X}}}{\to} T(\mathcal{X})) \in \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$ set

$$(\mathcal{X} \hookrightarrow \mathcal{X}') \coloneqq \mathrm{RealSqZ}(\gamma_{\mathcal{X}}) \in \mathrm{FormMod}_{\mathcal{X}/}.$$

We claim that the object

$$(\mathcal{X} \hookrightarrow \mathcal{X}') \in \mathrm{PreStk}_{\mathcal{X}/},$$

constructed above has a natural structure of an object of $\mathrm{SqZ}(\mathcal{X}, \mathcal{I})$.

It will be clear by unwinding the constructions that the two functors

$$\mathrm{SqZ}(\mathcal{X}, \mathcal{I}) \leftrightarrow \mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\mathcal{F}, T(\mathcal{X}))$$

are inverses of each other.

5.5.8.   Let S' be an object of $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, equipped with a map $f' : S' \to \mathcal{X}'$. Set

$$S \coloneqq S' \underset{\mathcal{X}'}{\times} \mathcal{X},$$

and let f denote the resulting map $S \to \mathcal{X}$. Denote $\mathcal{F}_S \coloneqq f^!(\mathcal{F}_{\mathcal{X}})$.

Note that Proposition 5.4.3 implies that $S \to S'$ has a canonical structure of square-zero extension by means of $\mathcal{I}_S \coloneqq \mathbb{D}^{\mathrm{Serre}}_S(\mathcal{F}_S)[-1]$. Hence, it remains to show that $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$.

5.5.9.   To prove that $S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, it is enough to show that $T^*(S)|_{\mathrm{red}S} \in \mathrm{Coh}(^{\mathrm{red}}S)^{\leq 0}$. We have an exact triangle

$$T^*(S/S')|_{\mathrm{red}S} \to T^*(S)|_{\mathrm{red}S} \to T^*(S')|_{\mathrm{red}S},$$

so it suffices to show that $T^*(S/S')|_{\mathrm{red}S} \in \mathrm{Coh}(^{\mathrm{red}}S)^{\leq 0}$.

We have:

$$T^*(S/S')|_{\mathrm{red}S} = \mathbb{D}^{\mathrm{Serre}}_{\mathrm{red}S}(T(S/S')|_{\mathrm{red}S}),$$

where $\mathbb{D}^{\mathrm{Serre}}$ is understood in the sense of Chapter 1, Corollary 4.3.8.

By Proposition 5.3.2, $T(S/S')|_{\mathrm{red}S}$ has a canonical filtration indexed by positive integers, with the d-th sub-quotient isomorphic to the d graded component $(\mathbf{oblv}_{\mathrm{Lie}} \circ \mathbf{free}_{\mathrm{Lie}}(\mathcal{F}_{\mathrm{red}S}))^d$ of $\mathbf{oblv}_{\mathrm{Lie}} \circ \mathbf{free}_{\mathrm{Lie}}(\mathcal{F}_{\mathrm{red}S})$.

The required assertion follows now from the fact that for every d,

$$\mathbb{D}^{\mathrm{Serre}}_{\mathrm{red}S}((\mathbf{oblv}_{\mathrm{Lie}} \circ \mathbf{free}_{\mathrm{Lie}}(\mathcal{F}_{\mathrm{red}S}))^d) \simeq \left(\mathrm{Lie}(d) \otimes \mathcal{I}_{\mathrm{red}S}[1]^{\otimes d}\right)^{\Sigma_d},$$

and hence lives in cohomological degrees $\leq -d$.

5.6. What is the general framework for the definition of Lie algebroids?
Here is a general categorical framework for the definition of 'broids' that our construction of Lie algebroids fits in.

5.6.1. Let $\mathcal{C}$ be an ∞-category with finite limits, and in particular, a final object $* \in \mathcal{C}$. Let $\mathcal{C}_*$ be the corresponding pointed category, i.e., $\mathcal{C}_* := \mathcal{C}_{*/}$.

Let $\mathcal{D}$ denote the stabilization of $\mathcal{C}_*$, i.e., the category of spectrum objects on $\mathcal{C}_*$. According to [**Lu2**, Corollary 1.4.2.17], this is a stable ∞-category. Let RealSplitSqZ denote the forgetful functor $\mathcal{D} \to \mathcal{C}_*$, i.e., what is usually denoted Ω^∞.

5.6.2. Consider the functor
$$\mathcal{D} \overset{\text{RealSplitSqZ}}{\longrightarrow} \mathcal{C}_* \to \mathcal{C},$$
where the second arrow is the forgetful functor.

Let us assume that this functor has a left adjoint, to be denoted coTan.

5.6.3. Note that for any $y \in \mathcal{C}$ we have a tautologically defined map $\text{coTan}(y) \to \text{coTan}(*)$.

Consider now the functor
$$(5.10) \quad \text{coTan}_{\text{rel}} : \mathcal{C} \to \mathcal{D}_{\text{coTan}(*)/}, \quad \text{coTan}_{\text{rel}}(y) := \text{coFib}(\text{coTan}(y) \to \text{coTan}(*)).$$

Assume that this functor also admits a left adjoint, to be denoted
$$\text{RealSqZ} : \mathcal{D}_{\text{coTan}(*)/} \to \mathcal{C}.$$

Consider the comonad
$$\text{coTan}_{\text{rel}} \circ \text{RealSqZ}$$
acting on $\mathcal{D}_{\text{coTan}(*)/}$.

The 'broids' that we have in mind are by definition objects of the category
$$(\text{coTan}_{\text{rel}} \circ \text{RealSqZ})\text{-comod}(\mathcal{D}_{\text{coTan}(*)/}).$$
The functor $\text{coTan}_{\text{rel}}$ of (5.10) upgrades to a functor
$$\text{coTan}_{\text{rel}}^{\text{enh}} : \mathcal{C} \to (\text{coTan}_{\text{rel}} \circ \text{RealSqZ})\text{-comod}(\mathcal{D}_{\text{coTan}(*)/}).$$

The above functor $\text{coTan}_{\text{rel}}^{\text{enh}}$ is *not* an equivalence in general, but it happens to be one in our particular example, see Sect. 5.6.5.

5.6.4. By contrast, the category of 'bras' is constructed as follows. We consider the functor
$$\text{RealSplitSqZ} \circ [1] : \mathcal{D} \to \mathcal{C}_*,$$
and its left adjoint
$$\mathcal{C}_* \to \mathcal{C} \overset{\text{coTan}_{\text{rel}}}{\longrightarrow} \mathcal{D};$$
we denote it by $\text{coTan}_{\text{rel}}$ by a slight abuse of notation.

The category of 'bras' is:
$$(\text{coTan}_{\text{rel}} \circ \text{RealSplitSqZ} \circ [1])\text{-comod}(\mathcal{D}).$$

The functor $\text{coTan}_{\text{rel}}$ upgrades to a functor
$$\text{coTan}_{\text{rel}}^{\text{enh}} : \mathcal{C}_* \to \text{RealSplitSqZ} \circ [1])\text{-comod}(\mathcal{D}).$$

This functor $\text{coTan}_{\text{rel}}^{\text{enh}}$ is also *not* an equivalence in general, but it happens to be one in the example of Sect. 5.6.5.

5.6.5.   In our case, we apply the above discussion to $\mathcal{C} = (\mathrm{FormMod}_{\mathcal{X}/})^{\mathrm{op}}$, so that

$$\mathcal{C}_* = \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}}).$$

Recall that by Chapter 7, Proposition 3.7.12, the functor

$$\mathrm{RealSplitSqZ} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$$

identifies $(\mathrm{IndCoh}(\mathcal{X}))^{\mathrm{op}}$ with the stabilization of $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})^{\mathrm{op}}$.

Now, we claim that the notion of 'broid' (resp. 'bra') defined above recovers the notion of Lie algebroid on $\mathcal{X}$ (resp., Lie algebra in $\mathrm{IndCoh}(\mathcal{X})$). Indeed, this follows from Corollary 5.2.5 (resp., Chapter 7, Corollary 3.7.6).

6. IndCoh **of a square-zero extension**

The goal of this section is to describe the category of ind-coherent sheaves on a square-zero extension.

First, we show that every ind-coherent sheaf on $\mathcal{X}$ has a canonical action of the Lie algebra inert_X. We then use this fact to give an algebraic description of the category of ind-coherent sheaves on a square-zero extension.

Subsequently, we show that the dualizing sheaf of a square-zero extension of $\mathcal{X}$ is naturally an extension of the direct image of the 'defining ideal' by the direct image image of the dualizing sheaf of $\mathcal{X}$.

6.1. Modules for the inertia Lie algebra. In this subsection we observe that any object of $\mathrm{IndCoh}(-)$ acquires a canonical action of the inertia Lie algebra.

6.1.1.   Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Recall the infinitesimal inertia group $\mathrm{Inert}_{\mathcal{X}}^{\mathrm{inf}}$ and its Lie algebra $\mathrm{inert}_{\mathcal{X}}$.

By Chapter 7, Sect. 5.2.1, we have:

$$\mathrm{inert}_{\mathcal{X}}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \simeq \mathrm{IndCoh}((\mathcal{X} \times \mathcal{X})^{\wedge}),$$

where the forgetful functor

$$\mathbf{oblv}_{\mathrm{inert}_{\mathcal{X}}} : \mathrm{inert}_{\mathcal{X}}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \to \mathrm{IndCoh}(\mathcal{X})$$

corresponds to

$$\Delta_{\mathcal{X}}^{!} : \mathrm{IndCoh}((\mathcal{X} \times \mathcal{X})^{\wedge}) \to \mathrm{IndCoh}(\mathcal{X}),$$

and the functor

$$\mathbf{triv}_{\mathrm{inert}_{\mathcal{X}}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{inert}_{\mathcal{X}}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))$$

corresponds to

$$p_s^{!} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}((\mathcal{X} \times \mathcal{X})^{\wedge}).$$

6.1.2. Note, however, that the functor

$$p_t^! : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}((\mathcal{X} \times \mathcal{X})^\wedge)$$

gives rise to another symmetric monoidal functor, denoted

$$\mathbf{can} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{inert}_\mathcal{X}\text{-}\mathrm{mod}(\mathrm{IndCoh}(\mathcal{X})),$$

equipped with an isomorphism

(6.1) $$\mathbf{oblv}_{\mathrm{inert}_\mathcal{X}} \circ \mathbf{can} = \mathrm{Id}_{\mathrm{IndCoh}(\mathcal{X})} .$$

The datum of the functor **can** and the isomorphism (6.1) is equivalent to a functorial assignment to any $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ of a structure of $\mathrm{inert}_\mathcal{X}$-module.

6.1.3. By construction, for $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$, a datum of isomorphism

$$\mathbf{can}(\mathcal{F}) \simeq \mathbf{triv}_{\mathrm{inert}_\mathcal{X}}(\mathcal{F}) \in \mathrm{inert}_\mathcal{X}\text{-}\mathrm{mod}(\mathrm{IndCoh}(\mathcal{X}))$$

is equivalent to that of an isomorphism

$$p_s^!(\mathcal{F}) \simeq p_t^!(\mathcal{F}) \in \mathrm{IndCoh}((\mathcal{X} \times \mathcal{X})^\wedge).$$

This datum is strictly weaker than that of descent of $\mathcal{F}$ with respect to the groupoid $(\mathcal{X} \times \mathcal{X})^\wedge$, i.e., a structure of crystal.

6.1.4. Assume for a moment that $\mathcal{F} = \Upsilon_\mathcal{X}(\mathcal{E})$ for $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}}$.

Consider the canonical map in $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$:

$$\mathrm{inert}_\mathcal{X} \to \mathrm{ker\text{-}anch}(\mathrm{At}(\mathcal{E})) \simeq \Upsilon_\mathcal{X}(\mathrm{End}(\mathcal{E})).$$

By Proposition 4.1.6, the datum of such a map is equivalent to that of structure of $\mathrm{inert}_\mathcal{X}$-module on $\Upsilon_\mathcal{X}(\mathcal{E})$. One can show that this is the same structure as given by the functor **can**, applied to $\Upsilon_\mathcal{X}(\mathcal{E})$.

6.2. The canonical split square-zero extension. In this section we observe that for any object $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ there exists a canonical (a.k.a. Atiyah) map $T(\mathcal{X})[-1] \overset{!}{\otimes} \mathcal{F} \to \mathcal{F}$.

We will see that this map is induced by the action of the Lie algebra $\mathrm{inert}_\mathcal{X}$ on $\mathcal{F}$, using the fact that $\mathbf{oblv}_{\mathrm{LieAlg}}(\mathrm{inert}_\mathcal{X}) = T(\mathcal{X})[-1]$.

6.2.1. Consider again the object

$$\Delta_\mathcal{X} : \mathcal{X} \to (\mathcal{X} \times \mathcal{X})^\wedge : p_s$$

in $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$. We have

$$T((\mathcal{X} \times \mathcal{X})^\wedge / \mathcal{X})_\mathcal{X} \simeq T(\mathcal{X}).$$

Hence, applying Chapter 7, Proposition 3.7.3 to the identity map $T(\mathcal{X}) \to T(\mathcal{X})$, we obtain a canonically defined map

$$\mathrm{RealSplitSqZ}(T(\mathcal{X})) \to (\mathcal{X} \times \mathcal{X})^\wedge,$$

such that the composition

$$\mathrm{RealSplitSqZ}(T(\mathcal{X})) \to (\mathcal{X} \times \mathcal{X})^\wedge \overset{p_s}{\to} \mathcal{X}$$

is the tautological projection $\mathrm{RealSplitSqZ}(T(\mathcal{X})) \to \mathcal{X}$.

6.2.2. Consider now the composition

$$\mathrm{RealSplitSqZ}(T(\mathcal{X})) \to (\mathcal{X} \times \mathcal{X})^{\wedge} \overset{p_t}{\to} \mathcal{X};$$

we denote it by $\mathfrak{d}$ (cf. Chapter 1, Sect. 4.5.1).

By Lemma 5.1.3, the map $\mathfrak{d}$ corresponds to a particular choice of the null-homotopy of the map

$$T(\mathcal{X})[-1] \overset{0}{\to} T(\mathcal{X}) \overset{\mathrm{id}}{\longrightarrow} T(\mathcal{X}).$$

Unwinding the definitions, the above null-homotopy is given by the identity map on $T(\mathcal{X})$.

6.2.3. Identifying

$$\mathrm{IndCoh}(\mathrm{RealSplitSqZ}(T(\mathcal{X}))) \simeq \mathbf{free}_{\mathrm{Lie}}(T(\mathcal{X})[-1])\text{-mod}(\mathrm{IndCoh}(X))$$

(see Chapter 7, Sect. 5.2.1), we obtain a functor

$$\mathrm{IndCoh}(X) \overset{\mathbf{can}}{\longrightarrow} \mathrm{IndCoh}((\mathcal{X} \times \mathcal{X})^{\wedge}) \to$$
$$\to \mathrm{IndCoh}(\mathrm{RealSplitSqZ}(T(\mathcal{X}))) \simeq \mathbf{free}_{\mathrm{Lie}}(T(\mathcal{X})[-1])\text{-mod}(\mathrm{IndCoh}(X)).$$

We denote this functor by

$$\mathbf{can_{free}} : \mathrm{IndCoh}(X) \to \mathbf{free}_{\mathrm{Lie}}(T(\mathcal{X})[-1])\text{-mod}(\mathrm{IndCoh}(X)).$$

Its composition with the forgetful functor

$$\mathbf{oblv_{free_{Lie}(T(\mathcal{X})[-1])}} : \mathbf{free}_{\mathrm{Lie}}(T(\mathcal{X})[-1])\text{-mod}(\mathrm{IndCoh}(X)) \to \mathrm{IndCoh}(\mathcal{X})$$

is the identity functor, i.e.,

$$(6.2) \qquad\qquad \mathbf{oblv_{free_{Lie}(T(\mathcal{X})[-1])}} \circ \mathbf{can_{free}} \simeq \mathrm{Id}_{\mathrm{IndCoh}(\mathcal{X})}.$$

6.2.4. The datum of the functor $\mathbf{can_{free}}$ and the isomorphism (6.2) is equivalent to a functorial assignment to any $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ of a map

$$(6.3) \qquad\qquad \alpha_{\mathcal{F}} : T(\mathcal{X})[-1] \overset{!}{\otimes} \mathcal{F} \to \mathcal{F}.$$

Note that by construction, for $\mathcal{F}' \in \mathrm{IndCoh}(\mathcal{X}_{\mathrm{dR}})$, the map

$$(6.4) \qquad \alpha_{(p_{\mathcal{X},\mathrm{dR}})^{!}(\mathcal{F}')} : T(\mathcal{X})[-1] \overset{!}{\otimes} (p_{\mathcal{X},\mathrm{dR}})^{!}(\mathcal{F}') \to (p_{\mathcal{X},\mathrm{dR}})^{!}(\mathcal{F}')$$

is canonically trivialized.

6.2.5. By construction, the map

$$\mathbf{free}_{\mathrm{Lie}}(T(\mathcal{X})[-1]) \to \mathrm{inert}_{\mathcal{X}}$$

coming from the identification $\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{inert}_{\mathcal{X}}) \simeq T(\mathcal{X})[-1]$, induces a commutative diagram

$$
\begin{array}{ccc}
\mathrm{IndCoh}(\mathcal{X}) & \overset{\mathbf{can}}{\longrightarrow} & \mathrm{inert}_{\mathcal{X}}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})) \\
{\scriptstyle \mathrm{Id}}\downarrow & & \downarrow \\
\mathrm{IndCoh}(\mathcal{X}) & \overset{\mathbf{can_{free}}}{\longrightarrow} & \mathbf{free}_{\mathrm{Lie}}(T(\mathcal{X})[-1])\text{-mod}(\mathrm{IndCoh}(X)) \\
{\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv_{free_{Lie}(T(\mathcal{X})[-1])}}} \\
\mathrm{IndCoh}(\mathcal{X}) & \overset{\mathrm{Id}}{\longrightarrow} & \mathrm{IndCoh}(\mathcal{X}).
\end{array}
$$

6.3. Description of IndCoh **of a square-zero extension.** In this subsection we will give an explicit description of the category $\mathrm{IndCoh}(-)$ on a square-zero extension.

6.3.1. Let $\gamma : \mathcal{F} \to T(\mathcal{X})$ be an object of $\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$. Consider the following category, denoted $\mathrm{Annul}(\mathcal{F}, \gamma)$:

It consists of objects $\mathcal{F}' \in \mathrm{IndCoh}(\mathcal{F})$, equipped with a null-homotopy for the map

$$\mathcal{F}[-1] \overset{!}{\otimes} \mathcal{F}' \to T(\mathcal{X})[-1] \overset{!}{\otimes} \mathcal{F}' \overset{\alpha'_{\mathcal{F}}}{\to} \mathcal{F}'.$$

We have a tautological forgetful functor

$$\mathrm{Annul}(\mathcal{F}, \gamma) \to \mathrm{IndCoh}(\mathcal{X}).$$

6.3.2. Consider now the object

$$\mathrm{RealSqZ}(\mathcal{F}, \gamma) \in \mathrm{FormMod}_{\mathcal{X}/}.$$

In this subsection we will prove (cf. Chapter 1, Sect. 5.1.1):

THEOREM 6.3.3. *There exists a canonically defined equivalence of categories*

$$\mathrm{Annul}(\mathcal{F}, \gamma) \simeq \mathrm{IndCoh}(\mathrm{RealSqZ}(\mathcal{F}, \gamma))$$

that commutes with the forgetful functors to $\mathrm{IndCoh}(\mathcal{X})$.

The rest of this subsection is devoted to the proof of Theorem 6.3.3.

6.3.4. *Step 1.* We first construct the functor

$$(6.5) \qquad \mathrm{IndCoh}(\mathrm{RealSqZ}(\mathcal{F}, \gamma)) \to \mathrm{Annul}(\mathcal{F}, \gamma).$$

Let $f : \mathcal{X} \to \mathcal{Y}$ be an object of $\mathrm{FormMod}_{\mathcal{X}/}$. It follows from the definitions, that for $\mathcal{F}' \in \mathrm{IndCoh}(\mathcal{Y})$, the map

$$T(\mathcal{X}/\mathcal{Y})[-1] \overset{!}{\otimes} f^!(\mathcal{F}') \to T(\mathcal{X})[-1] \overset{!}{\otimes} f^!(\mathcal{F}') \overset{\alpha_{f^!(\mathcal{F})}}{\longrightarrow} f^!(\mathcal{F}')$$

is equipped with a canonical null-homotopy.

Applying this to $\mathcal{Y} := \mathrm{RealSqZ}(\mathcal{F}, \gamma)$, and composing with the tautological map

$$\mathcal{F} \to T(\mathcal{X}/\mathrm{RealSqZ}(\mathcal{F}, \gamma)),$$

we obtain the desried functor (6.5).

6.3.5. *Step 2.* It is easy to see that the forgetful functor

$$\mathrm{Annul}(\mathcal{F}, \gamma) \to \mathrm{IndCoh}(\mathcal{X})$$

is monadic. Let $\mathsf{M}_{\mathcal{F}, \gamma}$ denote the corresponding monad.

By Step 1, we obtain a map of monads

$$(6.6) \qquad \mathsf{M}_{\mathcal{F}, \gamma} \to U(\mathbf{free}_{\mathrm{LieAlgbroids}}(\mathcal{F}, \gamma)).$$

To prove the proposition, it remains to show that the map (6.6) is an isomorphism.

6.3.6. *Step 3.* We claim that both sides in (6.6), and the map between them, can be naturally upgraded to the category

$$\mathrm{AssocAlg}\left((\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil}, \geq 0}\right).$$

Indeed, this enhancement corresponds to the $\mathbb{A}^1$-family that deforms γ to the 0 map, as in Sect. 5.3.5.

Since the functor ass. gr. is conservative on $(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}, \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil}, \geq 0}$, it suffices to show that the map (6.6) induces an isomorphism at the associated graded level.

This reduces the verification of the isomorphism (6.6) to the case when γ is the 0 map.

6.3.7. *Step 4.* Note that when $\gamma = 0$, the category $\mathrm{Annul}(\mathcal{F}, \gamma)$ identifies with that of objects $\mathcal{F}' \in \mathrm{IndCoh}(\mathcal{X})$, equipped with a map

$$\mathcal{F} \overset{!}{\otimes} \mathcal{F}' \to \mathcal{F}'.$$

I.e., $\mathrm{Annul}(\mathcal{F}, 0) \simeq \mathbf{free}_{\mathrm{Assoc}}(\mathcal{F})\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))$, and the monad $\mathsf{M}_{\mathcal{F}, \gamma}$ is given by tensor product with $\mathbf{free}_{\mathrm{Assoc}}(\mathcal{F})$.

Similarly, the monad $U(\mathbf{free}_{\mathrm{LieAlgbroids}}(\mathcal{F}, 0))$ is given by tensor product with $U(\mathbf{free}_{\mathrm{Lie}}(\mathcal{F}))$.

Unwinding the definitions, we obtain that the map (6.6) corresponds to the map

$$\mathbf{free}_{\mathrm{Assoc}}(\mathcal{F}) \to U(\mathbf{free}_{\mathrm{Lie}}(\mathcal{F})),$$

and hence is an isomorphism.

6.4. The dualizing sheaf of a square-zero extension. As a corollary of Theorem 6.3.3 we obtain the following fact that justifies the terminology 'square-zero extension'.

6.4.1. Let $\mathcal{X}, \mathcal{F}, \gamma$ be as above. Denote

$$\mathcal{X}' := \mathrm{RealSqZ}(\mathcal{F}, \gamma) \in \mathrm{FormMod}_{\mathcal{X}/},$$

Let $i : \mathcal{X} \to \mathcal{X}'$ denote the canonical map.

We claim:

PROPOSITION 6.4.2. *There is a canonical fiber sequence* $\mathrm{IndCoh}(\mathcal{X}')$

(6.7) $$i_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}'} \to i_*^{\mathrm{IndCoh}}(\mathcal{F})[1].$$

The rest of this subsection is devoted to the proof of the proposition.

6.4.3. *Step 1.* We will construct a fiber sequence
$$i_*^{\mathrm{IndCoh}}(\mathcal{F}) \to i_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}'}.$$

We interpret the category $\mathrm{IndCoh}(\mathcal{X}')$ as
$$\mathrm{Annul}(\mathcal{F}, \gamma) \simeq \mathsf{M}_{\mathcal{F},\gamma}\text{-mod}(\mathrm{IndCoh}(\mathcal{X})).$$

Under this identification, the functor i_*^{IndCoh} corresponds to $\mathbf{ind}_{\mathsf{M}_{\mathcal{F},\gamma}}$.

The object $\omega_{\mathcal{X}'}$ corresponds to $\omega_{\mathcal{X}} \in \mathrm{IndCoh}(\mathcal{X})$, where the null-homotopy for
$$\mathcal{F}[-1] \overset{!}{\otimes} \omega_{\mathcal{X}} \to T(\mathcal{X})[-1] \overset{!}{\otimes} \omega_{\mathcal{X}} \overset{\alpha_{\mathcal{X}}}{\longrightarrow} \omega_{\mathcal{X}}$$
comes from (6.4).

6.4.4. *Step 2.* The datum of a map $i_*^{\mathrm{IndCoh}}(\mathcal{F}) \to i_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}})$ is equivalent to that of a map
$$\mathcal{F} \to \mathsf{M}_{\mathcal{F},\gamma}(\omega_{\mathcal{X}})$$
in $\mathrm{IndCoh}(\mathcal{X})$.

Consider the canonical filtration on $\mathsf{M}_{\mathcal{F},\gamma}$, see Sect. 6.3.6. We have a fiber sequence
$$\omega_{\mathcal{X}} \to \mathsf{M}_{\mathcal{F},\gamma}^{\mathrm{Fil},\leq 1}(\omega_{\mathcal{X}}) \to \mathcal{F}.$$

Moreover, the composition
$$\omega_{\mathcal{X}} \to \mathsf{M}_{\mathcal{F},\gamma}^{\mathrm{Fil},\leq 1}(\omega_{\mathcal{X}}) \to \mathsf{M}_{\mathcal{F},\gamma}(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}},$$
(where the last arrow is obtained by adjunction from $i_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}'}$), is the identity map.

Hence, we obtain a splitting
$$\mathsf{M}_{\mathcal{F},\gamma}^{\mathrm{Fil},\leq 1}(\omega_{\mathcal{X}}) \simeq \mathcal{F} \oplus \omega_{\mathcal{X}},$$
and in particular a map
$$\mathcal{F} \to \mathsf{M}_{\mathcal{F},\gamma}(\omega_{\mathcal{X}}),$$
whose composition with $\mathsf{M}_{\mathcal{F},\gamma}(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}}$ is zero.

This gives rise to a map
$$\mathbf{ind}_{\mathsf{M}_{\mathcal{F},\gamma}}(\mathcal{F}) \to \mathbf{ind}_{\mathsf{M}_{\mathcal{F},\gamma}}(\omega_{\mathcal{X}})$$
in $\mathsf{M}_{\mathcal{F},\gamma}$-mod$(\mathrm{IndCoh}(\mathcal{X}))$, whose composition with the map
$$\mathbf{ind}_{\mathsf{M}_{\mathcal{F},\gamma}}(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}}$$
is zero.

6.4.5. *Step 3.* Thus, it remains to show that
$$\mathbf{oblv}_{\mathsf{M}_{\mathcal{F},\gamma}} \circ \mathbf{ind}_{\mathsf{M}_{\mathcal{F},\gamma}}(\mathcal{F}) \to \mathbf{oblv}_{\mathsf{M}_{\mathcal{F},\gamma}} \circ \mathbf{ind}_{\mathsf{M}_{\mathcal{F},\gamma}}(\omega_{\mathcal{X}}) \to \omega_{\mathcal{X}}$$
is an exact triangle.

It is enough to establish the exactness at the associated graded level. However, in this case, the maps in question identity with
$$(\mathbf{oblv}_{\mathrm{Assoc}} \circ \mathbf{free}_{\mathrm{Assoc}}(\mathcal{F})) \overset{!}{\otimes} \mathcal{F} \to (\mathbf{oblv}_{\mathrm{Assoc}} \circ \mathbf{free}_{\mathrm{Assoc}}(\mathcal{F})) \to \omega_{\mathcal{X}},$$
and the exactness is manifest.

7. Global sections of a Lie algebroid

In this section we address the following question: one expects that global sections of a Lie algebroid form a Lie algebra. This is done in two steps:

First for the tangent Lie algebroid and then in general. For the tangent Lie algebroid, the idea is that its global sections can be identified with the Lie algebra of the group of (formal) automorphisms of $\mathcal{X}$. To implement the second step, we relate actions of a free Lie algebra to free Lie algebroids.

7.1. Action of the free Lie algebra and Lie algebroids. In this subsection we show that the quotient of a prestack with respect to an action of a free Lie algebra is given by a square-zero extension of that prestack.

7.1.1.   For $V \in \text{Vect}$, consider $\textbf{free}_{\text{Lie}}(V) \in \text{LieAlg}(\text{Vect})$. Consider the corresponding object

$$\exp(\textbf{free}_{\text{Lie}}(V)) \in \text{Grp}(\text{FormMod}_{/\text{pt}}).$$

Let $\mathcal{X}$ be an object of $\text{PreStk}_{\text{laft-def}}$. Recall that according to Chapter 7, Theorem 6.1.5, the datum of an action of $\exp(\textbf{free}_{\text{Lie}}(V))$ on $\mathcal{X}$ is equivalent to that of map

$$V \otimes \omega_{\mathcal{X}} \to T(\mathcal{X})$$

in $\text{IndCoh}(\mathcal{X})$.

7.1.2.   Given an action of $\exp(\textbf{free}_{\text{Lie}}(V))$ on $\mathcal{X}$, consider

$$\exp(\textbf{free}_{\text{Lie}}(V)) \times \mathcal{X}$$

as a formal groupoid over $\mathcal{X}$.

Let

$$\mathcal{X} / \exp(\textbf{free}_{\text{Lie}}(V))$$

denote the corresponding object of $\text{FormMod}_{\mathcal{X}/}$.

We claim:

PROPOSITION 7.1.3.   *There is a canonical isomorphism in* $\text{FormMod}_{\mathcal{X}/}$

$$\mathcal{X} / \exp(\textbf{free}_{\text{Lie}}(V)) \simeq \text{RealSqZ}(V \otimes \omega_{\mathcal{X}} \to T(\mathcal{X})).$$

The above proposition can be reformulated as follows.

COROLLARY 7.1.4.   *The Lie algebroid corresponding to the formal groupoid* $\exp(\textbf{free}_{\text{Lie}}(V)) \times \mathcal{X}$ *identifies canonically with*

$$\textbf{free}_{\text{LieAlgbroid}}(V \otimes \omega_{\mathcal{X}} \to T(\mathcal{X})).$$

7.1.5. *Proof of Proposition 7.1.3.* Let $f : \mathcal{X} \to \mathcal{Y}$ be an object of $\text{FormMod}_{\mathcal{X}/}$. We need to show that the datum of a map

$$\mathcal{X} / \exp(\textbf{free}_{\text{Lie}}(V)) \to \mathcal{Y}$$

in $\text{FormMod}_{\mathcal{X}/}$ is canonically equivalent to that of a map

$$(V \otimes \omega_{\mathcal{X}} \to T(\mathcal{X})) \to (T(\mathcal{X}/\mathcal{Y}) \to T(\mathcal{X}))$$

in $\text{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$.

However, the latter follows from Chapter 7, Theorem 6.1.5, applied to $\mathcal{X}$, viewed as an object of $(\text{PreStk}_{\text{laft-def}})_{/\mathcal{Y}}$.

$\square$

7.2. The Lie algebra of vector fields. In this subsection we will show that global vector fields on prestack form a Lie algebra.

7.2.1. Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$.

Consider the (discontinuous) functor

$$(p_{\mathcal{X}}^{!})^{R} : \mathrm{IndCoh}(X) \to \mathrm{Vect},$$

right adjoint to $p_{\mathcal{X}}^{!}$.

REMARK 7.2.2. Note that when $\mathcal{X}$ is an eventually coconnective scheme X, the functor $(p_{X}^{!})^{R}$ is continuous and identifies with

$$\Gamma(X, -) \circ \Upsilon_{X}^{R},$$

where Υ_{X}^{R} is the right adjoint of the functor

$$\Upsilon_{X} : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X), \quad \mathcal{E} \mapsto \mathcal{E} \otimes \omega_{X}.$$

7.2.3. Consider the object $(p_{\mathcal{X}}^{!})^{R}(T(\mathcal{X})) \in \mathrm{Vect}$. We claim:

PROPOSITION-CONSTRUCTION 7.2.4. *The object $(p_{\mathcal{X}}^{!})^{R}(T(\mathcal{X}))$ can be canonically lifted to an object* $\mathrm{VF}(\mathcal{X}) \in \mathrm{LieAlg}(\mathrm{Vect})$.

PROOF. Recall the object

$$\mathrm{Aut}^{\mathrm{inf}}(\mathcal{X}) \in \mathrm{Grp}((\mathrm{FormMod}_{\mathrm{laft}})_{/\mathrm{pt}}),$$

see Chapter 7, Sect. 6.2.1.

Define

$$\mathrm{VF}(\mathcal{X}) := \mathrm{Lie}_{\mathrm{pt}}(\mathrm{Aut}^{\mathrm{inf}}(\mathcal{X})).$$

We need to show that

$$\mathbf{oblv}_{\mathrm{Lie}}(\mathrm{VF}(\mathcal{X})) \simeq (p_{\mathcal{X}}^{!})^{R}(T(\mathcal{X})).$$

This is equivalent to showing that for $V \in \mathrm{Vect}$,

$$\mathrm{Maps}_{\mathrm{LieAlg}(\mathrm{Vect})}(\mathbf{free}_{\mathrm{Lie}}(V), \mathrm{VF}(\mathcal{X})) \simeq \mathrm{Maps}_{\mathrm{Vect}}(V, (p_{\mathcal{X}}^{!})^{R}(T(\mathcal{X}))).$$

However, the latter follows from Chapter 7, Theorem 6.1.5.

$\square$

REMARK 7.2.5. Note that by the construction of $\mathrm{Aut}^{\mathrm{inf}}(\mathcal{X})$, for $\mathfrak{h} \in \mathrm{LieAlg}(\mathrm{Vect})$, the space

$$\mathrm{Maps}_{\mathrm{LieAlg}(\mathrm{Vect})}(\mathfrak{h}, \mathrm{VF}(\mathcal{X}))$$

identifies canonically with that of actions of the formal group $\exp(\mathfrak{h})$ on $\mathcal{X}$.

7.3. Construction of the Lie algebra structure. In this subsection we will finally construct a structure of Lie algebra on global sections of a Lie algebroid, see Proposition 7.3.3.

7.3.1.   Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. We define a functor

(7.1) $p^!_{\mathcal{X}} : \mathrm{LieAlg}(\mathrm{Vect})_{/\,\mathrm{VF}(\mathcal{X})} \to \mathrm{LieAlgbroid}(\mathcal{X})$

as follows.

By definition, we can think of an object

$$(\mathfrak{h} \to \mathrm{VF}(\mathcal{X})) \in \mathrm{LieAlg}(\mathrm{Vect})_{/\,\mathrm{VF}(\mathcal{X})}$$

as a datum of action of $\exp(\mathfrak{h})$ on $\mathcal{X}$.

We let $p^!_{\mathcal{X}}(\mathfrak{h} \to \mathrm{VF}(\mathcal{X})) \in \mathrm{LieAlgbroid}(\mathcal{X})$ be the Lie algebroid corresponding to the formal groupoid $\exp(\mathfrak{h}) \times \mathcal{X}$.

7.3.2.   We claim:

PROPOSITION 7.3.3.   *The functor $p^!_{\mathcal{X}}$ of* (7.1) *admits a right adjoint, denoted* $(p^!_{\mathcal{X}})^R_{/\,\mathrm{VF}(\mathcal{X})}$. *The composition*

$$\mathrm{LieAlgbroid}(\mathcal{X}) \overset{(p^!_{\mathcal{X}})^R/\mathrm{VF}(\mathcal{X})}{\longrightarrow} \mathrm{LieAlg}(\mathrm{Vect})_{/\,\mathrm{VF}(\mathcal{X})} \overset{\mathbf{oblv}_{\mathrm{Lie}}}{\longrightarrow} \mathrm{Vect}_{/(p^!_{\mathcal{X}})^R(T(\mathcal{X}))}$$

is the functor

$$\mathrm{LieAlgbroid}(\mathcal{X}) \overset{\mathbf{oblv}_{\mathrm{LieAlgbroid}/T}}{\longrightarrow} \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} \overset{(p^!_{\mathcal{X}})^R}{\longrightarrow} \mathrm{Vect}_{/(p^!_{\mathcal{X}})^R(T(\mathcal{X}))} \,.$$

PROOF.   Follows immediately from Corollary 7.1.4.   □

7.3.4.   Note that by construction, we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{LieAlgbroid}(\mathcal{X}) & \overset{(p^!_{\mathcal{X}})^R_{/\,\mathrm{VF}(\mathcal{X})}}{\longrightarrow} & \mathrm{LieAlg}(\mathrm{Vect})_{/\,\mathrm{VF}(\mathcal{X})} \\
{\scriptstyle \mathrm{ker\text{-}anch}} \downarrow & & \downarrow \\
\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) & \overset{(p^!_{\mathcal{X}})^R}{\longrightarrow} & \mathrm{LieAlg}(\mathrm{Vect})
\end{array}
$$

where the right vertical arrow is the functor

$$(\mathfrak{h} \overset{\gamma}{\to} \mathrm{VF}(\mathcal{X})) \mapsto \mathrm{Fib}(\gamma).$$

It is easy to see, however, that the diagram, obtained from the above one by passing to left adjoints along the vertical arrows, is also commutative:

(7.2)
$$
\begin{array}{ccc}
\mathrm{LieAlgbroid}(\mathcal{X}) & \overset{(p^!_{\mathcal{X}})^R_{/\,\mathrm{VF}(\mathcal{X})}}{\longrightarrow} & \mathrm{LieAlg}(\mathrm{Vect})_{/\,\mathrm{VF}(\mathcal{X})} \\
{\scriptstyle \mathrm{diag}} \uparrow & & \uparrow \\
\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) & \overset{(p^!_{\mathcal{X}})^R}{\longrightarrow} & \mathrm{LieAlg}(\mathrm{Vect}),
\end{array}
$$

where the right vertical arrow sends

$$\mathfrak{h} \mapsto (\mathfrak{h} \overset{0}{\to} \mathrm{VF}(\mathcal{X})).$$

7.3.5.   Let us denote by $(p_{\mathcal{X}}^!)^R$ the composition

$$\text{LieAlgbroid}(\mathcal{X}) \overset{(p_{\mathcal{X}}^!)^R_{/\text{VF}(\mathcal{X})}}{\longrightarrow} \text{LieAlg}(\text{Vect})_{/\text{VF}(\mathcal{X})} \to \text{LieAlg}(\text{Vect}),$$

where the second arrow is the forgetful functor.

From (7.3), we obtain a commutative diagram

(7.3)
$$\begin{array}{ccc}
\text{LieAlgbroid}(\mathcal{X}) & \overset{(p_{\mathcal{X}}^!)^R}{\longrightarrow} & \text{LieAlg}(\text{Vect}) \\
\text{diag} \uparrow & & \uparrow \text{id} \\
\text{LieAlg}(\text{IndCoh}(\mathcal{X})) & \overset{(p_{\mathcal{X}}^!)^R}{\longrightarrow} & \text{LieAlg}(\text{Vect}),
\end{array}$$

7.3.6.   Consider now the functor

(7.4) $\text{LieAlgbroid}(\mathcal{X}) \overset{\Omega^{\text{fake}}}{\longrightarrow} \text{LieAlg}(\text{IndCoh}(\mathcal{X})) \overset{(p_{\mathcal{X}}^!)^R}{\longrightarrow} \text{LieAlg}(\text{Vect}).$

We claim:

PROPOSITION 7.3.7. *The functor* (7.4) *identifies canonically with*

$\text{LieAlgbroid}(\mathcal{X}) \overset{\mathbf{oblv}_{\text{LieAlgbroid}}}{\longrightarrow} \text{IndCoh}(\mathcal{X}) \overset{(p_{\mathcal{X}}^!)^R}{\longrightarrow} \text{Vect} \overset{[-1]}{\longrightarrow} \text{Vect} \overset{\mathbf{triv}_{\text{Lie}}}{\longrightarrow} \text{LieAlg}(\text{Vect}).$

PROOF. Using (7.3), we rewrite the functor (7.4) as

$\text{LieAlgbroid}(\mathcal{X}) \overset{\Omega^{\text{fake}}}{\longrightarrow} \text{LieAlg}(\text{IndCoh}(\mathcal{X})) \overset{\text{diag}}{\longrightarrow} \text{LieAlgbroid}(\mathcal{X}) \overset{(p_{\mathcal{X}}^!)^R}{\longrightarrow} \text{LieAlg}(\text{Vect}).$

Using Sect. 3.1.4, we further rewrite this as

(7.5) $\text{LieAlgbroid}(\mathcal{X}) \to \text{LieAlgbroid}(\mathcal{X}) \overset{(p_{\mathcal{X}}^!)^R}{\longrightarrow} \text{LieAlg}(\text{Vect})$

where the first arrow is

$$\mathfrak{L} \mapsto 0 \underset{\mathfrak{L}}{\times} 0.$$

This the functor $\text{LieAlgbroid}(\mathcal{X}) \overset{(p_{\mathcal{X}}^!)^R}{\longrightarrow} \text{LieAlg}(\text{Vect})$ commutes with fiber products, the functor in (7.5) identifies with

(7.6) $\mathbf{oblv}_{\text{Grp}} \circ \Omega_{\text{Lie}} \circ (p_{\mathcal{X}}^!)^R.$

Now, recall that according to Chapter 6, Proposition 1.7.2, we have

$$\mathbf{oblv}_{\text{Grp}} \circ \Omega_{\text{Lie}} \simeq \mathbf{triv}_{\text{Lie}} \circ [-1] \circ \mathbf{oblv}_{\text{Lie}}.$$

Hence, (7.6) identifies with

$\mathbf{triv}_{\text{Lie}} \circ [-1] \circ \mathbf{oblv}_{\text{Lie}} \circ (p_{\mathcal{X}}^!)^R \simeq \mathbf{triv}_{\text{Lie}} \circ [-1] \circ \mathbf{oblv}_{\text{Lie}} \circ (p_{\mathcal{X}}^!)^R \circ \mathbf{oblv}_{\text{LieAlgbroid}},$

as required.

$\square$

REMARK 7.3.8. Propositions 7.3.3 and 7.3.7 can be summarized as follows: for a Lie algebroid $\mathfrak{L}$ on $\mathcal{X}$, consider the corresponding object $\mathbf{oblv}_{\text{LieAlgbroid}}(\mathfrak{L}) \in \text{IndCoh}(\mathcal{X})$. Of course, it does not have a structure of Lie algebra in $\text{IndCoh}(\mathcal{X})$. Yet, $(p_{\mathcal{X}}^!)^R(\mathbf{oblv}_{\text{LieAlgbroid}}(\mathfrak{L}))$ does have a structure of Lie algebra.

Now, $\mathbf{oblv}_{\text{LieAlgbroid}}(\mathfrak{L})[-1]$ does have a structure of Lie algebra, but it is not obtained by looping another object in $\text{LieAlg}(\text{IndCoh}(\mathcal{X}))$. Despite this, the Lie

algebra of global sections of $\mathbf{oblv}_{\text{LieAlgbroid}}(\mathfrak{L})[-1]$ is obtained by looping the Lie algebra of global sections of $\mathbf{oblv}_{\text{LieAlgbroid}}(\mathfrak{L})$.

8. Lie algebroids as modules over a monad

In this section we develop the idea borrowed from [**Fra**]:

Lie algebroids on $\mathcal{X}$ can be expressed as modules over a certain canonically defined monad acting on the category $\text{LieAlgbroid}(\text{IndCoh}(\mathcal{X}))$. This monad is given by the operation of 'semi-direct product' with the inertia Lie algebra $\text{inert}_{\mathcal{X}}$.

8.1. The inertia monad. In this subsection we will work in the category of spaces. Given a space X, we will define a monad acting on the category $\text{Grp}(\text{Spc}_{/X})$, modules for which 'almost' reproduce the category $\text{Grpoid}(X)$.

8.1.1.   For $X \in \text{Spc}$, consider the above pair of adjoint functors

$$\text{diag} : \text{Grp}(\text{Spc}_{/X}) \rightleftarrows \text{Grpoid}(X) : \text{Inert}.$$

It gives rise to a monad on $\text{Grp}(\text{Spc}_{/X})$ that we will denote by $\mathsf{M}_{\text{Inert}_X}$, and refer to it as the *inertia monad* on X.

8.1.2.   For $H \in \text{Grp}(\text{Spc}_{/X})$, the object $\mathsf{M}_{\text{Inert}_X}(H) \in \text{Grp}(\text{Spc}_{/X})$ has the following pieces of structure:

- We have a map $H \to \mathsf{M}_{\text{Inert}_X}(H)$, corresponding to the unit in $\mathsf{M}_{\text{Inert}_X}$;
- We have a map $\mathsf{M}_{\text{Inert}_X}(H) \to \text{Inert}_X$, corresponding to the map $H \to X$ and the identification

$$\mathsf{M}_{\text{Inert}_X}(X) = \text{Inert}(\text{diag}_X) = \text{Inert}_X;$$

- A right inverse $\text{Inert}_X \to \mathsf{M}_{\text{Inert}_X}(H)$ of the above map $\mathsf{M}_{\text{Inert}_X}(H) \to \text{Inert}_X$, corresponding to the map $X \to H$.

It is easy to see that the maps

$$H \to \mathsf{M}_{\text{Inert}_X}(H) \to \text{Inert}_X$$

form a fiber sequence in $\text{Grp}(\text{Spc}_{/X})$.

Monads having these properties will be axiomatized in Sect. 8.2 under the name *special monads*.

8.1.3.   Note that the fiber sequence and the section of the second arrow

$$H \to \mathsf{M}_{\text{Inert}_X}(H) \rightleftarrows \text{Inert}_X$$

makes $\mathsf{M}_{\text{Inert}_X}(H)$ look like a semi-direct product

$$\text{Inert}_X \ltimes H.$$

In particular, we obtain a canonically defined action of Inert_X on any $H \in \text{Grp}(\text{Spc}_{/X})$.

8.1.4. Consider the category

$$M_{\mathrm{Inert}_X}\text{-mod}(\mathrm{Grp}(\mathrm{Spc}_{/X})),$$

equipped with a pair of adjoint functors

$$\mathbf{ind}_{M_{\mathrm{Inert}_X}} : \mathrm{Grp}(\mathrm{Spc}_{/X}) \rightleftarrows M_{\mathrm{Inert}_X}\text{-mod}(\mathrm{Grp}(\mathrm{Spc}_{/X})) : \mathbf{oblv}_{M_{\mathrm{Inert}_X}}.$$

As we shall presently see, the category $M_{\mathrm{Inert}_X}\text{-mod}(\mathrm{Grp}(\mathrm{Spc}_{/X}))$ is 'almost equivalent' to $\mathrm{Grpoid}(X)$.

8.1.5. By construction, the functor Inert factors though a canonically defined functor

$$\mathrm{Inert}^{\mathrm{enh}} : \mathrm{Grpoid}(X) \rightarrow M_{\mathrm{Inert}_X}\text{-mod}(\mathrm{Grp}(\mathrm{Spc}_{/X})),$$

so that

$$\mathrm{Inert}(R) = \mathbf{oblv}_{M_{\mathrm{Inert}_X}}(\mathrm{Inert}^{\mathrm{enh}}(R)).$$

It is easy to see that the above functor $R \mapsto \mathrm{Inert}^{\mathrm{enh}}(R)$ admits a left adjoint; we will denote it by

$$\mathrm{diag}^{\mathrm{enh}} : M_{\mathrm{Inert}_X}\text{-mod}(\mathrm{Grp}(\mathrm{Spc}_{/X})) \rightarrow \mathrm{Grpoid}(X).$$

PROPOSITION 8.1.6. *The functor* $\mathrm{diag}^{\mathrm{enh}}$ *is fully faithful. Its essential image consists of those* $R \in \mathrm{Grpoid}(X)$, *for which the map*

$$\pi_0(\mathrm{Inert}(R)) \rightarrow \pi_0(R)$$

is surjective.

PROOF. First, we have the following general claim:

LEMMA 8.1.7. *Let* $\mathsf{F} : \mathbf{C} \rightleftarrows \mathbf{D} : \mathsf{G}$ *be a pair of adjoint functors between ∞-categories, where* G *commutes* G-*split geometric realizations. Then the resulting functor*

$$\mathsf{F}^{\mathrm{enh}} : (\mathsf{G} \circ \mathsf{F})\text{-mod}(\mathbf{C}) \rightarrow \mathbf{D}$$

is fully faithful.

The fact that $\mathrm{diag}^{\mathrm{enh}}$ is fully faithful follows immediately from the lemma. The essential image of $\mathrm{diag}^{\mathrm{enh}}$ lies in the specified subcategory of $\mathrm{Grpoid}(X)$ because this is so for diag, and because this subcategory is closed under colimits.

To prove the proposition it remains to show that the functor Inert is conservative on the specified subcategory of $\mathrm{Grpoid}(X)$ and commutes with geometric realizations. The former is straightforward. The latter follows from Chapter 5, Lemma 2.1.3.

$\square$

8.2. Special monads. In this subsection we introduce a certain class of monads that we call *special*. They will be useful in studying Lie algebroids. However, we believe that this notion has other applications as well.

8.2.1. *Assumption on the category.* Let $\mathfrak{T}$ be a pointed $(\infty, 1)$-category; denote its final/initial object by $* \in \mathfrak{T}$.

We shall make the following general assumptions:

- (i) $\mathfrak{T}$ admits limits;
- (ii) Sifted colimits in $\mathfrak{T}$ exist and are universal (=commute with base change);
- (iii) Groupoids in $\mathfrak{T}$ are universal (see [**Lu1**, Definition 6.1.2.14] for what this means).

Note that for any $\widetilde{t} \to t$, the map

(8.1) $$|\widetilde{t}^\bullet/t| \to t$$

is a *monomorphism* (here $\widetilde{t}^\bullet/t$ is the simplicial object of $\mathfrak{T}$ equal to the Čech nerve of $\widetilde{t} \to t$).

We shall say that $\widetilde{t} \to t$ is an *effective epimorphism* if the map (8.1) is an isomorphism. Let $(\mathfrak{T}_{/t})_{\mathrm{epi}}$ be the full subcategory of $\mathfrak{T}_{/t}$ spanned by effective epimorphisms.

8.2.2. We shall now make the following additional assumption on $\mathfrak{T}$:

For any $t \in \mathfrak{T}$, the functor

$$(\mathfrak{T}_{/t})_{\mathrm{epi}} \to \mathfrak{T}, \quad (\widetilde{t} \to t) \mapsto \widetilde{t} \underset{t}{\times} *$$

is conservative.

8.2.3. *Examples.* Here are two examples of this situation:

One is $\mathrm{Grp}(\mathrm{Spc}_{/X})$, where $X \in \mathrm{Spc}$.

Another is $\mathrm{LieAlg}(\mathbf{O})$, where $\mathbf{O}$ is a symmetric monoidal DG category.

8.2.4. One corollary of the property in Sect. 8.2.2 is that the inclusion

$$\mathrm{Grp}(\mathfrak{T}) \hookrightarrow \mathrm{Monoid}(\mathfrak{T})$$

is an equality.

Indeed, for $t \in \mathrm{Monoid}(\mathfrak{T})$, we need to show that the map

$$t \times t \overset{(\mathrm{id},\mathrm{mult})}{\longrightarrow} t \times t$$

is an isomorphism. However, the above map is a map on $(\mathfrak{T}_{/t})_{\mathrm{epi}}$, where both sides map to t via the first projection, while the base change of the above map with respect to $* \to t$ is the identity map.

8.2.5. *Definition of special monad.* Let $(\mathfrak{T}, *)$ be as above. Let $\mathrm{Monad}(\mathfrak{T})$ denote the category of all monads acting on $\mathfrak{T}$.

We let $\mathrm{Monad}(\mathfrak{T})^{\mathrm{spl}} \subset \mathrm{Monad}(\mathfrak{T})$ denote the full subcategory spanned by monads M satisfying the following condition:

For every $t \in \mathfrak{T}$, the maps

$$t \to \mathsf{M}(t) \to \mathsf{M}(*)$$

form a fiber sequence, i.e., the map

$$t \to \mathsf{M}(t) \underset{\mathsf{M}(*)}{\times} *$$

is an isomorphism.

Here $t \to \mathsf{M}(t)$ is given by the unit of the monad M, and $\mathsf{M}(t) \to \mathsf{M}(*)$ is given by the canonical map $t \to *$. We will refer to such monads as *special monads*.

8.2.6.   Note that for any $t \in \mathfrak{T}$, the above map

$$\mathsf{M}(t) \to \mathsf{M}(*)$$

admits a section, given by applying M to the canonical map $t \leftarrow *$. So, we have a diagram

(8.2) $$t \to \mathsf{M}(t) \rightleftarrows \mathsf{M}(*).$$

8.2.7. *Basic properties of special monads.* Note that (8.2) implies that for $t \in \mathfrak{T}$, the map $\mathsf{M}(t) \to \mathsf{M}(*)$ is an effective epimorphism. From here, we obtain:

LEMMA 8.2.8. *The monad* M, *considered as a mere endo-functor of* $\mathfrak{T}$, *commutes with sifted colimits.*

PROOF. We have to show that for a sifted family t_i the map

$$\operatorname{colim} \mathsf{M}(t_i) \to \mathsf{M}(\operatorname{colim} t_i)$$

is an isomorphism. By Sect. 8.2.2, it is enough to show that

$$(\operatorname{colim} \mathsf{M}(t_i)) \underset{\mathsf{M}(*)}{\times} * \to \mathsf{M}(\operatorname{colim} t_i) \underset{\mathsf{M}(*)}{\times} * \simeq \operatorname{colim} t_i$$

is an isomorphism. However, since sifted colimits in $\mathfrak{T}$ are universal,

$$(\operatorname{colim} \mathsf{M}(t_i)) \underset{\mathsf{M}(*)}{\times} * \simeq \operatorname{colim} \left(\mathsf{M}(t_i) \underset{\mathsf{M}(*)}{\times} * \right) \simeq \operatorname{colim} t_i,$$

as required. □

COROLLARY 8.2.9. *The category* $\mathsf{M}\text{-mod}(\mathfrak{T})$ *admits sifted colimits and the forgetful functor*

$$\mathbf{oblv}_\mathsf{M} : \mathsf{M}\text{-mod}(\mathfrak{T}) \to \mathfrak{T}$$

commutes with sifted colimits.

8.3. Infinitesimal inertia monad. We will now adapt the material in Sect. 8.1 to the setting of formal geometry.

8.3.1.   As in Sect. 8.1, the pair of adjoint functors

$$\operatorname{diag} : \operatorname{Grp}(\operatorname{FormMod}_{/\mathcal{X}}) \rightleftarrows \operatorname{FormGrpoid}(\mathcal{X}) : \operatorname{Inert}^{\operatorname{inf}}$$

defines a monad, denoted $\mathsf{M}_{\operatorname{Inert}^{\operatorname{inf}}_{\mathcal{X}}}$ on $\operatorname{Grp}(\operatorname{FormMod}_{/\mathcal{X}})$.

Moreover, is easy to see that $\mathsf{M}_{\operatorname{Inert}^{\operatorname{inf}}_{\mathcal{X}}}$ is *special.*

8.3.2.   Consider the resulting pair of adjoint functors

(8.3) $\operatorname{diag}^{\mathrm{enh}} : \mathsf{M}_{\mathrm{Inert}_{\mathcal{X}}^{\mathrm{inf}}}\text{-mod}\left(\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})\right) \rightleftarrows \mathrm{FormGrpoid}(\mathcal{X}) : \mathrm{Inert}^{\mathrm{inf},\mathrm{enh}}$.

We now claim:

PROPOSITION 8.3.3.   *The functor* $\operatorname{diag}^{\mathrm{enh}}$ *and* $\mathrm{Inert}^{\mathrm{inf},\mathrm{enh}}$ *of* (8.3) *are mutually inverse equivalences of categories.*

PROOF.   We need to show that the functor $\mathrm{Inert}^{\mathrm{inf}}$ satisfies the conditions of the Barr-Beck-Lurie theorem. The fact that the functor $\mathrm{Inert}^{\mathrm{inf}}$ commutes with sifted colimits (and, in particular, geometric realizations) follows from Chapter 5, Corollary 2.2.4. Hence, it remains to see that $\mathrm{Inert}^{\mathrm{inf}}$ is conservative. This follows, e.g., from the fact that the functor $\Omega_{\mathcal{X}}$ is conservative, via the fiber sequence (1.1).
□

8.4. The inertia monad on Lie algebras and Lie algebroids. In this sub-section we show that the category $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$ carries a canonical monad, given by semi-direct product with the inertia Lie algebra, and that Lie algebroids identify with the category of modules over this monad.

8.4.1.   Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. Recall the equivalence

$$\mathrm{Lie}_{\mathcal{X}} : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}}) \rightleftarrows \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) : \exp$$

of Chapter 7, Theorem 3.6.2.

Hence, the monad $\mathsf{M}_{\mathrm{Inert}_{\mathcal{X}}^{\mathrm{inf}}}$ acting on $\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}})$ defines a special monad, denoted $\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}$, on $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))$.

8.4.2.   From Proposition 8.3.3, we obtain:

COROLLARY 8.4.3.   *The category* $\mathrm{LieAlgbroid}(\mathcal{X})$, *equipped with the forgetful functor* $\ker . \mathrm{anch}$. *is canonically equivalent to the category* $\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}\text{-mod}(\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})))$, *equipped with the forgetful functor* $\mathbf{oblv}_{\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}}$.

8.4.4.   By adjunction, under the identification of Corollary 8.4.3, the functor

$$\operatorname{diag} : \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \to \mathrm{LieAlgbroid}(\mathcal{X})$$

identifies with

$$\mathbf{ind}_{\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}} : \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) \to \mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}\text{-mod}(\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))).$$

The zero Lie algebroid, i.e., the initial object of $\mathrm{LieAlgbroid}(\mathcal{X})$, corresponds to

$$\mathbf{ind}_{\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}}(0) \in \mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}\text{-mod}(\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))).$$

Under the identification of Corollary 8.4.3 the tangent algebroid $\mathcal{T}(\mathcal{X})$ (i.e., the final object in $\mathrm{LieAlgbroid}(\mathcal{X})$) corresponds to

$$0 \in \mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}\text{-mod}(\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X}))).$$

8.4.5. Note that

$$\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}(0) = \mathbf{oblv}_{\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}} \circ \mathbf{ind}_{\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}}(0) = \mathrm{inert}_{\mathcal{X}}.$$

As was mentioned already, the monad $\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}$ is special. Hence, for $\mathfrak{h} \in$ LieAlg(IndCoh($\mathcal{X}$)), from (8.2) we obtain a split fiber sequence

$$(8.4) \qquad\qquad \mathfrak{h} \to \mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}(\mathfrak{h}) \rightleftarrows \mathrm{inert}_{\mathcal{X}}.$$

Hence, we can think of $\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}(\mathfrak{h})$ as a semi-direct product

$$\mathrm{inert}_{\mathcal{X}} \ltimes \mathfrak{h}$$

for a canonically defined action of $\mathrm{inert}_{\mathcal{X}}$ on $\mathfrak{h}$.

REMARK 8.4.6. When we forget the Lie algebra structure on $\mathfrak{h}$, we recover the canonical action of $\mathrm{inert}_{\mathcal{X}}$ on objects of IndCoh($\mathcal{X}$) from Sect. 6.1.2.

Vice versa, since the functor **can** of Sect. 6.1.2 is symmetric monoidal, it defines an action of $\mathrm{inert}_{\mathcal{X}}$ on every $\mathfrak{h} \in$ LieAlg(IndCoh($\mathcal{X}$)), and one can show that this is the same action as defined above.

8.4.7. Recall the functor

$$\Omega^{\mathrm{fake}} : \mathrm{LieAlgbroid}(\mathcal{X}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})).$$

In terms of the equivalence of Corollary 8.4.3, it sends $\mathfrak{L} \in$ LieAlgbroid($\mathcal{X}$), to the fiber of the composite map

$$(8.5) \qquad\qquad \mathrm{inert}_{\mathcal{X}} \to \mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}(\mathfrak{h}) \to \mathfrak{h},$$

where the first arrow is the canonical splitting of (8.4), and the second arrow is given by the action of $\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}$ on $\mathfrak{h}$.

8.4.8. We have the following identifications

$$\mathrm{ker\text{-}anch} \circ \mathrm{diag}(\mathfrak{h}) \simeq \mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}(\mathfrak{h}) \simeq \mathrm{inert}_{\mathcal{X}} \ltimes \mathfrak{h};$$

$$\Omega^{\mathrm{fake}} \circ \mathrm{diag}(\mathfrak{h}) \simeq \Omega_{\mathrm{Lie}}(\mathfrak{h});$$

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} \circ \mathrm{diag}(\mathfrak{h}) \simeq (\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{h}) \overset{0}{\to} T(\mathcal{X})).$$

REMARK 8.4.9. Note that there are the following two ways to relate the category LieAlgbroid($\mathcal{X}$) to a more linear category.

One is given by Corollary 8.4.3, which implies that we can interpret LieAlgbroid($\mathcal{X}$) as $\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}$-mod(LieAlg(IndCoh($\mathcal{X}$))).

The other is as modules for the monad

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}} \simeq T(\mathcal{X}/-) \circ \mathrm{RealSqZ}$$

in the category IndCoh($\mathcal{X}$)$_{/T(\mathcal{X})}$.

This former has the advantage that the monad involved, i.e., $\mathsf{M}_{\mathrm{inert}_{\mathcal{X}}}$, is 'smaller': it is given by semi-direct product with $\mathrm{inert}_{\mathcal{X}}$.

The latter has the advantage that the recipient category, i.e., IndCoh($\mathcal{X}$)$_{/T(\mathcal{X})}$ is more elementary than LieAlg(IndCoh($\mathcal{X}$)).

9. Relation to classical Lie algebroids

In this section we let X be a *classical* scheme locally of finite type. Our goal is to show that Lie algebroids, as defined in Sect. 2.1, whose underlying object of IndCoh is 'classical' are the same as classical Lie algebroids.

9.1. Classical Lie algebroids. In this subsection we recall the notion of classical Lie algebroid on a classical scheme and state the main result of this section, Theorem 9.1.5.

9.1.1.   First, we introduce the object $T^{\mathrm{naive}}(X) \in \mathrm{QCoh}(X)^{\heartsuit}$ as follows.

Recall the functor

$$\Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X)$$

(see Volume I, Chapter 6, Sect. 3.2.5). Let Υ_X^R denote its right adjoint. [3]

We start with $T(X) \in \mathrm{IndCoh}(X)$, and consider the object

$$\Upsilon_X^R(T(X)) \in \mathrm{QCoh}(X).$$

It follows from the definitions that

$$\Upsilon_X^R(T(X)) \simeq \underline{\mathrm{Hom}}(T^*(X), \mathcal{O}_X),$$

where $\underline{\mathrm{Hom}}$ is internal Hom in the symmetric monoidal category $\mathrm{QCoh}(X)$.

In particular, $\Upsilon_X^R(T(X)) \in \mathrm{QCoh}(X)^{\geq 0}$. Finally, we set

$$T^{\mathrm{naive}}(X) := H^0(\Upsilon_X^R(T(X))).$$

I.e., $T^{\mathrm{naive}}(X)$ is the usual naive tangent sheaf of a classical scheme.

9.1.2.   Let us recall the notion of *classical* Lie algebroid over X (see [**BB**, Sect. 2]).

By definition, this is a data of

(1) $\mathfrak{L}^{\mathrm{cl}} \in \mathrm{QCoh}(X)^{\heartsuit}$;
(2)   a map $\mathrm{anch} : \mathfrak{L}^{\mathrm{cl}} \to T^{\mathrm{naive}}(X)$;
(3)   a Lie bracket on $\mathfrak{L}^{\mathrm{cl}}$, which is a differential operator of order 1,

such that

- The map anch is compatible with the Lie brackets;
- The $[\xi_1, f \cdot \xi_2] = f \cdot [\xi_1, \xi_2] + (\mathrm{anch}(\xi_1)(f)) \cdot \xi_2$.

9.1.3.   Let $\mathrm{LieAlgbroid}(X)^{\mathrm{cl}}$ denote the category of classical Lie algebroids on X. We have a tautological forgetful functor

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}/T^{\mathrm{naive}}} : \mathrm{LieAlgbroid}(X)^{\mathrm{cl}} \to (\mathrm{QCoh}(X)^{\heartsuit})_{/T^{\mathrm{naive}}(X)},$$

and it is easy to see that it admits a left adjoint, denoted $\mathbf{free}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}}$.

The pair

$$\mathbf{free}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}} : (\mathrm{QCoh}(X)^{\heartsuit})_{/T^{\mathrm{naive}}(X)} \rightleftarrows \mathrm{LieAlgbroid}(X)^{\mathrm{cl}} : \mathbf{oblv}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}/T^{\mathrm{naive}}}$$

is easily seen to be monadic.

---

[3]Since X is classical, and in particular, eventually coconnective, the functor Υ_X^R is continuous, see [**Ga1**, Corollary 9.6.3].

9.1.4. The goal of this section is to prove the following:

THEOREM 9.1.5. *There exists a canonical equivalence between* $\mathrm{LieAlgbroid}(X)^{\mathrm{cl}}$ *and the full subcategory of* $\mathrm{LieAlgbroid}(X)$ *that consists of those objects for which* $\mathbf{oblv}_{\mathrm{Algbroid}}(\mathfrak{L})$ *belongs to the essential image of* $\mathrm{QCoh}(X)^{\heartsuit}$ *under the (fully faithful) functor*

$$\Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X).$$

This equivalence makes the diagram

$$
\begin{array}{ccc}
\mathrm{LieAlgbroid}(X)^{\mathrm{cl}} & \longrightarrow & \mathrm{LieAlgbroid}(X) \\
{\scriptstyle \mathbf{oblv}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}/T^{\mathrm{naive}}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{LieAlgbroid}/T}} \\
(\mathrm{QCoh}(X)^{\heartsuit})_{/T^{\mathrm{naive}}(X)} & \xrightarrow{\ \Upsilon_X\ } & \mathrm{IndCoh}(X)_{/T(X)}
\end{array}
$$

commute.

9.2. The locally projective case. In this subsection we consider a special case of Theorem 9.1.5 where the groupoid corresponding to the algebroid in question is itself classical and formally smooth over $\mathcal{X}$.

9.2.1. Let $\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0} \subset \mathrm{QCoh}(X)^{\heartsuit}$ be the full subcategory consisting of objects that are Zariski-locally projective and countably generated.

As a first step towards the proof of Theorem 9.1.5 we will establish its particular case:

THEOREM 9.2.2. *The following four categories are naturally equivalent:*

(a) *The full subcategory of* $\mathrm{LieAlgbroid}(X)^{\mathrm{cl}}$, *consisting of those* $\mathfrak{L}^{\mathrm{cl}}$, *for which the object*

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}/T^{\mathrm{naive}}}(\mathfrak{L}^{\mathrm{cl}}) \in \mathrm{QCoh}(X)^{\heartsuit}$$

belongs to $\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0}$

(a') *The full subcategory of* $\mathrm{FormGrpoid}(X)$, *spanned by those objects* $\mathcal{R}$ *that:*

- $\mathcal{R}$ *is an indscheme, which is* classical *and* $\aleph_0$ (*see* [**GaRo1**, Sect. 1.4.11] *for what this means*);
- $\mathcal{R}$ *is classically formally smooth* (*see* [**GaRo1**, Defn. 8.1.1] *for what this means*) *relative to* X *with respect to the projection* $p_s : \mathcal{R} \to X$.

(b) *The full subcategory of* $\mathrm{LieAlgbroid}(X)$, *consisting of those objects* $\mathfrak{L}$, *for which*

$$\mathbf{oblv}_{\mathrm{Algbroid}}(\mathfrak{L}) \in \mathrm{IndCoh}(X),$$

belongs to the essential image under Υ_X *of the full subcategory*

$$\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0} \subset \mathrm{QCoh}(X).$$

(b') *The full subcategory of* $\mathrm{FormGrpoid}(X)$, *spanned by those objects* $\mathcal{R}$ *that:*

- $\mathcal{R}$ *is an indscheme, which is weakly* $\aleph_0$ (*see* [**GaRo1**, Sect. 1.4.11] *for what this means*);
- $\mathcal{R}$ *is formally smooth relative to* X (*see Chapter 1, Sect. 7.3.1 for what this means*) *with respect to the projection* $p_s : \mathcal{R} \to X$.

The rest of the subsection is devoted to the proof of Theorem 9.2.2.

9.2.3. *The equivalence of (a) and (a').* This is standard in the theory of classical Lie algebroids.

9.2.4. *The equivalence of (b) and (b').* Follows by combining Chapter 2, Corollary 3.3.5, [**GaRo1**, Corollary 8.3.6] and the following fact (see [**BD**, Proposition 7.12.6 and Theorem 7.12.8]):

LEMMA 9.2.5. *Let $\mathcal{F} \in \mathrm{QCoh}(X)^\heartsuit$ be Zariski-locally countably generated. Then the following conditions are equivalent:*

(i) *$\mathcal{F}$ is Zariski-locally projective.*

(ii) *The functor*
$$\mathrm{QCoh}(X)^\heartsuit \to \mathrm{Vect}^\heartsuit, \quad \mathcal{F}' \mapsto H^0(\Gamma(X, \mathcal{F} \otimes \mathcal{F}'))$$
can be written as
$$\operatorname*{colim}_{i \in \mathbb{Z}^{\geq 0}} \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}'),$$
where the maps $\mathcal{F}_i \to \mathcal{F}_j$ for $j \geq i$ are surjective.

9.2.6. *The equivalence of (a') and (b').* This is a relative version of [**GaRo1**, Corollary 9.1.7].

9.3. The general case. In this subsection we will finish the proof of Theorem 9.1.5 by reducing the general case to the projective one by a trick that involves monads.

9.3.1.   As will be evident from the proof, the assertion of Theorem 9.1.5 is Zariski-local on X. So, henceforth, we will assume that X is affine.

Consider the full subcategories
$$(\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0})_{/T^{\mathrm{naive}}(X)} \subset (\mathrm{QCoh}(X)^\heartsuit)_{/T^{\mathrm{naive}}(X)} \subset (\mathrm{QCoh}(X)^{\leq 0})_{/\Upsilon_X^R(T(X))}$$
and
$$(\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0}))_{/T(X)} \subset (\Upsilon_X(\mathrm{QCoh}(X)^\heartsuit))_{/T(X)} \subset (\Upsilon_X(\mathrm{QCoh}(X)^{\leq 0}))_{/T(X)} \subset$$
$$\subset \mathrm{IndCoh}(X)_{/T(X)}.$$

The functor Υ_X defines equivalences

$$
\begin{array}{ccc}
(\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0})_{/T^{\mathrm{naive}}(X)} & \xrightarrow{\ \sim\ } & (\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0}))_{/T(X)} \\
\downarrow & & \downarrow \\
(\mathrm{QCoh}(X)^\heartsuit)_{/T^{\mathrm{naive}}(X)} & \xrightarrow{\ \sim\ } & (\Upsilon_X(\mathrm{QCoh}(X)^\heartsuit))_{/T(X)} \\
\downarrow & & \downarrow \\
(\mathrm{QCoh}(X)^{\leq 0})_{/T^{\mathrm{naive}}(X)} & \xrightarrow{\ \sim\ } & (\Upsilon_X(\mathrm{QCoh}(X)^{\leq 0}))_{/T(X)}
\end{array}
$$

Note also that the inclusions
$$(\mathrm{QCoh}(X)^\heartsuit)_{/T^{\mathrm{naive}}(X)} \subset (\mathrm{QCoh}(X)^{\leq 0})_{/T^{\mathrm{naive}}(X)}$$
and
$$(\Upsilon_X(\mathrm{QCoh}(X)^\heartsuit))_{/T(X)} \subset (\Upsilon_X(\mathrm{QCoh}(X)^{\leq 0}))_{/T(X)}$$
admit left adjoints, given by truncation. We denote these functors in both contexts by $\tau_{\mathrm{QCoh}}^{\geq 0}$.

9.3.2.   Consider the monad $\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}$ acting on $\mathrm{IndCoh}(X)_{/T(X)}$. We have:

LEMMA 9.3.3.   *The monad* $\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}$ *preserves the full subcategories*

$$(\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0}))_{/T(X)} \subset (\Upsilon_X(\mathrm{QCoh}(X)^{\leq 0}))_{/T(X)} \subset \mathrm{IndCoh}(X)_{/T(X)}.$$

The map of functors

$$\tau_{\mathrm{QCoh}}^{\geq 0} \circ (\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}) \to$$
$$\to \tau_{\mathrm{QCoh}}^{\geq 0} \circ (\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}) \circ \tau_{\mathrm{QCoh}}^{\geq 0}$$

is an isomorphism.

PROOF.   Follows from Proposition 5.3.2.   □

9.3.4.   From Lemma 9.3.3 we obtain that the endo-functor

$$\tau_{\mathrm{QCoh}}^{\geq 0} \circ (\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}})$$

of

$$(\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit}))_{/T(X)} \to (\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit}))_{/T(X)}$$

has a natural structure of monad, and the category

$$(9.1) \quad \tau_{\mathrm{QCoh}}^{\geq 0} \circ (\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}})\text{-mod}((\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit}))_{/T(X)})$$

identifies canonically with the full subcategory of

$$(\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}})\text{-mod}((\Upsilon_X(\mathrm{QCoh}(X)^{\leq 0}))_{/T(X)}),$$

equal to the preimage of

$$(\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit}))_{/T(X)} \subset (\Upsilon_X(\mathrm{QCoh}(X)^{\leq 0}))_{/T(X)}$$

under the forgetful functor

$$(\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}})\text{-mod}((\Upsilon_X(\mathrm{QCoh}(X)^{\leq 0}))_{/T(X)}) \to$$
$$\to (\Upsilon_X(\mathrm{QCoh}(X)^{\leq 0}))_{/T(X)}.$$

Thus, we obtain that the full subcategory of $\mathrm{LieAlgbroid}(X)$ appearing in Theorem 9.1.5, identifies canonically with the category (9.1).

Hence, to prove Theorem 9.1.5, it suffices to show that under the equivalence (of ordinary (!) categories)

$$(\mathrm{QCoh}(X)^{\heartsuit})_{/T^{\mathrm{naive}}(X)} \simeq (\Upsilon_X(\mathrm{QCoh}(X)^{\heartsuit}))_{/T(X)},$$

the monad

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}/T^{\mathrm{naive}}} \circ \mathbf{free}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}}$$

identifies with the monad

$$\tau_{\mathrm{QCoh}}^{\geq 0} \circ (\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} \circ \mathbf{free}_{\mathrm{LieAlgbroid}}).$$

9.3.5. Note, however, that from Theorem 9.2.2, we obtain that the two monads are canonically identified when restricted to

$$(\text{QCoh}(X)^{\heartsuit,\text{proj},\aleph_0})_{/T^{\text{naive}}(X)} \simeq (\Upsilon_X(\text{QCoh}(X)^{\heartsuit,\text{proj},\aleph_0}))_{/T(X)}.$$

Moreover, it is easy to see that the monad $\mathbf{oblv}_{\text{LieAlgbroid}^{\text{cl}}/T^{\text{naive}}} \circ \mathbf{free}_{\text{LieAlgbroid}^{\text{cl}}}$ commutes with sifted colimits. The corresponding fact holds also for the monad

$$\tau_{\text{QCoh}}^{\geq 0} \circ (\mathbf{oblv}_{\text{LieAlgbroid}/T} \circ \mathbf{free}_{\text{LieAlgbroid}}),$$

by Corollary 8.2.9.

9.3.6. Now, the desired isomorphism of monads follows from the following fact: for any object $\gamma \in (\text{QCoh}(X)^{\heartsuit})_{/T^{\text{naive}}(X)}$, the category

$$\left((\text{QCoh}(X)^{\heartsuit,\text{proj},\aleph_0})_{/T^{\text{naive}}(X)}\right)_{/\gamma}$$

is sifted and the canonical map

$$\underset{\gamma' \in \left((\text{QCoh}(X)^{\heartsuit,\text{proj},\aleph_0})_{/T^{\text{naive}}(X)}\right)_{/\gamma}}{\text{colim}} \to \gamma$$

is an isomorphism.

9.4. Modules over classical Lie algebroids. In this subsection we compare we will compare the category $\mathfrak{L}\text{-mod}(\text{IndCoh}(X))$, as defined above, with the corresponding category for a classical Lie algebroid on a classical scheme.

9.4.1. Let X be a classical scheme of finite type, and let $\mathfrak{L}^{\text{cl}}$ be a classical Lie algebroid on X. Throughout this subsection we wil assume that $\mathfrak{L}^{\text{cl}}$ is *flat* as an $\mathcal{O}_X$-module.

Let

$$(\text{QCoh}(X \times X)_{\Delta_X})^{\heartsuit}_{\text{rel.flat}}$$

be the monoidal category introduced in Chapter 4, Sect. 4.1.1.

According to [**BB**, Sect. 2], to $\mathfrak{L}^{\text{cl}}$ one associates its universal enveloping algebra $U(\mathfrak{L}^{\text{cl}})$ which is an associative algebra object in $(\text{QCoh}(X \times X)_{\Delta_X})^{\heartsuit}_{\text{rel.flat}}$.

9.4.2. We have a canonically defined fully faithful monoidal functor

$$(\text{QCoh}(X \times X)_{\Delta_X})^{\heartsuit}_{\text{rel.flat}} \to \text{QCoh}(X \times X)$$

and a monoidal equivalence

$$\text{QCoh}(X \times X) \to \text{Funct}_{\text{cont}}(\text{QCoh}(X), \text{QCoh}(X)).$$

Composing, we obtain a fully faithful functor
(9.2)
$$\text{AssocAlg}((\text{QCoh}(X \times X)_{\Delta_X})^{\heartsuit}_{\text{rel.flat}}) \to \text{AssocAlg}\left(\text{Funct}_{\text{cont}}(\text{QCoh}(X), \text{QCoh}(X))\right).$$

Hence, we obtain that $U(\mathfrak{L}^{\text{cl}})$ gives rise to a monad acting on $\text{QCoh}(X)$. In particular, it makes sense to talk about the category

$$U(\mathfrak{L}^{\text{cl}})\text{-mod}(\text{QCoh}(X)).$$

This is, by definition, the category of modules over the classical Lie algebroid $\mathfrak{L}^{\text{cl}}$, denoted $\mathfrak{L}^{\text{cl}}\text{-mod}(\text{QCoh}(X))$.

REMARK 9.4.3. The category $\mathfrak{L}^{\mathrm{cl}}\text{-mod}(\mathrm{QCoh}(X))$ has a t-structure uniquely characterized by the property that the forgetful functor to $\mathrm{QCoh}(X)$ is t-exact. Now, as in [**GaRo2**, Proposition 4.7.3] one can show that if $\mathfrak{L}^{\mathrm{cl}}$ is flat as an object of $\mathrm{QCoh}(X)$, then the naturally defined functor

$$D((\mathfrak{L}^{\mathrm{cl}}\text{-mod}(\mathrm{QCoh}(X)))^\heartsuit) \to \mathfrak{L}^{\mathrm{cl}}\text{-mod}(\mathrm{QCoh}(X))$$

is an equivalence.

9.4.4. Let $\mathfrak{L}$ be the object of $\mathrm{LieAlgbroid}(X)$, corresponding to $\mathfrak{L}^{\mathrm{cl}}$ under the equivalence of Theorem 9.1.5.

The next assertion follows from Chapter 9, Theorem 6.1.2 (which will be proved independently):

LEMMA 9.4.5. *For $\mathfrak{L}^{\mathrm{cl}}$ flat as an $\mathcal{O}_X$-module, the endo-functor $\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L}))$ preserves the essential image of the (fully faithful) functor $\Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X)$.*

Hence, we obtain that $U(\mathfrak{L})$ defines a monad, denoted $U(\mathfrak{L})|_{\mathrm{QCoh}(X)}$, on $\mathrm{QCoh}(X)$. Moreover, the functor Υ_X gives rise to a fully faithful functor

$$U(\mathfrak{L})\text{-mod}(\mathrm{QCoh}(X)) \to U(\mathfrak{L})\text{-mod}(\mathrm{IndCoh}(X)) := \mathfrak{L}\text{-mod}(\mathrm{IndCoh}(X)).$$

9.4.6. We are going to prove:

THEOREM 9.4.7. *The monads $U(\mathfrak{L}^{\mathrm{cl}})$ and $U(\mathfrak{L})|_{\mathrm{QCoh}(X)}$ on $\mathrm{QCoh}(X)$ are canonically isomorphic.*

As a corollary, we obtain:

COROLLARY 9.4.8. *The category $\mathfrak{L}^{\mathrm{cl}}\text{-mod}(\mathrm{QCoh}(X))$ is canonically equivalent to the full subcategory of $\mathfrak{L}\text{-mod}(\mathrm{IndCoh}(X))$, consisting of objects, whose image under the forgetful functor*

$$\mathfrak{L}\text{-mod}(\mathrm{IndCoh}(X)) \to \mathrm{IndCoh}(X)$$

lies in the essential image of $\Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X)$.

9.4.9. *Proof of Theorem 9.4.7, Step 1.* First, the assumption on $\mathfrak{L}^{\mathrm{cl}}$ and Chapter 9, Theorem 6.1.2 imply that $U(\mathfrak{L})|_{\mathrm{QCoh}(X)}$ lies in the essential image of the functor (9.2).

Hence, the assertion of the the theorem is about comparison of associative algebras in the *ordinary* monoidal category $(\mathrm{QCoh}(X \times X)_{\Delta_X})^\heartsuit_{\mathrm{rel.flat}}$.

In particular, the assertion is Zariski-local on X, and hence we can assume that X is affine.

9.4.10. *Proof of Theorem 9.4.7, Step 2.* We claim that the stated isomorphism of associative algebras holds when

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}^{\mathrm{cl}}}(\mathfrak{L}^{\mathrm{cl}}) \in \mathrm{QCoh}(X)^{\heartsuit,\mathrm{proj},\aleph_0}.$$

Indeed, this follows by unwinding the construction of the equivalence in Theorem 9.2.2.

9.4.11. *Proof of Theorem 9.4.7, Step 3.* We claim that the assignments

$$\mathfrak{L} \rightsquigarrow U(\mathfrak{L}^{\mathrm{cl}}) \text{ and } \mathfrak{L} \rightsquigarrow U(\mathfrak{L})|_{\mathrm{QCoh}(X)}$$

commute with sifted colimits.

Indeed, for $U(\mathfrak{L}^{\mathrm{cl}})$ this follows from the construction. For $U(\mathfrak{L})|_{\mathrm{QCoh}(X)}$, this follows from Proposition 2.1.3(a) and Chapter 9, Theorem 6.1.2.

9.4.12. *Proof of Theorem 9.4.7, Step 4.* The required isomorphism follows from Step 3, since our $\mathfrak{L}^{\mathrm{cl}}$ can be written as a sifted colimit of Lie algebroids as in Step 2, see Sect. 9.3.6. □

A. An application: ind-coherent sheaves on push-outs

In this section we will use the material from Sect. 6.3 to show that the categories $\mathrm{IndCoh}(-)$ and $\mathrm{QCoh}(-)^{\mathrm{perf}}$ behave well with respect to push-outs of affine schemes.

A.1. Behavior of ind-coherent sheaves with respect to push-outs. In this subsection we will consider the case of IndCoh.

A.1.1. Let

$$
\begin{array}{ccc}
X_1' & \xrightarrow{\ f'\ } & X_2' \\
{\scriptstyle g_1}\big\uparrow & & \big\uparrow{\scriptstyle g_2} \\
X_1 & \xrightarrow{\ f\ } & X_2
\end{array}
$$

(A.1)

be a push-out diagram in $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, where the vertical maps are closed embeddings, and the horizontal maps are finite. Consider the corresponding commutative diagram of categories

(A.2)

$$
\begin{array}{ccc}
\mathrm{IndCoh}(X_1') & \xleftarrow{\ (f')^!\ } & \mathrm{IndCoh}(X_2') \\
{\scriptstyle g_1^!}\big\downarrow & & \big\downarrow{\scriptstyle g_2^!} \\
\mathrm{IndCoh}(X_1) & \xleftarrow{\ f^!\ } & \mathrm{IndCoh}(X_2).
\end{array}
$$

The goal of this subsection is to prove the following result:

THEOREM A.1.2. *The diagram* (A.2) *is a pullback square.*

The rest of this subsection is devoted to the proof of Theorem A.1.2.

A.1.3. *Reduction step 1.* Note that in Chapter 1, Proposition 1.4.5 we showed that the functor

(A.3) $\mathrm{IndCoh}(X_2') \to \mathrm{IndCoh}(X_1') \underset{\mathrm{IndCoh}(X_1)}{\times} \mathrm{IndCoh}(X_2)$

is fully faithful. So, it remains to show that the functor (A.3) is essentially surjective.

Let $\mathrm{IndCoh}(X_1')_{X_1} \subset \mathrm{IndCoh}(X_1')$ (resp., $\mathrm{IndCoh}(X_2')_{X_2} \subset \mathrm{IndCoh}(X_2')$) be the full subcategory consisting of objects with set-theoretic support on X_1 (resp., X_2). It is easy to see that it is sufficient to show that the corresponding functor

(A.4) $\mathrm{IndCoh}(X_2')_{X_2} \to \mathrm{IndCoh}(X_1')_{X_1} \underset{\mathrm{IndCoh}(X_1)}{\times} \mathrm{IndCoh}(X_2)$

is an equivalence.

Indeed, the essential surjectivity of (A.4) will imply the same property of (A.3), which follows from the localization sequences of DG categories

$$\mathrm{IndCoh}(X_2')_{X_2} \to \mathrm{IndCoh}(X_2') \to \mathrm{IndCoh}(X_2' \smallsetminus X_2)$$

and

$$\mathrm{IndCoh}(X_1')_{X_1} \underset{\mathrm{IndCoh}(X_1)}{\times} \mathrm{IndCoh}(X_2) \to \mathrm{IndCoh}(X_1') \underset{\mathrm{IndCoh}(X_1)}{\times} \mathrm{IndCoh}(X_2) \to \mathrm{IndCoh}(X_2' \smallsetminus X_2).$$

A.1.4. *Reduction step 2.* The formal completion of X_1 in X_1' can be written as a *filtered* colimit of schemes $X_{1,\alpha}'$, where each $X_1 \to X_{1,\alpha}'$ is a nilpotent embedding. Then the formal completion of X_2 in X_2' can be written as the colimit of the schemes

$$X_{2,\alpha}' := X_{1,\alpha}' \underset{X_1}{\sqcup} X_2,$$

see [**GaRo1**, Proposition 6.7.4].

The functors

$$\mathrm{IndCoh}(X_2')_{X_2} \to \lim_\alpha \mathrm{IndCoh}(X_{2,\alpha}') \text{ and } \mathrm{IndCoh}(X_1')_{X_1} \to \lim_\alpha \mathrm{IndCoh}(X_{1,\alpha}')$$

are both equivalences (see [**GaRo1**, Proposition 7.4.5]).

This reduces us to the case when $X_1 \to X_1'$ is a nilpotent embedding.

A.1.5. *Reduction step 3.* Using Chapter 1, Proposition 5.5.3 and the convergence property of IndCoh (see Volume I, Chapter 5, Proposition 6.4.3) we can further reduce to the case when the map

$$X_1 \to X_1'$$

has a structure of a square-zero extension.

A.1.6. *Proof in the case when $X_1 \to X_1'$ is a square-zero extension.* Let the square-zero extension $X_1 \to X_1'$ be given by a map

$$T^*(X_1) \to \mathcal{F}, \quad \mathcal{F}[-1] \in \mathrm{Coh}(X_1).$$

Then $X_2 \to X_2'$ is also a square-zero extension, given by

$$T^*(X_2) \xrightarrow{(df)^*} f_*(T^*(X_1)) \to f_*(\mathcal{F}).$$

Denote

$$\widetilde{\mathcal{F}}_1 := \mathbb{D}_{X_1}^{\mathrm{Serre}}(\mathcal{F}), \quad \widetilde{\mathcal{F}}_2 := \mathbb{D}_{X_2}^{\mathrm{Serre}}(f_*(\mathcal{F})).$$

Since f is finite, we have

$$f_*^{\mathrm{IndCoh}}(\widetilde{\mathcal{F}}_1) \simeq \widetilde{\mathcal{F}}_2.$$

According to Theorem 6.3.3, the category $\mathrm{IndCoh}(X_1')$ can be described as consisting of pairs $\mathcal{F}_1' \in \mathrm{IndCoh}(X_1)$, equipped with a null-homotopy of the composition

$$\widetilde{\mathcal{F}}_1[-1] \overset{!}{\otimes} \mathcal{F}_1' \to T(X_1)[-1] \overset{!}{\otimes} \mathcal{F}_1' \to \mathcal{F}_1',$$

and similarly for $\mathrm{IndCoh}(X_2')$.

Now, this makes the assertion of Theorem A.1.2 manifest: an object of the fiber product $\text{IndCoh}(X_1') \underset{\text{IndCoh}(X_1)}{\times} \text{IndCoh}(X_2)$ is an object $\mathcal{F}_2' \in \text{IndCoh}(X_2)$, equipped with a null-homotopy for the composition

$$\widetilde{\mathcal{F}}_1[-1] \overset{!}{\otimes} f^!(\mathcal{F}_2') \to T(X_1)[-1] \overset{!}{\otimes} f^!(\mathcal{F}_2') \to f^!(\mathcal{F}_2'),$$

which by adjunction is the same as a null-homotopy of the map

$$f_*^{\text{IndCoh}}(\widetilde{\mathcal{F}}_1[-1] \overset{!}{\otimes} f^!(\mathcal{F}_2')) \to f_*^{\text{IndCoh}}(T(X_1)[-1] \overset{!}{\otimes} f^!(\mathcal{F}_2')) \to \mathcal{F}_2',$$

while the latter, by the projection formula identifies with the map

$$f_*^{\text{IndCoh}}(\mathcal{F}_1)[-1] \overset{!}{\otimes} \mathcal{F}_2' \to f_*^{\text{IndCoh}}(T(X_1))[-1] \overset{!}{\otimes} \mathcal{F}_2' \to \mathcal{F}_2',$$

and the latter map identifies with

$$\widetilde{\mathcal{F}}_2[-1] \overset{!}{\otimes} \mathcal{F}_2' \to T(X_2)[-1] \overset{!}{\otimes} \mathcal{F}_2' \to \mathcal{F}_2'.$$

$\square$

A.2. Deformation theory for the functor $\text{QCoh}(-)^{\text{perf}}$.

In this subsection we will study the behavior of the category $\text{QCoh}(-)^{\text{perf}}$ with respect to push-outs.

A.2.1.   First, we claim that Theorem A.1.2 admits the following corollary:

COROLLARY A.2.2. *Under the assumptions of Theorem A.1.2, the diagram*

(A.5)
$$\begin{array}{ccc}
\text{QCoh}(X_1')^{\text{perf}} & \overset{(f')^*}{\longleftarrow} & \text{QCoh}(X_2')^{\text{perf}} \\
{\scriptstyle g_1^*}\downarrow & & \downarrow{\scriptstyle g_2^*} \\
\text{QCoh}(X_1)^{\text{perf}} & \overset{f^*}{\longleftarrow} & \text{QCoh}(X_2)^{\text{perf}}
\end{array}$$

is a pullback square.

PROOF. Follows from the fact that we have a commutative diagram of *symmetric monoidal* functors

$$\begin{array}{ccc}
\text{QCoh}(X_2') & \longrightarrow & \text{QCoh}(X_1') \underset{\text{QCoh}(X_1)}{\times} \text{QCoh}(X_2) \\
{\scriptstyle \Upsilon}\downarrow & & \downarrow{\scriptstyle \Upsilon} \\
\text{IndCoh}(X_2') & \longrightarrow & \text{IndCoh}(X_1') \underset{\text{IndCoh}(X_1)}{\times} \text{IndCoh}(X_2),
\end{array}$$

combined with Volume I, Chapter 6, Lemma 3.3.7:

Indeed, Theorem A.1.2 implies that the bottom horizontal arrow identifies the category of dualizable objects in $\text{IndCoh}(X_2')$ with

$$\text{IndCoh}(X_1')^{\text{dualizable}} \underset{\text{IndCoh}(X_1)^{\text{dualizable}}}{\times} \text{IndCoh}(X_2)^{\text{dualizable}}.$$

$\square$

A.2.3.   We now claim that the diagram (A.5) is a pullback square for any diagram of affine schemes (A.2), in which the vertical arrows are closed embeddings and horizontal maps finite.

Indeed, the laft property of the functor $\mathrm{QCoh}(-)^{\mathrm{perf}}$, we reduce the assertion to the case when X_1, X_1' and X_2 belong to $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$.

A.2.4.   We are now ready to finish the proof of the fact that the prestack Perf admits deformation theory.

From Corollary A.2.2 it follows that Perf admits pro-cotangent spaces and is infinitesimally cohesive. Hence, it remains to show that it admits a pro-cotangent complex.

By Chapter 1, Lemma 4.2.4(b), it suffices to prove the following. Let $f : X_1 \to X_2$ be a map in $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}}$, and let $\mathcal{F}_1$ be an object of $\mathrm{Coh}(X_1)^{\leq 0}$, and let $(X_1)_{\mathcal{F}_1}$ denote the corresponding split square-zero extension of X_1.

For every $\mathcal{F}_2 \in \mathrm{Coh}(X_2)^{\leq 0}$ equipped with a map $f^*(\mathcal{F}_2) \to \mathcal{F}_1$, consider the map

$$(X_1)_{\mathcal{F}_1} \to (X_2)_{\mathcal{F}_2},$$

and the corresponding functor

$$\mathrm{QCoh}((X_2)_{\mathcal{F}_2})^{\mathrm{perf}} \to \mathrm{QCoh}((X_1)_{\mathcal{F}_1})^{\mathrm{perf}} \underset{\mathrm{QCoh}(X_1)^{\mathrm{perf}}}{\times} \mathrm{QCoh}(X_2)^{\mathrm{perf}}.$$

We need to show that the functor

$$\underset{\mathcal{F}_2 \in \mathrm{Coh}(X_2)^{\leq 0}, f^*(\mathcal{F}_2) \to \mathcal{F}_1}{\mathrm{colim}} \mathrm{QCoh}((X_2)_{\mathcal{F}_2})^{\mathrm{perf}} \to \mathrm{QCoh}((X_1)_{\mathcal{F}_1})^{\mathrm{perf}} \underset{\mathrm{QCoh}(X_1)^{\mathrm{perf}}}{\times} \mathrm{QCoh}(X_2)^{\mathrm{perf}}$$

is an equivalence.

We will deduce this from Theorem 6.3.3 and Volume I, Chapter 6, Lemma 3.3.7.

A.2.5.   We rewrite the category $\mathrm{QCoh}((X_1)_{\mathcal{F}_1})^{\mathrm{perf}}$ as consisting of pairs $\mathcal{F}' \in \mathrm{QCoh}(X_1)^{\mathrm{perf}}$, equipped with a map

$$\mathbb{D}^{\mathrm{Serre}}_{X_1}(\mathcal{F}_1) \overset{!}{\otimes} \Upsilon_{X_1}(\mathcal{F}') \to \Upsilon_{X_1}(\mathcal{F}')$$

in $\mathrm{IndCoh}(X_1)$, which is equivalent to a map

$$\mathrm{End}(\mathcal{F}') \to \mathcal{F}_1,$$

in $\mathrm{QCoh}(X_1)$, and similarly for $\mathrm{QCoh}((X_2)_{\mathcal{F}_2})^{\mathrm{perf}}$.

For a given $\mathcal{F}' \in \mathrm{QCoh}(X_2)^{\mathrm{perf}}$, denote $\mathcal{E} := \mathrm{End}(\mathcal{F}') \in \mathrm{QCoh}(X_2)^{\mathrm{perf}}$. Thus, we have to show that the map

(A.6) $$\underset{\mathcal{F}_2 \in \mathrm{Coh}(X_2)^{\leq 0}, f^*(\mathcal{F}_2) \to \mathcal{F}_1}{\mathrm{colim}} \mathrm{Maps}_{\mathrm{QCoh}(X_2)}(\mathcal{E}, \mathcal{F}_2) \to$$

$$\to \mathrm{Maps}_{\mathrm{QCoh}(X_1)}(f^*(\mathcal{E}), \mathcal{F}_1) \simeq \mathrm{Maps}_{\mathrm{QCoh}(X_2)}(\mathcal{E}, f_*(\mathcal{F}_1))$$

is an isomorphism.

A.2.6.   We note that the index category
$$\mathcal{F}_2 \in \mathrm{Coh}(X_2)^{\leq 0}, f^*(\mathcal{F}_2) \to \mathcal{F}_1$$
that appears in the above formula identifies by adjunction with
$$\mathcal{F}_2 \in \mathrm{Coh}(X_2)^{\leq 0}, \mathcal{F}_2 \to f_*(\mathcal{F}_1),$$
i.e., with $(\mathrm{Coh}(X_2)^{\leq 0})_{/f_*(\mathcal{F}_1)}$.

Since $f_*(\mathcal{F}_1) \in \mathrm{QCoh}(X_2)^{\geq 0}$, this category is filtered and the map
$$\operatorname*{colim}_{\mathcal{F}_2 \in \mathrm{Coh}(X_2)^{\leq 0}, \mathcal{F}_2 \to f_*(\mathcal{F}_1)} \mathcal{F}_2 \to f_*(\mathcal{F}_1)$$
is an isomorphism.

Now, the isomorphism in (A.6) follows from the fact that $\mathcal{E} \in \mathrm{QCoh}(X_2)$ is compact.

CHAPTER 9

Infinitesimal differential geometry

Introduction

0.1. What do we mean by 'infinitesimal differential geometry'? The goal of this chapter is to make sense in the context of derived algebraic geometry of a number of notions of *differential nature* that are standard when working with schemes. These notions include:

- Deformation to the normal cone of a closed embedding;
- The notion of the n-th infinitesimal neighborhood of a scheme embedded into another one;
- The PBW filtration on the universal enveloping algebra of a Lie algebroid (over a smooth scheme);
- The Hodge filtration (a.k.a. de Rham resolution) of the dualizing D-module (again, over a smooth scheme).

A feature of the above objects in the setting of classical schemes is that they are constructed by explicit formulas.

For example, the PBW filtration on the universal enveloping algebra of a Lie algebroid is defined by letting the n-th term of the filtration be generated by n-fold products of sections of the Lie algebroid, a notion that is hard to make sense in the context of higher algebra, and hence derived algebraic geometry.

The de Rham resolution

$$\omega_X \underset{\mathcal{O}_X}{\otimes} \Lambda^n(T(X)) \to ... \to \omega_X \underset{\mathcal{O}_X}{\otimes} T(X) \to \omega_X$$

is also defined by explicitly writing down the differential, something that we cannot do in higher algebra.

But our task is even harder: not only do we want to have the above notions for derived schemes, but we want to have them for objects (and maps) in the category $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. So, an altogether different method is needed to define these objects.

0.1.1. Continuing with the example of $U(\mathfrak{L})$ for a Lie algebroid $\mathfrak{L}$, the initial idea of how to produce a filtration is pretty clear: the category of filtered objects in Vect identifies with $\mathrm{QCoh}(\mathbb{A}^1)^{\mathbf{G}_m}$, and similarly, for a DG category $\mathbf{C}$, the category $\mathbf{C}^{\mathrm{Fil}}$ of filtered objects in $\mathbf{C}$ identifies with

$$(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbf{G}_m}.$$

Now, the category $\mathbf{C}^{\mathrm{Fil}, \geq 0}$ of *non-negatively* filtered objects identifies with

$$(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}},$$

where the superscript $\mathbb{A}^1_{\text{left-lax}}$ stands for the structure of left-lax equivariance with respect to the monoid $\mathbb{A}^1$; see Sect. 1.2.3 where this notion is introduced.

For $\mathfrak{L} \in \text{LieAlgbroid}(\mathcal{X})$, we regard $U(\mathfrak{L})$ as an algebra object in the monoidal category $\text{Funct}_{\text{cont}}(\text{IndCoh}(\mathcal{X}), \text{IndCoh}(\mathcal{X}))$, and we wish to lift it to an object
(0.1)
$$U(\mathfrak{L})^{\text{Fil}} \in \text{AssocAlg}\Big(\big(\text{Funct}_{\text{cont}}(\text{IndCoh}(\mathcal{X}), \text{IndCoh}(\mathcal{X})) \otimes \text{QCoh}(\mathbb{A}^1)\big)^{\mathbb{A}^1_{\text{left-lax}}}\Big).$$

We shall now explain how to produce such a $U(\mathfrak{L})^{\text{Fil}}$, and this will bring us to the idea of deformation to the normal cone (rather, normal bundle in the present context), central for this chapter.

0.1.2. Our main construction is the following. For $\mathcal{X} \in \text{PreStk}_{\text{laft-def}}$ and $\mathcal{Y} \in \text{FormMod}_{\mathcal{X}/}$ we construct a family

$$\mathcal{Y}_{\text{scaled}} \in \text{FormMod}_{\mathcal{X} \times \mathbb{A}^1 // \mathcal{Y} \times \mathbb{A}^1},$$

i.e., a family of objects of $\text{FormMod}_{\mathcal{X} // \mathcal{Y}}$ parameterized by points of $\mathbb{A}^1$.

The fiber $\mathcal{Y}_\lambda$ of this family over $0 \neq \lambda \in \mathbb{A}^1$ is be (canonically) isomorphic to the initial $\mathcal{Y}$. Its fiber $\mathcal{Y}_0$ over $0 \in \mathbb{A}^1$ identifies canonically with the vector-prestack $\text{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1])$ (see Chapter 7, Sect. 1.4), where we can think of $T(\mathcal{X}/\mathcal{Y})[1]$ as the normal to $\mathcal{X}$ in $\mathcal{Y}$.

Crucially, the above $\mathbb{A}^1$-family has the following extra structure: it is *left-lax equivariant* with respect to the monoid $\mathbb{A}^1$ acting on itself by multiplication. Concretely, this means that for $\lambda, a \in \mathbb{A}^1$, we have a system of maps

$$\mathcal{Y}_{a \cdot \lambda} \to \mathcal{Y}_\lambda$$

that satisfy a natural associativity condition.

We will denote the resulting object of $(\text{FormMod}_{\mathcal{X} \times \mathbb{A}^1 // \mathcal{Y} \times \mathbb{A}^1})^{\mathbb{A}^1_{\text{left-lax}}}$ by $\mathcal{Y}_{\text{scaled}, \mathbb{A}^1_{\text{left-lax}}}$. It is the existence of this object that will allow us to carry out the 'differential' constructions mentioned earlier.

0.1.3. Here is how the deformation $\mathcal{Y}_{\text{scaled}, \mathbb{A}^1_{\text{left-lax}}}$ can be used in order to produce the object (0.1).

The datum of $U(\mathfrak{L})$ is encoded by the category $\mathfrak{L}\text{-mod}(\text{IndCoh}(\mathcal{X}))$, equipped with the forgetful functor $\mathbf{oblv}_{\mathfrak{L}} : \mathfrak{L}\text{-mod}(\text{IndCoh}(\mathcal{X})) \to \text{IndCoh}(\mathcal{X})$.

As will be explained in Sects. 6.2 and 6.3, constructing $U(\mathfrak{L})^{\text{Fil}}$ is equivalent to finding a *right-lax* equivariant extension of the pair $(\mathfrak{L}\text{-mod}(\text{IndCoh}(\mathcal{X})), \mathbf{oblv}_{\mathfrak{L}})$.

Let $(\mathcal{X} \overset{f}{\to} \mathcal{Y}) \in \text{FormMod}_{\mathcal{X}/}$ be the formal moduli problem corresponding to $\mathfrak{L}$. According to Chapter 8, Sect. 4.1.2, we have an identification

$$\mathfrak{L}\text{-mod}(\text{IndCoh}(\mathcal{X})) \simeq \text{IndCoh}(\mathcal{Y})$$

under which the functor $\mathbf{oblv}_{\mathfrak{L}}$ corresponds to $f^!$.

We define the sought-for left-lax equivariant extension for $\text{IndCoh}(\mathcal{Y})$ to be the category $\text{IndCoh}(\mathcal{Y}_{\text{scaled}})$, and for the functor $f^!$ to be the pullback along

$$\mathcal{X} \times \mathbb{A}^1 \to \mathcal{Y}_{\text{scaled}}.$$

The right-lax equivariant structure on $\text{IndCoh}(\mathcal{Y}_{\text{scaled}})$ is given by the left-lax equivariant structure on $\mathcal{Y}_{\text{scaled}}$, given by $\mathcal{Y}_{\text{scaled}, \mathbb{A}^1_{\text{left-lax}}}$.

0.2. The n-th infinitesimal neighborhood and the Hodge filtration. The deformation $\mathcal{Y} \rightsquigarrow \mathcal{Y}_{\mathrm{scaled}, \mathbb{A}^1_{\mathrm{left\text{-}lax}}}$ is used also for the construction of n-th infinitesimal neighborhoods and of the Hodge filtration on the dualizing D-module (crystal).

0.2.1.   The idea of infinitesimal neighborhoods

$$(0.2) \qquad \qquad \mathcal{X} = \mathcal{X}^{(0)} \to \mathcal{X}^{(1)} \to \ldots \to \mathcal{X}^{(n)} \to \ldots \to \mathcal{Y}$$

is that each $\mathcal{X}^{(n)}$ is a square-zero extension of $\mathcal{X}^{(n-1)}$ by means of the object of $\mathrm{IndCoh}(\mathcal{X}^{(n-1)})$ equal to the direct image of $\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])$ under $\mathcal{X} \to \mathcal{X}^{(n-1)}$. We remind that $T(\mathcal{X}/\mathcal{Y})[1]$ should be thought of as the normal bundle to $\mathcal{X}$ inside $\mathcal{Y}$.

To specify such an extension we need to specify a map

$$(0.3) \qquad \qquad \mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1]) \to T(\mathcal{X}^{(n-1)})/\mathcal{Y})|_{\mathcal{X}}.$$

For example, for $n = 1$, the map (0.3) is the identity. However, for $n \geq 2$ we encounter a problem: which map should it be?

Here the deformation $\mathcal{Y} \rightsquigarrow \mathcal{Y}_{\mathrm{scaled}, \mathbb{A}^1_{\mathrm{left\text{-}lax}}}$ comes to our rescue.

0.2.2.   We modify the problem, and instead of the system (0.2), we want to construct its filtered version

$$(0.4) \qquad \mathcal{X} \times \mathbb{A}^1 = \mathcal{X}^{(0)}_{\mathrm{scaled}} \to \mathcal{X}^{(1)}_{\mathrm{scaled}} \to \ldots \to \mathcal{X}^{(n)}_{\mathrm{scaled}} \to \ldots \to \mathcal{Y}_{\mathrm{scaled}}$$

in $(\mathrm{FormMod}_{\mathcal{X} \times \mathbb{A}^1 // \mathcal{Y} \times \mathbb{A}^1})^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}$.

In particular, instead of the map (0.3), we now need to construct the map

$$(0.5) \qquad \qquad \mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1]) \to T(\mathcal{X}^{(n-1)}_{\mathrm{scaled}})/\mathcal{Y})|_{\mathcal{X} \times \mathbb{A}^1},$$

in

$$\mathrm{IndCoh}(\mathcal{X} \times \mathbb{A}^1)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil}, \geq 0},$$

where $\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])$ is placed in degree n.

Now, the point is that one can prove that $T(\mathcal{X}^{(n-1)}_{\mathrm{scaled}})/\mathcal{Y})|_{\mathcal{X} \times \mathbb{A}^1}$ belongs to

$$\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil}, \geq n} \subset \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil}, \geq 0},$$

and its n-th associated graded is isomorphic precisely to $\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])$, and this gives rise to the desired map (0.5).

0.2.3.   When $\mathcal{X} \to \mathcal{Y}$ is a closed embedding, it is intuitively clear what the n-th infinitesmal neighborhood $\mathcal{X}^{(n)}$ of $\mathcal{X}$ in $\mathcal{Y}$ is doing.

But we can apply our construction to any map between objects of $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. In particular, we can take the map

$$p_{\mathcal{X}, \mathrm{dR}} : \mathcal{X} \to \mathcal{X}_{\mathrm{dR}}.$$

What is the n-th infinitesmal neighborhood of $\mathcal{X}$ in $\mathcal{X}_{\mathrm{dR}}$?

A concrete version of this question is the following: consider the filtration on $\omega_{\mathcal{X}_{\mathrm{dR}}}$, whose n-th term is the direct image of $\omega_{\mathcal{X}^{(n)}}$ under $\mathcal{X}^{(n)} \to \mathcal{X}_{\mathrm{dR}}$.

This filtration is the Hodge filtration on $\omega_{\mathcal{X}_{\mathrm{dR}}}$. Its n-th associated graded is

$$\mathbf{ind}_{\mathrm{dR}, \mathcal{X}}(\mathrm{Sym}^n(T(\mathcal{X})[1])) \in \mathrm{IndCoh}(\mathcal{X}_{\mathrm{dR}}) \simeq \mathrm{Crys}(\mathcal{X}).$$

If $\mathcal{X} = X$ is a smooth scheme, this filtration incarnates the de Rham resolution of the dualizing D-module.

0.3. Constructing the deformation. We now address the question of how the object

$$\mathcal{Y}_{\text{scaled},\mathbb{A}^1_{\text{left-lax}}} \in (\text{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1})^{\mathbb{A}^1_{\text{left-lax}}}$$

is constructed.

0.3.1.   To construct $\mathcal{Y}_{\text{scaled},\mathbb{A}^1_{\text{left-lax}}}$ we will use the equivalence of Chapter 5, Theorem 2.3.2, and will instead construct the corresponding ($\mathbb{A}^1$ left-lax equivariant) $\mathbb{A}^1$-family of formal groupoids over $\mathcal{X}$.

This $\mathbb{A}^1$-family of groupoids, denoted $\mathcal{R}^\bullet_{\text{scaled}}$, is constructed by a certain universal procedure, explained to us by J. Lurie.

0.3.2.   Namely, the prestacks $\mathcal{R}^n_{\text{scaled}}$ are obtained as mapping spaces from a certain universal family of affine schemes $\text{Bifurc}^n_{\text{scaled}}$ over $\mathbb{A}^1$, i.e., for a point $\lambda \in \mathbb{A}^1$, we have

$$\mathcal{R}^n_\lambda = \mathbf{Maps}((\text{Bifurc}^n_{\text{scaled}})_\lambda, \mathcal{X}) \underset{\mathbf{Maps}((\text{Bifurc}^n_{\text{scaled}})_\lambda, \mathcal{Y})}{\times} \mathcal{Y}.$$

The simplicial structure on the assignment $n \mapsto \mathcal{R}^n_{\text{scaled}}$ comes from the structure on the assignment

$$n \mapsto \text{Bifurc}^n_{\text{scaled}}$$

of simplicial object $\text{Bifurc}^\bullet_{\text{scaled}}$ in the category $((\text{Sch}^{\text{aff}})_{/\mathbb{A}^1})^{\text{op}}$.

0.3.3.   Once said in the above way, it is clear what $\text{Bifurc}^\bullet_{\text{scaled}}$ must be. For $n = 0$ we have $\text{Bifurc}^0_{\text{scaled}} = \mathbb{A}^1$, because we want $\mathcal{X}^0_\lambda$ to be just $\mathcal{X}$ for any $\lambda \in \mathbb{A}^1$.

For $0 \neq \lambda \in \mathbb{A}^1$ we want $\mathcal{X}^\bullet_\lambda$ to be the Čech nerve of the map $\mathcal{X} \to \mathcal{Y}$. So, $(\text{Bifurc}^\bullet_{\text{scaled}})_\lambda$ is the groupoid in $(\text{Sch}^{\text{aff}})^{\text{op}}$ given by

$$(\text{Bifurc}^n_{\text{scaled}})_\lambda = \underbrace{\text{pt} \sqcup ... \sqcup \text{pt}}_{n+1}.$$

I.e., this is the Čech nerve of the map $\varnothing \to \text{pt}$ in $(\text{Sch}^{\text{aff}})^{\text{op}}$.

Now, since we want $\mathcal{X}^1_0$ to be $\text{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1])$, we want $(\text{Bifurc}^1_{\text{scaled}})_0$ to be the scheme of dual numbers. From here, it is easy to to guess that all of $\text{Bifurc}^1_{\text{scaled}}$ should be

$$\text{Spec}(k[u,\lambda]/(u-\lambda)\cdot(u+\lambda)),$$

where the variable u corresponds to the projection $\text{Bifurc}^1_{\text{scaled}} \to \mathbb{A}^1$.

The structure on $(\text{Bifurc}^1_{\text{scaled}})_\lambda$ of groupoid in $(\text{Sch}^{\text{aff}})^{\text{op}}$ is completely determined by what it is when localized away from $0 = \lambda \in \mathbb{A}^1$.

0.4. What else is done in this chapter?
0.4.1.   In Sect. 2 we perform the main construction of this chapter–that of the deformation

$$\mathcal{Y}_{\text{scaled},\mathbb{A}^1_{\text{left-lax}}} \in (\text{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1})^{\mathbb{A}^1_{\text{left-lax}}}.$$

0.4.2. In Sect. 3 we translate the construction $\mathcal{Y} \rightsquigarrow \mathcal{Y}_{\text{scaled}, \mathbb{A}^1_{\text{left-lax}}}$ to the language of Lie algebroids.

We obtain that any Lie algebroid $\mathfrak{L}$ canonically gives rise to a *non-negatively filtered Lie algebroid*, denoted $\mathfrak{L}^{\text{Fil}}$, which technically means an object of

$$\text{LieAlgbroid}(\mathcal{X} \times \mathbb{A}^1/\mathbb{A}^1),$$

equipped with a structure of left-lax equivariance with respect to $\mathbb{A}^1$.

The associated graded of $\mathfrak{L}^{\text{Fil}}$, i.e., the fiber of the above family over $0 \in \mathbb{A}^1$ is the *trivial* Lie algebroid corresponding to the object $\mathbf{oblv}_{\text{LieAlgbroid}}(\mathfrak{L}) \in \text{IndCoh}(\mathcal{X})$.

We show that the construction $\mathfrak{L} \rightsquigarrow \mathfrak{L}^{\text{Fil}}$ is compatible with the forgetful functor

$$\mathbf{oblv}_{\text{LieAlgbroid}/T} : \text{LieAlgbroid}(\mathcal{X}) \to \text{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}.$$

Namely, the object

$$\mathbf{oblv}_{\text{LieAlgbroid}/T}(\mathfrak{L}^{\text{Fil}}) \in (\text{IndCoh}(\mathcal{X})^{\text{Fil}, \geq 0})_{/T(\mathcal{X})} \simeq (\text{IndCoh}(\mathcal{X} \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X} \times \mathbb{A}^1}})^{\mathbb{A}^1_{\text{left-lax}}}$$

is the $\mathbb{A}^1$-family, whose value at $\lambda \in \mathbb{A}^1$ is obtained by scaling the original anchor map

$$\mathbf{oblv}_{\text{LieAlgbroids}}(\mathfrak{L}) \to T(\mathcal{X})$$

by λ.

0.4.3. In Sect. 4 we prove the following result: let $\mathcal{H}$ be a formal group over $\mathcal{X}$, and consider the corresponding pointed formal moduli problem $B_{\mathcal{X}}(\mathcal{H})$.

On the one hand, the procedure of deformation to the normal bundle yields the family $(B_{\mathcal{X}}(\mathcal{H}))_{\text{scaled}, \mathbb{A}^1_{\text{left-lax}}}$, which by functoriality is an object of

$$\text{Ptd}\left((\text{FormMod}_{/\mathcal{X} \times \mathbb{A}^1})^{\mathbb{A}^1_{\text{left-lax}}}\right).$$

On the other hand, using the equivalence

$$\text{Grp}(\text{FormMod}_{/\mathcal{X}}) \simeq \text{LieAlg}(\text{IndCoh}(\mathcal{X}))$$

and using the canonical deformation of any Lie algebra $\mathfrak{h} \rightsquigarrow \mathfrak{h}^{\text{Fil}}$ (see Chapter 6, Sect. 1.5), we obtain an object

$$\mathcal{H}_{\text{scaled}, \mathbb{A}^1_{\text{left-lax}}} \in \text{Grp}\left((\text{FormMod}_{/\mathcal{X} \times \mathbb{A}^1})^{\mathbb{A}^1_{\text{left-lax}}}\right).$$

We prove that there is a canonical isomorphism:

$$B_{\mathcal{X} \times \mathbb{A}^1}(\mathcal{H}_{\text{scaled}, \mathbb{A}^1_{\text{left-lax}}}) \simeq (B_{\mathcal{X}}(\mathcal{H}))_{\text{scaled}, \mathbb{A}^1_{\text{left-lax}}}.$$

I.e., the procedure of deforming a moduli problem, which was defined geometrically via the schemes $\text{Bifurc}^\bullet_{\text{scaled}, \mathbb{A}^1_{\text{left-lax}}}$, reproduces the procedure of scaling the Lie algebra.

0.4.4.   In Sect. 5 we carry out the constriction of the n-th infinitesimal neighborhood of a nil-isomorphism $\mathcal{X} \to \mathcal{Y}$.

We show that the natural map

$$\operatorname*{colim}_n \mathcal{X}^{(n)} \to \mathcal{Y}$$

is an isomorphism.

We use the above isomorphism to construct a filtration on $\omega_{\mathcal{Y}}$ whose n-th term is

$$(f_n)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(n)}}),$$

where f_n denotes the map $\mathcal{X}^{(n)} \to \mathcal{Y}$. When we interpret $\mathrm{IndCoh}(\mathcal{Y})$ as $\mathfrak{L}\text{-mod}(\mathrm{IndCoh}(\mathcal{X}))$ for the corresponding Lie algebroid $\mathfrak{L}$, the n-th associated graded of the above filtration is

$$\mathbf{ind}_{\mathfrak{L}}(\mathrm{Sym}^n(\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[1])).$$

When $\mathcal{Y} = \mathcal{X}_{\mathrm{dR}}$ we recover the Hodge filtration.

0.4.5.   In Sect. 6 we construct the filtration on the universal enveloping algebra of a Lie algebroid.

We also show that the n-th term of the filtration is given by pull-push along

$$\mathcal{X} \xleftarrow{p_s} {}'\mathcal{X}^{(n)} \xrightarrow{p_t} \mathcal{X},$$

where ${}'\mathcal{X}^{(n)}$ denotes the n-th infinitesimal neighborhood of $\mathcal{X}$ under the unit map $\mathcal{X} \to \mathcal{R}$, where $\mathcal{R}$ is the total space of the groupoid corresponding to $\mathfrak{L}$.

0.4.6.   Finally, in Sect. 7.1 we apply some elements of the theory developed above to the study of regular embeddings.

We say that a map $f : \mathcal{X} \to \mathcal{Y}$ between objects of $\mathrm{PreStk}_{\mathrm{def}}$ is a regular embedding of relative dimension n if

$$T^*(\mathcal{X}/\mathcal{Y})[-1] \in \mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)$$

belongs to $\mathrm{QCoh}(\mathcal{X})^-$ and is a vector bundle of rank n.

We show that for a regular embedding of relative dimension n, the functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{Y})$$

admits a *left* adjoint, to be denoted $f^{\mathrm{IndCoh},*}$, and we prove Grothendiek's formula

$$f^{\mathrm{IndCoh},*} \simeq \mathrm{Sym}^n(T^*(\mathcal{X}/\mathcal{Y})) \otimes f^!,$$

where we note that $\mathrm{Sym}^n(T^*(\mathcal{X}/\mathcal{Y})) \in \mathrm{QCoh}(\mathcal{X})$ is a line bundle placed in cohomological degree n.

As a corollary, we deduce that for a *schematic smooth map* $g : \mathcal{X} \to \mathcal{Z}$ of relative dimension n, the functor

$$g^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{X}),$$

left adjoint to g_*^{IndCoh}, is defined and we have

$$g^! \simeq \mathrm{Sym}^n(T^*(\mathcal{X}/\mathcal{Z})[1]) \otimes g^{\mathrm{IndCoh},*}.$$

1. Filtrations and the monoid $\mathbb{A}^1$

Let $\mathcal{C}$ be a functor
$$(\mathrm{PreStk})^{\mathrm{op}} \to 1\text{-}\mathrm{Cat}\,.$$
For example, $\mathcal{C}(\mathcal{X}) = \mathrm{QCoh}(\mathcal{X})$ or $\mathcal{C}(\mathcal{X}) = \mathrm{LieAlg}(\mathrm{QCoh}(\mathcal{X}))$.

Suppose now that a prestack $\mathcal{X}$ is acted on by a monoid $\mathcal{G}$. In this section we introduce the notion of what it means for an object $c \in \mathcal{C}(\mathcal{X})$ to be left-lax (resp., right-lax) equivariant with respect to $\mathcal{G}$. This notion generalizes the much more well-known one when $\mathcal{G}$ is a group (and when instead of lax equivariance we have the usual equivariance).

Taking $\mathcal{X} = \mathbb{A}^1$ and $\mathcal{G} = \mathbb{A}^1$, acting on itself by multiplication, we will see that the category
$$(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}$$
(here $\mathbf{C}$ is an arbitrary DG category) is equivalent to that of non-negatively filtered objects in $\mathbf{C}$.

This observation produces a mechanism of creating non-negatively filtered objects from algebraic geometry, as long as we can replace the initial geometric object by an $\mathbb{A}^1$-family, which is left-lax equivariant with respect to the action of $\mathbb{A}^1$ on itself.

1.1. Equivariance with respect to a monoid.
The notion of equivariance with respect to a group-action is completely standard. The situation with monoids may be less familiar: in fact, there are three different notions of equivariance: right-lax equivariance, left-lax equivariance and just (or strict) equivariance.

1.1.1. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two ∞-categories, each equipped with an action of a monoid-object of Spc, denoted G. Let $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ be a functor.

Informally, a structure of *right-lax equivariance* (resp., *left-lax equivariance*) on Φ with respect to G is a homotopy-coherent system of assignments for every point $g \in G$ of a natural transformation $g \circ \Phi \to \Phi \circ g$ (resp., $\Phi \circ g \to g \circ \Phi$), in a way compatible with the monoid structure.

A structure of (strict) *equivariance* is when the above maps are isomorphisms.

If G is a group, then the above maps are automatically isomorphisms.

1.1.2. Formally, the notion of *right-lax equivariance* falls into the paradigm of right-lax module functors between two module categories over a given monoidal category: we can view G as a monoidal ∞-category.

Equivalently, this definition can be formalized as follows. Consider the corresponding simplicial object $G^\bullet$ in Spc, and consider the ∞-category
$$BG := \mathfrak{L}(G^\bullet),$$
see Volume I, Chapter 10, Sect. 1.3.2 for the notation.

I.e., BG is a category with one object, whose monoid of endomorphisms is identified with G.

The datum of action of G on an ∞-category $\mathcal{C}$ is equivalent to that of a co-Cartesian fibration
$$\mathcal{C}_{BG} \to BG,$$

that fits into a pullback diagram

$$
\begin{CD}
\mathcal{C} @>>> \mathcal{C}_{BG} \\
@VVV @VVV \\
* @>>> BG.
\end{CD}
$$

A structure right-lax equivariance on Φ with respect to G is by definition a datum of extension of Φ to a functor

$$\Phi_{BG} : (\mathcal{C}_1)_{BG} \to (\mathcal{C}_2)_{BG}$$

over BG.

A structure of right-lax equivariance on Φ is a structure of (strict) equivariance if Φ_{BG} sends co-Cartesian arrows to co-Cartesian arrows. Equivalently, this is a natural transformation between the functors

$$BG \to 1\text{-Cat},$$

classifying the co-Cartesian fibrations $(\mathcal{C}_1)_{BG}$ and $(\mathcal{C}_2)_{BG}$, respectively.

A structure of left-lax equivariance on Φ is a structure of right-lax equivariance on the functor

$$\Phi^{\mathrm{op}} : \mathcal{C}_1^{\mathrm{op}} \to \mathcal{C}_2^{\mathrm{op}}.$$

1.1.3. It is clear that the composition of functors, each endowed with a structure of right-lax (resp., left-lax) equivariance, has a structure of right-lax (resp., left-lax) equivariance.

It is also clear that if a functor Φ has a structure of right-lax (resp., left-lax) equivariance, then its left (resp., right) adjoint, if it exists, has a natural structure of left-lax (resp., right-lax) equivariance.

1.2. Equivariance in algebraic geometry. In this subsection we will adapt the notion of equivariant functor, where instead of just ∞-categories we consider contravariant functors on $\mathrm{Sch}^{\mathrm{aff}}$ with values in ∞-categories.

As a particular case, we will obtain the notion of left-lax or right-lax equivariant quasi-coherent sheaf on a prestack, equipped with an action of a monoid.

1.2.1. Let $\mathcal{C}$ be a presheaf of categories, i.e., a functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to 1\text{-Cat}.$$

Let $\mathcal{G}$ be a monoidal prestack, i.e., a monoid-object in PreStk, equivalently, a functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Monoid}(\mathrm{Spc}).$$

Informally, an action of $\mathcal{G}$ on $\mathcal{C}$ is by definition a system of actions of $\mathcal{G}(S)$ on $\mathcal{C}(S)$ for $S \in \mathrm{Sch}^{\mathrm{aff}}$, compatible with pullbacks.

Formally, an action of $\mathcal{G}$ on $\mathcal{C}$ is a datum of a functor

$$\mathcal{C}_{B\mathcal{G}} : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to 1\text{-Cat},$$

equipped with a natural transformation $\mathcal{C}_{B\mathcal{G}} \to B\mathcal{G}$ and a pullback diagram

$$
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & \mathcal{C}_{B\mathcal{G}} \\
\downarrow & & \downarrow \\
\mathrm{pt} & \longrightarrow & B\mathcal{G},
\end{array}
$$

such that for every $S \in \mathrm{Sch}^{\mathrm{aff}}$, the corresponding functor

$$\mathcal{C}_{B\mathcal{G}}(S) \to B\mathcal{G}(S)$$

is a co-Cartesian fibration.

1.2.2.   Let $\mathcal{C}_1, \mathcal{C}_2$ be two presheaves of categories, each equipped with an action of $\mathcal{G}$.

For a natural transformation $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$, a datum of *right-lax equivariance* (resp., *left-lax equivariance*) with respect to $\mathcal{G}$ is a compatible system of structures of right-lax equivariance (resp., left-lax equivariance) on the functors

$$\Phi_S : \mathcal{C}_1(S) \to \mathcal{C}_2(S).$$

1.2.3.   A particular case of this situation is when $\mathcal{C}_1 = \mathcal{X} \in \mathrm{PreStk}$, i.e., is a functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}.$$

In this case we can think of a functor Φ from $\mathcal{X}$ to $\mathcal{C} = \mathcal{C}_2$ as a section $p \in \mathcal{C}(\mathcal{X})$, where we regard $\mathcal{C}$ as a functor

$$\mathrm{PreStk}^{\mathrm{op}} \to \text{1-Cat}$$

by right Kan-extending the original $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \overset{\mathcal{C}}{\to} \text{1-Cat}$ along $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow \mathrm{PreStk}^{\mathrm{op}}$.

Thus, we obtain the notion of a section $p \in \mathcal{C}(\mathcal{X})$ to be *right-lax equivariant* (resp., *left-lax equivariant*, *(strictly) equivariant*) with respect to $\mathcal{G}$.

We will denote the resulting categories by

$$\mathcal{C}(\mathcal{X})^{\mathcal{G}_{\mathrm{right\text{-}lax}}}, \quad \mathcal{C}(\mathcal{X})^{\mathcal{G}_{\mathrm{left\text{-}lax}}} \text{ and } \mathcal{C}(\mathcal{X})^{\mathcal{G}},$$

respectively. We have the fully faithful embeddings

$$\mathcal{C}(\mathcal{X})^{\mathcal{G}_{\mathrm{right\text{-}lax}}} \hookleftarrow \mathcal{C}(\mathcal{X})^{\mathcal{G}} \hookrightarrow \mathcal{C}(\mathcal{X})^{\mathcal{G}_{\mathrm{left\text{-}lax}}}.$$

1.2.4.   An example of the situation in Sect. 1.2.3 is when

$$\mathcal{C} = \mathrm{QCoh}^*,$$

where the action of $\mathcal{G}$ on QCoh^* is trivial.

Thus, for $\mathcal{X} \in \mathrm{PreStk}$ and $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})$, a datum of right-lax equivariance (resp., left-lax equivariance) with respect to $\mathcal{G}$ assigns to every $x \in \mathrm{Maps}(S, \mathcal{X})$ and $g \in \mathrm{Maps}(S, \mathcal{G})$ a map

$$x^*(\mathcal{F}) \to (g \cdot x)^*(\mathcal{F}) \quad (\text{resp., } (g \cdot x)^*(\mathcal{F}) \to x^*(\mathcal{F})),$$

in a way compatible with products of g's and pullbacks $S_1 \to S_2$.

We let $\mathrm{QCoh}(\mathcal{X})^{\mathcal{G}_{\mathrm{right\text{-}lax}}}$ (resp., $\mathrm{QCoh}(\mathcal{X})^{\mathcal{G}_{\mathrm{left\text{-}lax}}}$) denote the category of objects in $\mathrm{QCoh}(\mathcal{X})$, equipped with a structure of right-lax (resp., left-lax) equivariance with respect to $\mathcal{G}$. We let $\mathrm{QCoh}(\mathcal{X})^{\mathcal{G}}$ be the category of objects equipped with a structure of (strict) $\mathcal{G}$-equivariance.

1.2.5.   Here are several other examples of presheaves $\mathcal{C}$ that we will use (for now, all of them have the trivial action of $\mathcal{G}$):

(i) Take $\mathcal{C}(S) = \mathbf{C} \otimes \mathrm{QCoh}(S)$ for a fixed DG category $\mathbf{C}$.

(ii) Take $\mathcal{C}(S) := \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O} \otimes \mathrm{QCoh}(S))$ for a fixed symmetric monoidal DG category $\mathbf{O}$ and an operad $\mathcal{P}$.

(iii) Take $\mathcal{C}(S) := \mathrm{QCoh}(S)\text{-}\mathbf{mod}$, the category of module categories over $\mathrm{QCoh}(S)$.

(iv) Take $\mathcal{C}(S) := \mathrm{PreStk}_{/S}$. In this case, for a prestack $\mathcal{X}$, a map $p : \mathcal{X} \to \mathcal{C}$ amounts to a prestack $\mathcal{Y}$ over $\mathcal{X}$. Given a $\mathcal{G}$-action on $\mathcal{X}$, a datum of right-lax equivariance on p is equivalent to that of a lift of the given $\mathcal{G}$-action on $\mathcal{X}$ to a $\mathcal{G}$-action on $\mathcal{Y}$. This structure is a (strict) equivariance if and only if the square

$$
\begin{array}{ccc}
\mathcal{G} \times \mathcal{Y} & \xrightarrow{\ \text{action}\ } & \mathcal{Y} \\
\downarrow & & \downarrow \\
\mathcal{G} \times \mathcal{X} & \xrightarrow{\ \text{action}\ } & \mathcal{X}
\end{array}
$$

is Cartesian.

1.3. The category of filtered objects. It is well-known that the formalism of equivariance with respect to the multiplicative group allows to give an algebro-geometric interpretation to the notion of filtered object in a given DG category.

We will review this construction in the present subsection.

1.3.1.   Let $\mathbf{C}$ be a DG category. Recall the notation

$$\mathbf{C}^{\mathrm{Fil}} := \mathrm{Funct}(\mathbb{Z}, \mathbf{C}),$$

see Chapter 6, Sect. 1.3.

1.3.2.   Consider the presheaf of categories $\mathbf{C} \otimes \mathrm{QCoh}(-)$ from Example (i) in Sect. 1.2.5.

We will take our group-prestack $\mathcal{G}$ to be $\mathbb{G}_m$. We take $\mathcal{X} = \mathbb{A}^1$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by multiplication.

PROPOSITION-CONSTRUCTION 1.3.3. *There is a canonical equivalence*

$$
\tag{1.1}
\mathbf{C}^{\mathrm{Fil}} \simeq \left(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1) \right)^{\mathbb{G}_m}.
$$

PROOF.   By [**Ga3**, Theorem 2.2.2], the natural functor

$$\mathbf{C} \otimes \left(\mathrm{QCoh}(\mathbb{A}^1) \right)^{\mathbb{G}_m} \to \left(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1) \right)^{\mathbb{G}_m}$$

is an equivalence. It is equally easy to see that the functor

$$\mathbf{C} \otimes \mathrm{Vect}^{\mathrm{Fil}} \to \mathbf{C}^{\mathrm{Fil}}$$

is an equivalence. Hence, it is sufficient to treat the case $\mathbf{C} = \mathrm{Vect}$.

We construct the functor

$$\left(\mathrm{QCoh}(\mathbb{A}^1) \right)^{\mathbb{G}_m} \to \mathrm{Vect}^{\mathrm{Fil}}$$

as follows. Given $\mathcal{F} \in \left(\mathrm{QCoh}(\mathbb{A}^1) \right)^{\mathbb{G}_m}$ we define the corresponding functor $\mathbb{Z} \to \mathrm{Vect}$ by

$$n \mapsto \Gamma\left(\mathbb{A}^1, \mathcal{F}(n \cdot \{0\}) \right)^{\mathbb{G}_m},$$

where the superscript $\mathbb{G}_m$ stands for taking $\mathbb{G}_m$-invariants, and $\mathcal{F}(n \cdot \{0\})$ means twisting $\mathcal{F}$ by the corresponding Cartier divisor.

The fact that this functor is an equivalence is a straightforward check.

$\square$

1.3.4. Consider the functor
$$\left(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{G}_m} \to \mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1) \to \mathbf{C},$$
given by restriction to $\{1\} \in \mathbb{A}^1$. Under the identification (1.1), this functor corresponds to the functor of 'forgetting the filtration'
$$\mathbf{oblv}_{\mathrm{Fil}} : \mathbf{C}^{\mathrm{Fil}} \to \mathbf{C}.$$

1.4. The category of graded objects. In this subsection we will consider a variant of the material in Sect. 1.3, where instead of filtered objects we consider graded ones.

1.4.1. Consider the category
$$\mathbf{C}^{\mathrm{gr}} := \mathbf{C}^{\mathbb{Z}}.$$

As in Proposition 1.3.3, we have:
$$\mathbf{C}^{\mathbb{Z}} \simeq \mathbf{C}^{\mathbb{G}_m},$$
i.e., this is the $\mathbb{G}_m$-equivariant category for the presheaf of categories $\mathbf{C} \otimes \mathrm{QCoh}(-)$ over $\mathcal{X} = \mathrm{pt}$.

1.4.2. The forgetful functor
$$\mathbf{C}^{\mathbb{G}_m} \to \mathbf{C}$$
corresponds to the functor of 'forgetting the grading'
$$\mathbf{oblv}_{\mathrm{gr}} : \mathbf{C}^{\mathrm{gr}} \to \mathbf{C}.$$

1.4.3. The adjoint functors
$$(\mathrm{gr} \to \mathrm{Fil}) : \mathbf{C}^{\mathrm{gr}} \rightleftarrows \mathbf{C}^{\mathrm{Fil}} : \mathrm{Rees}$$
(Chapter 6, Sect. 1.3.3) correspond to the functors of pullback and push-forward along the projection $\mathbb{A}^1 \to \mathrm{pt}$.

1.4.4. Consider the functor of the 'associated graded'
$$\mathrm{ass\text{-}gr} : \mathbf{C}^{\mathrm{Fil}} \to \mathbf{C}^{\mathrm{gr}},$$
see Chapter 6, Sect. 1.3.4.

In terms of the identification

(1.2) $$\mathbf{C}^{\mathrm{Fil}} \simeq \left(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{G}_m},$$

the functor ass-gr corresponds to
$$\mathcal{F} \mapsto (\mathrm{Id}_{\mathbf{C}} \otimes i_0)^*(\mathcal{F}),$$
where $i_0 : \{0\} \to \mathbb{A}^1$.

1.5. Positive and negative filtrations. It turns out that if in the discussion in Sect. 1.3, we replace the group $\mathbb{G}_m$ by the monoid $\mathbb{A}^1$, the corresponding extra structure will single out non-negative filtrations (in the case of left-lax equivariance) or non-positive filtrations (in the case of right-lax equivariance).

1.5.1.   Consider again the presheaf of categories from Example (i) in Sect. 1.2.5.

Let us now take the monoid-prestack $\mathcal{G} = \mathbb{A}^1$, where $\mathbb{A}^1$ is a monoid with respect to the operation of *multiplication*. We take $\mathcal{X} = \mathbb{A}^1$, where $\mathbb{A}^1$ acts on itself by multiplication. We consider $\mathbb{G}_m$ as a sub-monoid of $\mathbb{A}_1$.

We have:

LEMMA 1.5.2.

(a) *The forgetful functor*

$$\left(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \to \left(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{G}_m}$$

is fully faithful and its essential image identifies with $\mathbf{C}^{\mathrm{Fil},\geq 0} \subset \mathbf{C}^{\mathrm{Fil}}$.

(b) *The forgetful functor*

$$\left(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{A}^1_{\mathrm{right\text{-}lax}}} \to \left(\mathbf{C} \otimes \mathrm{QCoh}(\mathbb{A}^1)\right)^{\mathbb{G}_m}$$

is fully faithful and its essential image identifies with $\mathbf{C}^{\mathrm{Fil},\leq 0} \subset \mathbf{C}^{\mathrm{Fil}}$.

1.5.3.   We also have the following graded analog of Lemma 1.5.2:

LEMMA 1.5.4.

(a) *The forgetful functor*

$$\mathbf{C}^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \to \mathbf{C}^{\mathbb{G}_m}$$

is fully faithful and its essential image identifies with $\mathbf{C}^{\mathrm{gr},\geq 0} \subset \mathbf{C}^{\mathrm{gr}}$.

(b) *The forgetful functor*

$$\mathbf{C}^{\mathbb{A}^1_{\mathrm{right\text{-}lax}}} \to \mathbf{C}^{\mathbb{G}_m}$$

is fully faithful and its essential image identifies with $\mathbf{C}^{\mathrm{gr},\leq 0} \subset \mathbf{C}^{\mathrm{gr}}$.

1.6. Scaling the structure of a $\mathcal{P}$-algebra. In this subsection we make a digression and explain that the construction in Chapter 6, Sect. 1.4 of endowing an algebra B over an operad $\mathcal{P}$ with a filtration can be viewed as the operation of 'scaling' the structure maps $\mathcal{P}(n) \otimes B^{\otimes n} \to B$.

1.6.1.   Let $\mathbf{O}$ be a symmetric monoidal category, and let $\mathcal{P}$ be an operad. Recall the presheaf of categories

$$\mathcal{C}(S) := \mathbf{O} \otimes \mathrm{QCoh}(S),$$

endowed with the trivial action of a monoid $\mathcal{G}$.

The operad $\mathcal{P}$ defines an algebra object in the monoidal category of $\mathcal{G}$-equivariant endomorphisms of $\mathcal{C}$. It follows that for a prestack $\mathcal{X}$, equipped with an action of $\mathcal{G}$, the forgetful functor

$$(\mathcal{P}\text{-mod}(\mathcal{C}(\mathcal{X})))^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \to (\mathcal{C}(\mathcal{X}))^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}$$

is monadic, with the corresponding monad being given by the action of $\mathcal{P}$ on $(\mathcal{C}(\mathcal{X}))^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}$ as a symmetric monoidal DG category. Hence, we obtain an identification

$$(\mathcal{P}\text{-mod}(\mathcal{C}(\mathcal{X})))^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \simeq \mathcal{P}\text{-mod}(\mathcal{C}(\mathcal{X})^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}).$$

1.6.2. We apply this to $\mathcal{G} = \mathcal{X} = \mathbb{A}^1$, acting on itself by multiplication. Thus, we obtain an identification

$$(\mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathbb{A}^1)))^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \simeq \mathcal{P}\text{-Alg}((\mathbf{O} \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}).$$

Using Lemma 1.5.2(a), we identify

$$\mathcal{P}\text{-Alg}((\mathbf{O} \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}) \simeq \mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{Fil},\geq 0}).$$

Thus, we obtain a canonical equivalence:

$$(1.3) \qquad (\mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathbb{A}^1)))^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \simeq \mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{Fil},\geq 0}).$$

1.6.3. Recall the functor

$$\mathrm{AddFil} : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O}^{\mathrm{Fil},\geq 0}).$$

see Chapter 6, Sect. 1.4.2.

We obtain that it gives rise to a functor

$$\mathrm{Scale}^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} : \mathcal{P}\text{-Alg}(\mathbf{O}) \to (\mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathbb{A}^1)))^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}.$$

Composing with the forgetful functor

$$(\mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathbb{A}^1)))^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \to \mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathbb{A}^1)),$$

we obtain a functor

$$\mathrm{Scale} : \mathcal{P}\text{-Alg}(\mathbf{O}) \to \mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathbb{A}^1)).$$

Sometimes we will use the short-hand notation

$$B_{\mathrm{scaled}} := \mathrm{Scale}(B) \text{ and } B_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}} := \mathrm{Scale}^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}(B).$$

1.6.4. The functor Scale has the following properties:
- $\mathbf{oblv}_{\mathcal{P}}(B_{\mathrm{scaled}}) \simeq \mathbf{oblv}_{\mathcal{P}}(B) \otimes \mathcal{O}_{\mathbb{A}^1}$;
- $i_\lambda^*(B_{\mathrm{scaled}}) \simeq B$ for any $0 \neq \lambda \in \mathbb{A}^1$;
- $i_0^*(B_{\mathrm{scaled}}) \simeq \mathbf{triv}_{\mathcal{P}} \circ \mathbf{oblv}_{\mathcal{P}}(B)$.

REMARK 1.6.5. One can endow the functor Scale with a structure of *associativity* with respect to the monoid structure on $\mathbb{A}^1$. This gives rise to a *non-trivial* action of the monoid $\mathbb{A}^1$ on presheaf of categories $\mathcal{P}\text{-Alg}(\mathbf{O} \otimes \mathrm{QCoh}(-))$.

2. Deformation to the normal bundle

In this section we introduce a key construction that deforms a nil-isomorphism $\mathcal{X} \to \mathcal{Y}$ to its normal bundle. It is a derived analog of the deformation of a closed embedding to its normal cone.

In subsequent sections, this procedure will give rise to naturally defined filtrations on various objects constructed out of Lie algebroids (e.g., the universal enveloping algebra of a Lie algebroid).

The geometric input into the main construction in this section, explained in Sect. 2.2, was suggested to us by J. Lurie.

2.1. The idea. Before we give the formal construction, let us explain its idea. It is the following: given a nil-closed embedding $\mathcal{X} \to \mathcal{Y}$ we will construct the deformation of $\mathcal{Y}$ to the normal bundle by deforming the corresponding groupoid $\mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X}$.

The sought-for $\mathbb{A}^1$-family of groupoids of $\mathcal{X}$ will obtained by mapping into the original $\mathcal{X}$ a particular $\mathbb{A}^1$-family of groupoids in the category $({}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$, denoted

$$(\mathrm{Bifurc}^{\bullet}_{\mathrm{scaled}})_{\lambda}, \quad \lambda \in \mathbb{A}^1.$$

In this subsection we will informally describe what this family looks like.

We draw the reader's attention to the fact that groupoid objects in the category $({}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$ are not very familiar gadgets.

2.1.1.   For any $\lambda \in \mathbb{A}^1$, the scheme $(\mathrm{Bifurc}^0_{\mathrm{scaled}})_{\lambda}$ of objects in $(\mathrm{Bifurc}^{\bullet}_{\mathrm{scaled}})_{\lambda}$ is just pt. The scheme $(\mathrm{Bifurc}^1_{\mathrm{scaled}})_{\lambda}$ of 1-morphisms is described as follows: if $\lambda \neq 0$, then

$$(\mathrm{Bifurc}^1_{\mathrm{scaled}})_{\lambda} = \{\lambda\} \sqcup \{-\lambda\} \subset \mathbb{A}^1.$$

I.e., $(\mathrm{Bifurc}^{\bullet}_{\mathrm{scaled}})_{\lambda}$ is the free groupoid in $({}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$ with the scheme of objects being pt.

When $\lambda = 0$, then $(\mathrm{Bifurc}^1_{\mathrm{scaled}})_0$ is the scheme of dual numbers $\mathrm{Spec}(k[\epsilon]/\epsilon^2)$. I.e., we should think of $\mathrm{Spec}(k[\epsilon]/\epsilon^2)$ the limit of $\{\lambda\} \cup \{-\lambda\}$ as $\lambda \to 0$.

2.1.2.   For any $\lambda \in \mathbb{A}^1$ and $\mathcal{X} \in \mathrm{PreStk}$, we obtain a groupoid object in PreStk by considering the prestack of maps from $(\mathrm{Bifurc}^{\bullet}_{\mathrm{scaled}})_{\lambda}$ into it:

$$\mathbf{Maps}((\mathrm{Bifurc}^{\bullet}_{\mathrm{scaled}})_{\lambda}, \mathcal{X}).$$

Note that for $\lambda \neq 0$, the groupoid $\mathbf{Maps}((\mathrm{Bifurc}^{\bullet}_{\mathrm{scaled}})_{\lambda}, \mathcal{X})$ is just $\mathcal{X} \times \mathcal{X} \rightrightarrows \mathcal{X}$.

However, for $\lambda = 0$, the groupoid $\mathbf{Maps}((\mathrm{Bifurc}^{\bullet}_{\mathrm{scaled}})_{\lambda}, \mathcal{X})$ is the total space of the tangent complex on $\mathcal{X}$.

2.2. A family of co-groupoids. We will now spell out the construction described above in a more formal way.

2.2.1.   In the category of connective DG algebras, consider the Čech nerve, denoted $A^{\bullet}$, of the map

$$k \to 0.$$

Explicitly, $A^0 = k$, $A^1 = k \oplus k$ and, in general, $A^i = \underbrace{k \oplus \dots \oplus k}_{i+1}$. In particular, all A^i are classical.

2.2.2.   We now claim that the groupoid $A^{\bullet}$ in classical commutative algebras can be naturally lifted to one in the category of *non-negatively filtered* classical commutative algebras, denoted $(A^{\mathrm{Fil}})^{\bullet}$.

Indeed, we define the filtration on A^i to be

$$(A^i)_n = \begin{cases} A \text{ for } n \geq 1 \\ k \text{ for } n = 0. \end{cases}$$

In particular, $(A^0)_n = A^0$ for all n.

REMARK 2.2.3. Note that $(A^{\mathrm{Fil}})^1$ is the the filtered algebra that we used in Chapter 6, Sect. 1.4.1 in order to construct a canonical filtration on algebras over operads.

2.2.4. Note that

$$(2.1) \qquad \mathrm{ass\text{-}gr}((A^{\mathrm{Fil}})^\bullet) \simeq k \oplus \epsilon(B^\bullet(k)), \quad \epsilon^2 = 0, \deg(\epsilon) = 1.$$

In other words, $\mathrm{ass\text{-}gr}((A^{\mathrm{Fil}})^\bullet)$ is the classifying space simplicial object in the category of classical augmented commutative algebra, corresponding to the commutative group object

$$k[\epsilon]/\epsilon^2, \quad \deg(\epsilon) = 1.$$

2.2.5. Applying the equivalence of (1.3), we turn the groupoid $(A^{\mathrm{Fil}})^\bullet$ in the category of non-negatively filtered commutative connective DG algebras into a groupoid, denoted

$$A^\bullet_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}}$$

in the category of commutative connective algebras in $\mathrm{QCoh}(\mathbb{A}^1)$, equipped with a structure of left-lax equivariance with respect to $\mathbb{A}^1$.

Denote by $A^\bullet_{\mathrm{scaled}}$ the groupoid in the category of commutative connective algebras in $\mathrm{QCoh}(\mathbb{A}^1)$, obtained from $A^\bullet_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}}$ by forgetting the left-lax equivariance with respect to $\mathbb{A}^1$.

Explicitly,

$$A^0_{\mathrm{scaled}} = k[u],$$

and

$$A^1_{\mathrm{scaled}} = \mathrm{Spec}(k[u,\epsilon]/(u-\epsilon)\cdot(u+\epsilon)).$$

The two maps

$$A^1_{\mathrm{scaled}} \rightrightarrows A^0_{\mathrm{scaled}}$$

are given by

$$\epsilon \mapsto u \text{ and } \epsilon \mapsto -u,$$

respectively.

The degeneracy map $A^0_{\mathrm{scaled}} \to A^1_{\mathrm{scaled}}$ is $u \mapsto u$. The inverse for the groupoid is the map

$$A^1_{\mathrm{scaled}} \to A^1_{\mathrm{scaled}}, \quad u \mapsto u, \epsilon \mapsto -\epsilon.$$

2.2.6. Passing to spectra (in the sense of algebraic geometry), we obtain a groupoid object, denoted

$$\mathrm{Bifurc}^\bullet_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{right\text{-}lax}}}$$

in the category

$$\left(((\mathrm{Sch}^{\mathrm{aff}})_{/\mathbb{A}^1})^{\mathbb{A}^1_{\mathrm{right\text{-}lax}}}\right)^{\mathrm{op}}.$$

Let

$$s,t : \mathrm{Bifurc}^0_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{right\text{-}lax}}} \rightrightarrows \mathrm{Bifurc}^1_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{right\text{-}lax}}}$$

denote the two face maps (source and target, respectively).

Let $\mathrm{Bifurc}^\bullet_{\mathrm{scaled}}$ denote the groupoid object in the category

$$\left((\mathrm{Sch}^{\mathrm{aff}})_{/\mathbb{A}^1}\right)^{\mathrm{op}},$$

obtained from $\mathrm{Bifurc}^\bullet_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{right\text{-}lax}}}$ by forgetting the the left-lax equivariance with respect to $\mathbb{A}^1$.

Note that by construction, for any $0 \neq \lambda \in \mathbb{A}^1$, we have

$$(\mathrm{Bifurc}^i_{\mathrm{scaled}})_\lambda = \underbrace{\mathrm{pt} \sqcup ... \sqcup \mathrm{pt}}_{i+1},$$

and the groupoid structure is that of the Čech nerve of the map

$$\varnothing \to \mathrm{pt},$$

viewed as a morphism in $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$.

REMARK 2.2.7. Note that according to Sect. 1.2.5, Example (iv), the datum of upgrading of $\mathrm{Bifurc}^\bullet_{\mathrm{scaled}}$ to $\mathrm{Bifurc}^\bullet_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{right\text{-}lax}}}$ amounts simply to the action of the monoid $\mathbb{A}^1$ on $\mathrm{Bifurc}^\bullet_{\mathrm{scaled}}$, compatible with the projection to $\mathbb{A}^1$.

2.3. The canonical deformation of a groupoid. We will now use $\mathrm{Bifurc}^\bullet_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}}$ to deform the Čech nerve of a nil-isomorphism in $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$ to the total space of its relative tangent complex.

2.3.1. Let $\mathcal{X} \to \mathcal{Y}$ be a map in PreStk, and let $\mathcal{R}^\bullet$ be its Čech nerve. Consider the $\mathbb{A}^1$-family of simplicial objects of PreStk equal to

$$(2.2) \qquad \mathrm{Weil}^{\mathrm{Bifurc}^\bullet_{\mathrm{scaled}}}_{\mathbb{A}^1}(\mathcal{X} \times \mathrm{Bifurc}^\bullet_{\mathrm{scaled}}) \underset{\mathrm{Weil}^{\mathrm{Bifurc}^\bullet_{\mathrm{scaled}}}_{\mathbb{A}^1}(\mathcal{Y} \times \mathrm{Bifurc}^\bullet_{\mathrm{scaled}})}{\times} (\mathcal{Y} \times \mathbb{A}^1),$$

where the notation Weil is as in Sect. A.1.

I.e., for an affine scheme S, a point of the space of maps from S to i-simplices of (2.2) consists of the data of:

- a map $S \to \mathbb{A}^1$;
- a map $y : S \to \mathcal{Y}$;
- a map $S \underset{\mathbb{A}^1}{\times} \mathrm{Bifurc}^i_{\mathrm{scaled}} \to \mathcal{X}$;
- an identification of the composition $S \underset{\mathbb{A}^1}{\times} \mathrm{Bifurc}^i_{\mathrm{scaled}} \to \mathcal{X} \to \mathcal{Y}$ with the map

$$S \underset{\mathbb{A}^1}{\times} \mathrm{Bifurc}^i_{\mathrm{scaled}} \to S \overset{y}{\to} \mathcal{Y}.$$

Note that, by construction, the simplicial object (2.2) in $(\mathrm{PreStk})_{/\mathbb{A}^1}$ is augmented by $\mathcal{Y}$.

By Sects. A.2.3 and A.2.1, we have:

LEMMA 2.3.2. *Assume that $\mathcal{X}$ and $\mathcal{Y}$ belong to* $\mathrm{PreStk}_{\mathrm{laft}}$ *(resp.,* $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$*). Then the same will be true for the terms of the simplicial prestack* (2.2).

2.3.3.   Assume now that $\mathcal{X}$ and $\mathcal{Y}$ belong to $\mathrm{PreStk}_{\text{laft-def}}$ and that $\mathcal{X} \to \mathcal{Y}$ is a nil-isomorphism. Denote the simplicial prestack (2.2) by $\mathcal{R}^{\bullet}_{\text{scaled}}$.

Denote
$$\mathcal{R}_{\text{scaled}} := \mathcal{R}^1_{\text{scaled}}.$$

We have a canonical (unit) map $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{R}^{\bullet}_{\text{scaled}}$, and it is clear that this map is a nil-isomorphism. We claim:

LEMMA 2.3.4.   *The simplicial object $\mathcal{R}^{\bullet}_{\text{scaled}}$ is a groupoid object of* $(\mathrm{PreStk}_{\text{laft-def}})_{/\mathbb{A}^1}$.

PROOF.   We need to show that for any $n \geq 2$, the canonical map

(2.3)
$$\mathcal{R}^n_{\text{scaled}} \to \underbrace{\mathcal{R}_{\text{scaled}} \underset{\mathcal{X} \times \mathbb{A}^1}{\times} \cdots \underset{\mathcal{X} \times \mathbb{A}^1}{\times} \mathcal{R}_{\text{scaled}}}_{n \text{ times}}$$

is an isomorphism.

By Chapter 1, Proposition 8.3.2, it is enough to show that the map in question induces an isomorphism of the tangent spaces along the unit section.

By (A.4), for an affine scheme S and a point

$$S \xrightarrow{x,\lambda} \mathcal{X} \times \mathbb{A}^1,$$

the pullback of the tangent space of the left-hand side of (2.3) relative to $\mathcal{Y}$ identifies with

$$T_x(\mathcal{X}/\mathcal{Y}) \underset{k[u]}{\otimes} \Gamma(\mathrm{Bifurc}^n_{\text{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^n_{\text{scaled}}}),$$

while that of the right-hand side with

$$T_x(\mathcal{X}/\mathcal{Y}) \underset{k[u]}{\otimes} \Gamma(\mathrm{Bifurc}^1_{\text{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\text{scaled}}}) \underset{T_x(\mathcal{X}/\mathcal{Y}) \otimes k[u]}{\times} \cdots$$
$$\cdots \underset{T_x(\mathcal{X}/\mathcal{Y}) \otimes k[u]}{\times} T_x(\mathcal{X}/\mathcal{Y}) \underset{k[u]}{\otimes} \Gamma(\mathrm{Bifurc}^1_{\text{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\text{scaled}}}).$$

Now, the required assertion follows from the fact that

$$\Gamma(\mathrm{Bifurc}^n_{\text{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^n_{\text{scaled}}}) \simeq \Gamma(\mathrm{Bifurc}^1_{\text{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\text{scaled}}}) \underset{k[u]}{\times} \cdots \underset{k[u]}{\times} \Gamma(\mathrm{Bifurc}^1_{\text{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\text{scaled}}}).$$

$\square$

2.3.5.   Let us calculate the fiber $\mathcal{R}^{\bullet}_0$ of $\mathcal{R}^{\bullet}_{\text{scaled}}$ over $0 \in \mathbb{A}^1$. First, by Sect. 2.2.4, the groupoid $\mathcal{R}^{\bullet}_0$ is actually a group, and furthermore a commutative group-object in $\mathrm{FormMod}_{/\mathcal{X}}$.

We now claim:

PROPOSITION 2.3.6.   *The commutative group-object $\mathcal{R}^{\bullet}_0 \in \mathrm{FormMod}_{/\mathcal{X}}$ identifies canonically with* $\mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y}))$.

PROOF.   We will consider both sides as functors on the category $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$.

The commutative group-object $\mathcal{R}_0^\bullet \in \mathrm{FormMod}_{/\mathcal{X}}$ assigns to $\mathcal{Z} \in \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$ the space equal to the fiber of the restriction map

(2.4)
$$* \underset{\mathrm{Maps}(\mathcal{Z},\mathcal{X})}{\times} \mathrm{Maps}(\mathcal{Z} \times \mathrm{Spec}(k[\lambda]/\lambda^2), \mathcal{X}) \underset{\mathrm{Maps}(\mathcal{Z} \times \mathrm{Spec}(k[\lambda]/\lambda^2),\mathcal{Y})}{\times} \mathrm{Maps}(\mathcal{Z}, \mathcal{Y}) \to$$
$$\to * \underset{\mathrm{Maps}(\mathcal{X},\mathcal{X})}{\times} \mathrm{Maps}(\mathcal{X} \times \mathrm{Spec}(k[\lambda]/\lambda^2), \mathcal{X}) \underset{\mathrm{Maps}(\mathcal{X} \times \mathrm{Spec}(k[\lambda]/\lambda^2),\mathcal{Y})}{\times} \mathrm{Maps}(\mathcal{X}, \mathcal{Y}),$$

where the map in the above formula is given by restriction along $\mathcal{X} \to \mathcal{Z}$, and where the commutative group structure coming from the structure of commutative group on

$$\mathrm{Spec}(k[\lambda]/\lambda^2) \in ((\mathrm{Sch}^{\mathrm{aff}})_{\mathrm{pt}/})^{\mathrm{op}}.$$

By Chapter 7, Corollary 3.6.7, the commutative group-object $\mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y}))$ in the category $\mathrm{FormMod}_{/\mathcal{X}}$ assigns to

$$(\mathcal{Z} \overset{\pi}{\to} \mathcal{X}) \in \mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$$

the space

(2.5) $$\mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}\big(\mathrm{coFib}(\omega_{\mathcal{X}} \to \pi_*^{\mathrm{IndCoh}}(\omega_{\mathcal{Z}})), T(\mathcal{X}/\mathcal{Y})\big).$$

Note that
$$\mathcal{Z} \times \mathrm{Spec}(k[\lambda]/\lambda^2) \simeq \mathrm{RealSplitSqZ}(\omega_{\mathcal{Z}}),$$
see Chapter 7, Sect. 3.7 for the notation.

Hence, we can rewrite the fiber of the map (2.4) as

$$\mathrm{Fib}\big(\mathrm{Maps}_{\mathcal{Z}//\mathcal{Y}}(\mathrm{RealSplitSqZ}(\omega_{\mathcal{Z}}), \mathcal{X}) \to \mathrm{Maps}_{\mathcal{X}//\mathcal{Y}}(\mathrm{RealSplitSqZ}(\omega_{\mathcal{X}}), \mathcal{X})\big),$$

and further as

$$\mathrm{Fib}\big(\mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{Z})}(\omega_{\mathcal{Z}}, T(\mathcal{X}/\mathcal{Y})|_{\mathcal{Z}}) \to \mathrm{Maps}_{\mathrm{IndCoh}(\mathcal{X})}(\omega_{\mathcal{X}}, T(\mathcal{X}/\mathcal{Y}))\big),$$

identifies with (2.5), as required.

$\square$

2.4. Deformation of a formal moduli problem to the normal bundle. We will now use the deformation

$$\mathcal{R}^\bullet \rightsquigarrow \mathcal{R}^\bullet_{\mathrm{scaled}}$$

to construct the deformation

$$\mathcal{Y} \rightsquigarrow \mathcal{Y}^\bullet_{\mathrm{scaled}}.$$

2.4.1.   Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$ and let $\mathcal{Y}$ be an object of $\mathrm{FormMod}_{\mathcal{X}/}$.

Consider the formal groupoid $\mathcal{R}^\bullet_{\mathrm{scaled}}$ over $\mathcal{X}$ (and relative to $\mathcal{Y} \times \mathbb{A}^1$). Applying Chapter 5, Theorem 2.3.2, we obtain an object

$$\mathcal{Y}_{\mathrm{scaled}} \in \mathrm{FormMod}_{\mathcal{X} \times \mathbb{A}^1//\mathcal{Y} \times \mathbb{A}^1},$$

i.e., an $\mathbb{A}^1$-family of objects $\mathcal{Y}_{\mathrm{scaled}} \in \mathrm{FormMod}_{\mathcal{X}/}$.

2.4.2. By construction, the fiber $\mathcal{Y}_{\{\lambda\}}$ at $0 \neq \lambda \in \mathbb{A}^1$ identifies with the original $\mathcal{Y}$.

On the other hand, by Proposition 2.3.6 the fiber $\mathcal{Y}_0$ at $0 \in \mathbb{A}^1$ identifies canonically with

$$\mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1]),$$

where $T(\mathcal{X}/\mathcal{Y})[1] =: N(\mathcal{X}/\mathcal{Y})$ can be thought of as the normal bundle to $\mathcal{X}$ in $\mathcal{Y}$.

2.4.3. *Example.* Let us take $\mathcal{Y} = \mathcal{X}_{\mathrm{dR}}$. Then the object

$$(\mathcal{X}_{\mathrm{dR}})_{\mathrm{scaled}} \in \mathrm{FormMod}_{\mathcal{X} \times \mathbb{A}^1 // \mathcal{Y} \times \mathbb{A}^1}$$

is the Dolbeault degeneration of $\mathcal{X}_{\mathrm{dR}}$ to $\mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X})[1])$.

REMARK 2.4.4. One can show that when $\mathcal{X} = X$ is a classical scheme, and $\mathcal{Y}$ is the obtained as the formal completion of a classical scheme Y along a *regular* closed embedding $X \to Y$, then $\mathcal{Y}_{\mathrm{scaled}}$ is a nil-schematic ind-scheme (i.e., formal scheme) equal to the formal completion along $X \times \mathbb{A}^1$ of the scheme given by the usual deformation of Y to the normal cone.

2.5. The action of the monoid $\mathbb{A}^1$. Above to any $\mathcal{Y} \in \mathrm{FormMod}_{\mathcal{X}/}$ we have assigned an $\mathbb{A}^1$-family $\mathcal{Y}_{\mathrm{scaled}}$ of objects of $\mathrm{FormMod}_{\mathcal{X}/}$. However, this family possesses an extra structure: that of left-lax equivariance with respect to the monoid $\mathbb{A}^1$.

According to Sect. 1.5, this is exactly the kind of structure that allows to endow linear objects attached to $\mathcal{Y}$ with a non-negative filtration. The latter observation will be extensively used in the sequel.

2.5.1. Recall that by construction, the groupoid

$$\mathrm{Bifurc}^\bullet_{\mathrm{scaled}} \in \left((\mathrm{Sch}^{\mathrm{aff}})_{/\mathbb{A}^1}\right)^{\mathrm{op}}$$

could be naturally upgraded to

$$\mathrm{Bifurc}^\bullet_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{right\text{-}lax}}} \in \left(((\mathrm{Sch}^{\mathrm{aff}})_{/\mathbb{A}^1})^{\mathbb{A}^1_{\mathrm{right\text{-}lax}}}\right)^{\mathrm{op}}.$$

Consider now the functor

$$\mathrm{FormMod}_{\mathcal{X} \times - // \mathcal{Y} \times -} : (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \to 1\text{-Cat}, \quad S \mapsto \mathrm{FormMod}_{\mathcal{X} \times S // \mathcal{Y} \times S}.$$

By transport of structure, we obtain that for $\mathcal{Y} \in \mathrm{FormMod}_{\mathcal{X}/}$, the object

$$\mathcal{Y}_{\mathrm{scaled}} \in \mathrm{FormMod}_{\mathcal{X} \times \mathbb{A}^1 // \mathcal{Y} \times \mathbb{A}^1},$$

viewed as a natural transformation

$$\mathbb{A}^1 \to \mathrm{FormMod}_{\mathcal{X} \times - // \mathcal{Y} \times -}$$

has a natural structure of *left-lax equivariance* with respect to $\mathbb{A}^1$, where the target presheaf of categories $\mathrm{FormMod}_{\mathcal{X} \times - // \mathcal{Y} \times -}$ is endowed with the trivial action of $\mathbb{A}^1$.

Thus, we obtain a well-defined object

$$\mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \in \left(\mathrm{FormMod}_{\mathcal{X} \times \mathbb{A}^1 // \mathcal{Y} \times \mathbb{A}^1}\right)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}.$$

2.5.2.   Restricting along $\{0\} \to \mathbb{A}^1$, we obtain the object

$$\mathcal{Y}_{0,\mathbb{A}^1_{\text{left-lax}}} \in \left(\text{FormMod}_{\mathcal{X}/}\right)^{\mathbb{A}^1_{\text{left-lax}}},$$

that according to Proposition 2.3.6 identifies with

$$\text{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1]),$$

where $T(\mathcal{X}/\mathcal{Y})$ is regarded as an object of

$$\text{IndCoh}(\mathcal{X}) \simeq \text{IndCoh}(\mathcal{X})^{\text{gr},=1} \subset \text{IndCoh}(\mathcal{X})^{\text{gr},\geq 0} \simeq \text{IndCoh}(\mathcal{X})^{\mathbb{A}^1_{\text{left-lax}}}.$$

3. The canonical filtration on a Lie algebroid

In this section we will show that any Lie algebroid on $\mathcal{X}$ gives rise, in a canonical way, to a *filtered Lie algebroid*. This construction is a generalization of the construction in Sect. 1.6 that assigns to a Lie algebra in $\text{IndCoh}(\mathcal{X})$ (or any symmetric monoidal DG category) a filtered Lie algebra.

The associated graded of this filtration will yield the trivial Lie algebroid, and this fact will be subsequently used to establish various properties of formal moduli problems.

When working in the setting of classical algebraic geometry, the above filtered structure can be constructed 'by hand'. However, in the context of derived algebraic geometry we will use the deformation to the normal bundle to produce it.

3.1. Deformation to the normal bundle and Lie algebroids. In this subsection we will adapt the material of Sect. 2.5 to the language of Lie algebroids.

3.1.1.   Consider the presheaf of categories

$$\text{LieAlgbroid}(\mathcal{X} \times -/-) : (\text{Sch}^{\text{aff}}_{\text{aft}})^{\text{op}} \to \text{1-Cat}, \quad S \mapsto \text{LieAlgbroid}(\mathcal{X} \times S/S).$$

We obtain that for any $\mathfrak{L} \in \text{LieAlgbroid}(\mathcal{X})$ there is a canonically defined object

$$\mathfrak{L}^{\text{Fil}} \in \left(\text{LieAlgbroid}(\mathcal{X} \times \mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}}.$$

Moreover, this assignment is functorial in $\mathfrak{L}$. We denote the resulting functor

$$\text{LieAlgbroid}(\mathcal{X}) \to \left(\text{LieAlgbroid}(\mathcal{X} \times \mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}}$$

by AddFil.

3.1.2.   Let us denote by ass-gr the functor

$$\left(\text{LieAlgbroid}(\mathcal{X} \times \mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}} \to \left(\text{LieAlgbroid}(\mathcal{X})\right)^{\mathbb{A}^1_{\text{left-lax}}},$$

given by taking the fiber at $0 \in \mathbb{A}^1$.

By Sect. 2.4.2, the composite functor

$$\text{ass-gr} \circ \text{AddFil} : \text{LieAlgbroid}(\mathcal{X}) \to \text{LieAlgbroid}(\mathcal{X})$$

equals the composition

$$\text{LieAlgbroid}(\mathcal{X}) \overset{\mathbf{oblv}_{\text{LieAlgbroid}}}{\longrightarrow} \text{IndCoh}(\mathcal{X}) \overset{\deg=1}{\longrightarrow} \text{IndCoh}(\mathcal{X})^{\text{gr}} \simeq \text{IndCoh}(\mathcal{X})^{\mathbb{A}^1_{\text{left-lax}}} \to$$

$$\overset{\mathbf{triv}_{\text{Lie}}}{\longrightarrow} \text{LieAlg}(\text{IndCoh}(\mathcal{X})^{\mathbb{A}^1_{\text{left-lax}}}) \simeq \text{LieAlg}(\text{IndCoh}(\mathcal{X}))^{\mathbb{A}^1_{\text{left-lax}}} \overset{\text{diag}}{\longrightarrow} \text{LieAlgbroid}(\mathcal{X})^{\mathbb{A}^1_{\text{left-lax}}}.$$

REMARK 3.1.3. As in Remark 1.6.5, one can show that the above functor

$$\mathrm{AddFil} : \mathrm{LieAlgbroid}(\mathcal{X}) \to \big(\mathrm{LieAlgbroid}(\mathcal{X} \times \mathbb{A}^1/\mathbb{A}^1)\big)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}$$

is part of a richer structure. Namely, the functor

$$\mathrm{LieAlgbroid}(\mathcal{X} \times -/-) : (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \to 1\text{-}\mathrm{Cat}$$

carries a canonical action of the monoid $\mathbb{A}^1$.

3.2. Compatibility with the forgetful functor. We shall now study how the above canonical filtration on a Lie algebroid is compatible with the forgetful functor

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} : \mathrm{LieAlgbroid}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}.$$

3.2.1. Consider presheaf of categories

$$\mathrm{IndCoh}(X \times -)_{/T(\mathcal{X})|_{\mathcal{X}\times-}}, \quad (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}})^{\mathrm{op}} \to 1\text{-}\mathrm{Cat}, \quad S \mapsto \mathrm{IndCoh}(X \times S)_{/T(\mathcal{X})|_{\mathcal{X}\times S}}.$$

The functor $\mathbf{oblv}_{\mathrm{LieAlgbroid}/T}$ defines a natural transformation

$$\mathrm{LieAlgbroid}(\mathcal{X} \times -/-) \to \mathrm{IndCoh}(X \times -)_{/T(\mathcal{X})|_{\mathcal{X}\times-}}.$$

We endow $\mathrm{IndCoh}(X \times -)_{/T(\mathcal{X})|_{\mathcal{X}\times-}}$ with the trivial action of the monoid $\mathbb{A}^1$, and the above natural transformation is (obviously) $\mathbb{A}^1$-equivariant.

In particular, we obtain a functor

$$\mathbf{oblv}_{\mathrm{LieAlgbroid}/T} : \big(\mathrm{LieAlgbroid}(\mathcal{X} \times \mathbb{A}^1/\mathbb{A}^1)\big)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \to$$
$$\big(\mathrm{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}}\big)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}.$$

3.2.2. Note that by Lemma 1.5.2(a), the category $\big(\mathrm{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}}\big)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}$ identifies with

$$(3.1) \qquad\qquad\qquad (\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0})_{/T(\mathcal{X})}.$$

Above we view $T(\mathcal{X})$ as an object of $\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0}$ via the functor $(\mathrm{gr} \to \mathrm{Fil}) \circ (\deg = 0)$, i.e.,

$$\mathrm{IndCoh}(\mathcal{X}) \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0,\leq 0} \subset \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0}.$$

3.2.3. Recall now that in Chapter 8, Sect. 5.3.5, we defined a functor

$$\mathrm{IndCoh}(X)_{/T(\mathcal{X})} \to \big(\mathrm{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}}\big)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}.$$

Namely, for $\mathcal{F} \overset{\gamma}{\to} T(\mathcal{X}) \in \mathrm{IndCoh}(X)_{/T(\mathcal{X})}$, the underlying object $(\mathcal{F} \overset{\gamma}{\to} T(\mathcal{X}))_{\mathrm{scaled}}$ of $\mathrm{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1}}$ is given by

$$\mathcal{F}|_{\mathcal{X}\times\mathbb{A}^1} \overset{\gamma_{\mathrm{scaled}}}{\longrightarrow} T(\mathcal{X})|_{\mathcal{X}\times\mathbb{A}^1},$$

where the value of γ_{scaled} over $\lambda \in \mathbb{A}^1$ equals $\lambda \cdot \gamma$.

The structure of left-lax $\mathbb{A}^1$-equivariance on $(\mathcal{F} \overset{\gamma}{\to} T(\mathcal{X}))_{\mathrm{scaled}}$ is defined naturally. Denote this functor by AddFil.

3.2.4.   In terms of the identification (3.1), the functor AddFil sends $\mathcal{F} \overset{\gamma}{\to} T(\mathcal{X})$ to

$$(\text{gr} \to \text{Fil}) \circ (\text{deg} = 1)(\mathcal{F}) \overset{\gamma}{\to} (\text{gr} \to \text{Fil}) \circ (\text{deg} = 0)(T(\mathcal{X})).$$

Here for $i \geq 0$, we recall that $(\text{gr} \to \text{Fil}) \circ (\text{deg} = i)$ denotes the functor

$$\text{IndCoh}(\mathcal{X}) \simeq \text{IndCoh}(\mathcal{X})^{\text{Fil}, \geq i, \leq i} \subset \text{IndCoh}(\mathcal{X})^{\text{Fil}, \geq i} \subset \text{IndCoh}(\mathcal{X})^{\text{Fil}, \geq 0}$$

(i.e., we take an object of $\text{IndCoh}(\mathcal{F})$ and place it in degree i).

3.2.5.   The goal of this section is to establish the following:

PROPOSITION 3.2.6.   *The following diagram of functors commutes:*

$$
\begin{array}{ccc}
\text{LieAlgbroid}(\mathcal{X}) & \overset{\text{AddFil}}{\longrightarrow} & \left(\text{LieAlgbroid}(\mathcal{X} \times \mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}} \\
(3.2)\;{\scriptsize\mathbf{oblv}_{\text{LieAlgbroid}/T}} \downarrow & & \downarrow {\scriptsize\mathbf{oblv}_{\text{LieAlgbroid}/T}} \\
\text{IndCoh}(X)_{/T(\mathcal{X})} & \overset{\text{AddFil}}{\longrightarrow} & \left(\text{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X} \times \mathbb{A}^1}}\right)^{\mathbb{A}^1_{\text{left-lax}}} .
\end{array}
$$

REMARK 3.2.7.   Note that by adjunction from the commutative diagram (3.2), we obtain a diagram that commutes *up to a natural transformation*:

$$
\begin{array}{ccc}
\text{LieAlgbroid}(\mathcal{X}) & \overset{\text{AddFil}}{\longrightarrow} & \left(\text{LieAlgbroid}(\mathcal{X} \times \mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}} \\
(3.3)\;\;{\scriptsize\mathbf{free}_{\text{LieAlgbroid}}} \uparrow & & \uparrow {\scriptsize\mathbf{free}_{\text{LieAlgbroid}}} \\
\text{IndCoh}(X)_{/T(\mathcal{X})} & \overset{\text{AddFil}}{\longrightarrow} & \left(\text{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X} \times \mathbb{A}^1}}\right)^{\mathbb{A}^1_{\text{left-lax}}} .
\end{array}
$$

We note, however, that the above natural transformation is *not* an isomorphism. Indeed, the two circuits give a different result even after applying the functor ass-gr.

I.e., the structure of filtered Lie algebroid on filtration on $\mathbf{free}_{\text{LieAlgbroid}}(\mathcal{F} \overset{\gamma}{\to} T(\mathcal{X}))$, given by the construction in Chapter 8, Sect. 5.3 is *different* from the canonical filtration that exists on an arbitrary Lie algebroid, given by the construction in Sect. 3.1.

3.3. Proof of Proposition 3.2.6.

3.3.1.   For $\mathfrak{L} \in \text{LieAlgbroid}(\mathcal{X})$ the object

$$\mathbf{oblv}_{\text{LieAlgbroid}/T} \circ \text{AddFil}(\mathfrak{L}) \in \left(\text{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X} \times \mathbb{A}^1}}\right)^{\mathbb{A}^1_{\text{left-lax}}}$$

can be described as follows.

Let $\mathcal{Y}$ be the object of $\text{FormMod}_{\mathcal{X}/}$, corresponding to $\mathfrak{L}$. Consider the corresponding prestack

$$\mathcal{R}_{\text{scaled}} := \text{Weil}_{\mathbb{A}^1}^{\text{Bifurc}^1_{\text{scaled}}}(\mathcal{X} \times \text{Bifurc}^1_{\text{scaled}}) \underset{\text{Weil}_{\mathbb{A}^1}^{\text{Bifurc}^1_{\text{scaled}}}(\mathcal{Y} \times \text{Bifurc}^1_{\text{scaled}})}{\times} (\mathcal{Y} \times \mathbb{A}^1),$$

equipped with a structure of left-lax equivariance with respect to $\mathbb{A}^1$.

Consider the object

$$T(\mathcal{R}_{\text{scaled}}/\mathcal{X} \times \mathbb{A}^1)|_{\mathcal{X} \times \mathbb{A}^1} \in \text{IndCoh}(\mathcal{X} \times \mathbb{A}^1),$$

where $\mathcal{R}_{\mathrm{scaled}} \to \mathcal{X} \times \mathbb{A}^1$ is induced by the map $t : \mathbb{A}^1 \to \mathrm{Bifurc}^1_{\mathrm{scaled}}$. The map $s : \mathbb{A}^1 \to \mathrm{Bifurc}^1_{\mathrm{scaled}}$ induces a map

$$T(\mathcal{R}_{\mathrm{scaled}}/\mathcal{X} \times \mathbb{A}^1)|_{\mathcal{X} \times \mathbb{A}^1} \to T(\mathcal{X})|_{\mathcal{X} \times \mathbb{A}^1},$$

and the resulting object of $\mathrm{IndCoh}(\mathcal{X} \times \mathbb{A}^1)_{/\mathrm{IndCoh}(X \times \mathbb{A}^1)/T(\mathcal{X})|_{\mathcal{X} \times \mathbb{A}^1}}$ naturally lifts to

$$\left(\mathrm{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X} \times \mathbb{A}^1}}\right)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}},$$

which is our $\mathbf{oblv}_{\mathrm{LieAlgbroid}} \circ \mathrm{AddFil}(\mathfrak{L})$.

We need to show that the above object is obtained from the tautological map

$$T(\mathcal{X}/\mathcal{Y}) \to T(\mathcal{X})$$

by the scaling procedure of Sect. 3.2.3.

3.3.2.   We identify

$$T(\mathcal{R}_{\mathrm{scaled}}/\mathcal{X} \times \mathbb{A}^1)|_{\mathcal{X} \times \mathbb{A}^1} \simeq \mathrm{Fib}\left(T(\mathcal{R}_{\mathrm{scaled}}/\mathcal{Y} \times \mathbb{A}^1)|_{\mathcal{X} \times \mathbb{A}^1} \to T(\mathcal{X} \times \mathbb{A}^1/\mathcal{Y} \times \mathbb{A}^1)\right),$$

which, by (A.4), identifies with

$$T(\mathcal{X}/\mathcal{Y}) \otimes \mathrm{Fib}\left(\Gamma(\mathrm{Bifurc}^1_{\mathrm{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\mathrm{scaled}}}) \xrightarrow{t^*} \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1})\right),$$

and its map to

$$\mathrm{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X} \times \mathbb{A}^1}} \simeq T(\mathcal{X}) \otimes \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1})$$

identifies with

$$T(\mathcal{X}/\mathcal{Y}) \otimes \mathrm{Fib}\left(\Gamma(\mathrm{Bifurc}^1_{\mathrm{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\mathrm{scaled}}}) \xrightarrow{t^*} \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1})\right) \to$$

$$\to T(\mathcal{X}/\mathcal{Y}) \otimes \Gamma(\mathrm{Bifurc}^1_{\mathrm{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\mathrm{scaled}}}) \xrightarrow{s^*}$$

$$\to T(\mathcal{X}/\mathcal{Y}) \otimes \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) \to T(\mathcal{X}) \otimes \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}).$$

3.3.3.   Thus, we need to show that the composite arrow

$$\mathrm{Fib}\left(\Gamma(\mathrm{Bifurc}^1_{\mathrm{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\mathrm{scaled}}}) \xrightarrow{t^*} \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1})\right) \to$$

$$\to \Gamma(\mathrm{Bifurc}^1_{\mathrm{scaled}}, \mathcal{O}_{\mathrm{Bifurc}^1_{\mathrm{scaled}}}) \xrightarrow{s^*} \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}),$$

viewed as an object of

$$(\mathrm{QCoh}(\mathbb{A}^1)_{/\mathcal{O}_{\mathbb{A}^1}})^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}},$$

is obtained by the scaling procedure of Sect. 3.2.3 from the identity map $\mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}_{\mathbb{A}^1}$.

3.3.4.   We identify the above map with

$$\mathrm{Fib}((A^{\mathrm{Fil}})^1 \to k) \to (A^{\mathrm{Fil}})^1 \to k,$$

where $(A^{\mathrm{Fil}})^1$ is the filtered algebra from Sect. 2.2.2. Here the two maps $k \oplus k \simeq A^1 \rightrightarrows k \oplus k$ are the projection on the first and the second copy of k

The resulting map in $\mathrm{Vect}^{\mathrm{Fil}, \geq 0}$ is

$$(\mathrm{gr} \to \mathrm{Fil}) \circ (\deg = 1)(k) \to (\mathrm{gr} \to \mathrm{Fil}) \circ (\deg = 0)(k),$$

as required.

$\square$

4. The case of groups

In this section we will show that assignment $\mathfrak{L} \rightsquigarrow \mathfrak{L}^{\mathrm{Fil}}$ of Sect. 3.1 reproduces in the case of Lie algebras the construction of scaling the Lie algebra structure

$$\mathfrak{h} \rightsquigarrow \mathfrak{h}^{\mathrm{Fil}},$$

see Chapter 6, Sect. 1.4.3.

This is not altogether obvious, since the filtration in the case of Lie algebras was produced purely algebraically, and in the case of Lie algebroids we used geometry (specifically, the co-simplicial scheme $\mathrm{Bifurc}_{\mathrm{scaled}}^{\bullet}$).

4.1. Deformation to the normal cone in the *pointed* case. In thus subsection we will consider the deformation

$$\mathcal{Y} \rightsquigarrow \mathcal{Y}_{\mathrm{scaled}, \mathbb{A}^1_{\mathrm{left\text{-}lax}}}$$

when $\mathcal{Y}$ is an object of $\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}})$. Denote

$$\mathcal{H} := \Omega_{\mathcal{X}}(\mathcal{Y}).$$

In this case, by functoriality, $\mathcal{Y}_{\mathrm{scaled}, \mathbb{A}^1_{\mathrm{left\text{-}lax}}}$ is an object of

$$\left(\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X} \times \mathbb{A}^1}) \right)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}.$$

4.1.1.   Consider the corresponding object

$$\mathcal{H}_{\mathrm{scaled}} := \Omega_{\mathcal{X}}(\mathcal{Y}_{\mathrm{scaled}})$$

in $\left(\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X} \times \mathbb{A}^1}) \right)$.

By functoriality, it lifts to an object

$$\mathcal{H}_{\mathrm{scaled}, \mathbb{A}^1_{\mathrm{left\text{-}lax}}} \in \left(\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X} \times \mathbb{A}^1}) \right)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}.$$

4.1.2.   Applying the functor

$$\mathrm{Lie}_{\mathcal{Z}} : \mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{Z}}) \to \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{Z}))$$

(see Chapter 7, Sect. 3.6), we obtain an object

$$\mathrm{Lie}_{\mathcal{X} \times \mathbb{A}^1}(\mathcal{H}_{\mathrm{scaled}, \mathbb{A}^1_{\mathrm{left\text{-}lax}}}) \in \left(\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X} \times \mathbb{A}^1)) \right)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}.$$

Using the equivalence of (1.3), we regard it as an object, denoted

$$\mathrm{Lie}_{\mathcal{X}}(\mathcal{H}^{\mathrm{Fil}}) \in \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil} \geq 0}).$$

We claim:

THEOREM 4.1.3. *The above object* $\mathrm{Lie}_{\mathcal{X}}(\mathcal{H}^{\mathrm{Fil}}) \in \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil} \geq 0})$ *identifies canonically with the object* $(\mathrm{Lie}_{\mathcal{X}}(\mathcal{H}))^{\mathrm{Fil}}$ *of Chapter 6, Sect. 1.4.3.*

4.1.4. It will follow from the proof that the isomorphism in Theorem 4.1.3 is compatible with the corresponding forgetful functors.

Namely, it follows from Proposition 3.2.6 that there is a canonical isomorphism

$$(4.1) \qquad \mathbf{oblv}_{\mathrm{Lie}}(\mathrm{Lie}_{\mathcal{X}}(\mathcal{H}^{\mathrm{Fil}})) \simeq (\mathrm{gr} \to \mathrm{Fil}) \circ (\deg = 1) \circ \mathbf{oblv}_{\mathrm{Lie}}(\mathrm{Lie}_{\mathcal{X}}(\mathcal{H})).$$

In addition, by construction,

$$(4.2) \qquad \mathbf{oblv}_{\mathrm{Lie}}((\mathrm{Lie}_{\mathcal{X}}(\mathcal{H}))^{\mathrm{Fil}}) \simeq (\mathrm{gr} \to \mathrm{Fil}) \circ (\deg = 1) \circ \mathbf{oblv}_{\mathrm{Lie}}(\mathrm{Lie}_{\mathcal{X}}(\mathcal{H})).$$

Now, the isomorphisms (4.1) and (4.2) are compatible via the isomorphism of Theorem 4.1.3.

4.1.5. Translating to the language of Lie algebroids we obtain:

COROLLARY 4.1.6. *The following diagram canonically commutes*

$$
\begin{array}{ccc}
\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) & \xrightarrow{\ \mathrm{AddFil}\ } & (\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X} \times \mathbb{A}^1))^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \\
{\scriptstyle \mathrm{diag}}\downarrow & & \downarrow{\scriptstyle \mathrm{diag}} \\
\mathrm{LieAlgebroids}(\mathcal{X}) & \xrightarrow{\ \mathrm{AddFil}\ } & \left(\mathrm{LieAlgbroid}(\mathcal{X} \times \mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}.
\end{array}
$$

4.1.7. The compatibility in Sect. 4.1.4 amounts to the fact that the data of commutativity of the outer square in

$$
\begin{array}{ccc}
\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) & \xrightarrow{\ \mathrm{AddFil}\ } & (\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X} \times \mathbb{A}^1))^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \\
{\scriptstyle \mathrm{diag}}\downarrow & & \downarrow{\scriptstyle \mathrm{diag}} \\
\mathrm{LieAlgebroids}(\mathcal{X}) & \xrightarrow{\ \mathrm{AddFil}\ } & \left(\mathrm{LieAlgbroid}(\mathcal{X} \times \mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \\
{\scriptstyle \mathbf{oblv}_{\mathrm{LieAlgrboid}/T(\mathcal{X})}}\downarrow & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{LieAlgrboid}/T(\mathcal{X})}} \\
\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} & \xrightarrow{\ \mathrm{AddFil}\ } & \left(\mathrm{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X} \times \mathbb{A}^1}}\right)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}},
\end{array}
$$

equals one in the outer square of

$$
\begin{array}{ccc}
\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})) & \xrightarrow{\ \mathrm{AddFil}\ } & (\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X} \times \mathbb{A}^1))^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \\
\downarrow & & \downarrow \\
\mathrm{IndCoh}(\mathcal{X}) & \xrightarrow{(\mathrm{gr}\to\mathrm{Fil})\circ(\deg=1)} & \mathrm{IndCoh}(\mathcal{X} \times \mathbb{A}^1)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \\
\downarrow & & \downarrow \\
\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})} & \xrightarrow{\ \mathrm{AddFil}\ } & \left(\mathrm{IndCoh}(X \times \mathbb{A}^1)_{/T(\mathcal{X})|_{\mathcal{X} \times \mathbb{A}^1}}\right)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}},
\end{array}
$$

where the lower vertical arrows are given by

$$\mathcal{F} \mapsto (\mathcal{F} \xrightarrow{0} T(\mathcal{X})).$$

REMARK 4.1.8. By adjunction, the next diagram commutes *up to a natural transformation*:

$$\text{LieAlg}(\text{IndCoh}(\mathcal{X})) \xrightarrow{\text{AddFil}} (\text{LieAlg}(\text{IndCoh}(\mathcal{X} \times \mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}}$$

$$\Big\uparrow \text{ker-anch} \qquad\qquad\qquad\qquad\qquad \Big\uparrow \text{ker-anch}$$

$$\text{LieAlgebroids}(\mathcal{X}) \xrightarrow{\text{AddFil}} \left(\text{LieAlgbroid}(\mathcal{X} \times \mathbb{A}^1/\mathbb{A}^1)\right)^{\mathbb{A}^1_{\text{left-lax}}}.$$

We note, however, that this natural transformation is *not* an isomorphism.

4.1.9.   The rest of this section is devoted to the proof of Theorem 4.1.3.

4.2. A digression: category objects and group-objects. We will now explain a general categorical paradigm that will be used in the proof of Theorem 4.1.3.

4.2.1.   Let $\mathcal{C}$ be a pointed category with finite limits. Let $c^\bullet$ be a Segal-object (a.k.a., category-object) of $\mathcal{C}$; see Volume I, Chapter 5, Sect. 5.1.1 for what this means.

On the one hand, we consider the simplicial object of $\mathcal{C}$ equal to

$$(4.3) \qquad\qquad\qquad\qquad {}'c^\bullet := * \underset{c^\bullet}{\times} c^0,$$

where $c^0 \to c^n$ is given by the degeneracy map.

It is easy to see that ${}'c^\bullet$ is a groupoid-object in $\mathcal{C}$ with ${}'c^0 = *$, i.e., it defines a structure of group-object on ${}'d := {}'c^1$.

4.2.2.   On the other hand, consider the group-objects $\Omega(c^1)$ and $\Omega(c^0)$. The 'target' map $t : c^1 \to c^0$ defines a homomorphism $\Omega(c^1) \to \Omega(c^0)$. Define

$${}''d := \text{Fib}(\Omega(c^1) \to \Omega(c^0)).$$

We claim:

PROPOSITION 4.2.3. *Under the above circumstances, there is a canonical isomorphism of group-objects in* $\mathcal{C}$

$${}'d \simeq {}''d.$$

PROOF. Consider the category-object in $\text{Grp}(\mathcal{C})$ given by $\Omega(c^\bullet)$. We can regard it as a group-object in the category of category-objects in $\mathcal{C}$ and as such it acts on ${}'c^\bullet$. This action defines an action of the group-object $\Omega(c^1)$ on the object of $\mathcal{C}$ underlying ${}'c^1$.

The action of the group $\text{Fib}(\Omega(c^1) \to \Omega(c^0))$ on ${}'c^1$ has an additional structure: it commutes with the action of the group-object ${}'c^1$ on itself by *right* translations.

This defines a homomorphism ${}''d \to {}'d$. At the level of the underlying objects of $\mathcal{C}$, this map is the map

$$* \underset{*\underset{c^0}{\times}*}{\times} (* \underset{c^1}{\times} *) \to * \underset{c^1}{\times} * \to * \underset{c^1}{\times} c^0,$$

which is an isomorphism since the degeneracy map $c^0 \to c^1$ is a right inverse to $t : c^1 \to c^0$.

$\square$

4.3. Proof of Theorem 4.1.3.

4.3.1. *Step 1.* We claim that the object $\mathcal{H}_{\text{scaled},\mathbb{A}^1_{\text{left-lax}}} \in \big(\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{X}\times\mathbb{A}^1})\big)^{\mathbb{A}^1_{\text{left-lax}}}$ is given by

$$(4.4) \qquad \mathrm{Weil}^{\mathcal{X}\times\mathrm{Bifurc}^1_{\text{scaled}}}_{\mathcal{X}\times\mathbb{A}^1}(\mathcal{H}\times\mathrm{Bifurc}^1_{\text{scaled}}) \underset{\mathcal{H}\times\mathbb{A}^1}{\times} (\mathcal{X}\times\mathbb{A}^1),$$

with its natural left-lax equivariant structure with respect to $\mathbb{A}^1$, and the map

$$\mathrm{Weil}^{\mathcal{X}\times\mathrm{Bifurc}^1_{\text{scaled}}}_{\mathcal{X}\times\mathbb{A}^1}(\mathcal{H}\times\mathrm{Bifurc}^1_{\text{scaled}}) \to \mathrm{Weil}^{\mathcal{X}\times\mathrm{Bifurc}^0_{\text{scaled}}}_{\mathcal{X}\times\mathbb{A}^1}(\mathcal{H}\times\mathrm{Bifurc}^0_{\text{scaled}}) = \mathcal{H}\times\mathbb{A}^1$$

is induced by $t: \mathrm{Bifurc}^0_{\text{scaled}} \to \mathrm{Bifurc}^1_{\text{scaled}}$.

Indeed, we apply the setting of Sect. 4.2 to the category

$$\mathcal{C} := \big(\mathrm{Ptd}(\mathrm{FormMod}_{/\mathcal{X}\times\mathbb{A}^1})\big)^{\mathbb{A}^1_{\text{left-lax}}}$$

and

$$c^\bullet := \mathrm{Weil}^{\mathcal{X}\times\mathrm{Bifurc}^1_{\text{scaled}}}_{\mathcal{X}\times\mathbb{A}^1}(\mathcal{Y}\times\mathrm{Bifurc}^1_{\text{scaled}}).$$

Then the object $'d$ of Sect. 4.2.1 identifies with

$$\mathrm{Weil}^{\mathrm{Bifurc}^\bullet_{\text{scaled}}}_{\mathbb{A}^1}(\mathcal{X}\times\mathrm{Bifurc}^\bullet_{\text{scaled}}) \underset{\mathrm{Weil}^{\mathrm{Bifurc}^\bullet_{\text{scaled}}}_{\mathbb{A}^1}(\mathcal{Y}\times\mathrm{Bifurc}^\bullet_{\text{scaled}})}{\times} (\mathcal{Y}\times\mathbb{A}^1) = \mathcal{H}_{\text{scaled}}.$$

This is while the object $''d$ of Sect. 4.2.1 identifies with (4.4).

Note that the group structure on (4.4) is induced by that on $\mathcal{H}$ (i.e., the groupoid structure on $\mathrm{Bifurc}^1_{\text{scaled}}$ is not involved).

4.3.2. *Step 2.* From the commutativity of the diagram (A.7), we obtain a canonical identification of objects of objects of $\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})\otimes\mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1_{\text{left-lax}}}$

$$\mathrm{Lie}_{\mathcal{X}\times\mathbb{A}^1}\big(\mathrm{Weil}^{\mathcal{X}\times\mathrm{Bifurc}^1_{\text{scaled}}}_{\mathcal{X}\times\mathbb{A}^1}(\mathcal{H}\times\mathrm{Bifurc}^1_{\text{scaled}}) \underset{\mathcal{H}\times\mathbb{A}^1}{\times} (\mathcal{X}\times\mathbb{A}^1)\big) \simeq$$

$$\simeq \mathrm{Fib}\big(\mathrm{Weil}^{\mathcal{X}\times\mathrm{Bifurc}^1_{\text{scaled}}}_{\mathcal{X}\times\mathbb{A}^1}(\mathrm{Lie}_{\mathcal{X}}(\mathcal{H})|_{\mathcal{X}\times\mathrm{Bifurc}^1_{\text{scaled}}}) \to \mathrm{Lie}_{\mathcal{X}}(\mathcal{H})|_{\mathcal{X}\times\mathbb{A}^1}\big).$$

Now, it follows from Remark 2.2.3 that the latter expression is canonically isomorphic to $(\mathrm{Lie}_{\mathcal{X}}(\mathcal{H}))^{\mathrm{Fil}}$. $\qquad\square$

5. Infinitesimal neighborhoods

Let $X \to Y$ be a closed embedding of classical schemes. In this case we can consider the n-infinitesimal neighborhood of X inside Y (it corresponds to the n-th power of the defining X in Y).

However, the derived version of this construction is not so evident (what do we mean by the n-th power of an ideal?).

In this section we will define the corresponding derived version in the general context of formal moduli problems. The key tool will be deformation to the normal bundle from Sect. 2.

5.1. The n-th infinitesimal neighborhood. Let $\mathcal{X} \to \mathcal{Y}$ be a map objects of $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$. In this subsection we will construct a sequence of objects

$$\mathcal{X} = \mathcal{X}^{(0)} \to \mathcal{X}^{(1)} \to \dots \to \mathcal{X}^{(n)} \to \dots \to \mathcal{Y},$$

with $\mathcal{X}^{(n)} \in \mathrm{FormMod}_{/\mathcal{X}}$.

The prestacks $\mathcal{X}^{(n)}$ will generalize the construction of the 'n-th infinitesimal neighborhood of X in Y' for a closed embedding of classical schemes $X \to Y$. (In the case when the embedding is regular, the derived construction will agree with the classical construction. However, in general, even if both X and Y are classical, the derived n-th infinitesimal neighborhood will have a non-trivial derived structure.)

It will follow from the construction that $\mathcal{X}^{(1)}$ is the square-zero extension corresponding to the map $T(\mathcal{X}/\mathcal{Y}) \to T(\mathcal{X})$, i.e.,

$$\mathrm{RealSqZ}(T(\mathcal{X}/\mathcal{Y}) \to T(\mathcal{X})),$$

see Chapter 8, Sect. 5.1.1.

5.1.1.   We will construct the objects $\mathcal{X}^{(n)}$ inductively, starting from $n = 0$. In fact, we will construct their filtered enhancements, denoted

$$\mathcal{X}^{(n)}_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \in \left(\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}\right)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}.$$

Let

$$\mathcal{X}^{(n)}_{\mathrm{scaled}} \in \mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}$$

be the object obtained from $\mathcal{X}^{(n)}_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}}$ by forgetting the structure of left-lax equivariance with respect to $\mathbb{A}^1$, so that $\mathcal{X}^{(n)}$ is the fiber of $\mathcal{X}^{(n)}_{\mathrm{scaled}}$ at $1 \in \mathbb{A}^1$.

Set $\mathcal{X}^{(0)}_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}} := \mathcal{X} \times \mathbb{A}^1$. Assume that $\mathcal{X}^{(n-1)}_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}}$, equipped with a map

$$\mathcal{X}^{(n-1)}_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \to \mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}}$$

has been constructed.

5.1.2.   Consider the object

$$T(\mathcal{X}^{(n-1)}/\mathcal{Y})|_{\mathcal{X}} \in \mathrm{IndCoh}(\mathcal{X}).$$

It canonically lifts to an object in $\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0} \in \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil}}$, denoted $(T(\mathcal{X}^{(n-1)}/\mathcal{Y})|_{\mathcal{X}})^{\mathrm{Fil}}$. Namely, we consider

$$T(\mathcal{X}^{(n-1)}_{\mathrm{scaled}}/\mathcal{Y}_{\mathrm{scaled}})|_{\mathcal{X}\times\mathbb{A}^1} \in \mathrm{IndCoh}(\mathcal{X} \times \mathbb{A}^1),$$

equipped with the natural structure of left-lax equivariance with respect to $\mathbb{A}^1$, and thus giving rise to the sought-for

$$(T(\mathcal{X}^{(n-1)}/\mathcal{Y})|_{\mathcal{X}})^{\mathrm{Fil}} \in \mathrm{IndCoh}(\mathcal{X} \times \mathbb{A}^1)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0}.$$

We will prove:

THEOREM 5.1.3.   *The object $(T(\mathcal{X}^{(n-1)}/\mathcal{Y})|_{\mathcal{X}})^{\mathrm{Fil}}$ belongs to $\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq n}$, and the n-th term of the filtration identifies canonically with*

$$\mathrm{Sym}^n\left(T(\mathcal{X}/\mathcal{Y})[1]\right)[-1].$$

Here, by a slight abuse of notation, we denote by $\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])[-1]$ the object of

$$\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq n} \subset \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0}$$

that should properly be denoted

$$(\mathrm{gr} \to \mathrm{Fil}) \circ (\deg = n)(\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])[-1]).$$

5.1.4. Let i_{n-1} denote the map $\mathcal{X} \to \mathcal{X}^{(n-1)}$, and let $i_{n-1,\mathrm{scaled}}$ denote the map

$$\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}^{(n-1)}_{\mathrm{scaled}}.$$

Assuming Theorem 5.1.3, we obtain a canonically defined map

$$(i_{n-1,\mathrm{scaled}})^{\mathrm{IndCoh}}_*(\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])[-1]) \to T(\mathcal{X}^{(n-1)}_{\mathrm{scaled}}/\mathcal{Y}_{\mathrm{scaled}}).$$

5.1.5. We let $\mathcal{X}^{(n)}_{\mathrm{scaled}}$ denote the square-zero extension of $\mathcal{X}^{(n-1)}_{\mathrm{scaled}}$ corresponding to the composite map

$$(i_{n-1,\mathrm{scaled}})^{\mathrm{IndCoh}}_*(\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])[-1]) \to T(\mathcal{X}^{(n-1)}_{\mathrm{scaled}}/\mathcal{Y}_{\mathrm{scaled}}) \to T(\mathcal{X}^{(n-1)}_{\mathrm{scaled}}).$$

By transport of structure, the object

$$\mathcal{X}^{(n)}_{\mathrm{scaled}} \in \mathrm{FormMod}_{\mathcal{X} \times \mathbb{A}^1 // \mathcal{Y} \times \mathbb{A}^1}$$

lifts to an object

$$\mathcal{X}^{(n)}_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \in \left(\mathrm{FormMod}_{\mathcal{X} \times \mathbb{A}^1 // \mathcal{Y} \times \mathbb{A}^1}\right)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}},$$

the map $\mathcal{X}^{(n-1)}_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \to \mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}}$ is equipped with an extension to a map

$$\mathcal{X}^{(n)}_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \to \mathcal{Y}_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}}.$$

5.2. Computing the colimit. In this subsection we will show that the colimit of the n-th infinitesimal neighborhoods recovers the ambient prestack.

5.2.1. Recall that according to Chapter 5, Corollary 2.3.6, the category $\mathrm{FormMod}_{\mathcal{X}/}$ admits sifted, and in particular, filtered colimits. Consider the object

$$\underset{n}{\mathrm{colim}}\, \mathcal{X}^{(n)} \in \mathrm{FormMod}_{\mathcal{X}/},$$

which is equipped with a canonically defined map to $\mathcal{Y}$.

PROPOSITION 5.2.2. *The map*

$$\underset{n}{\mathrm{colim}}\, \mathcal{X}^{(n)} \to \mathcal{Y}$$

is an isomorphism in $\mathrm{FormMod}_{\mathcal{X}/}$.

PROOF. By Chapter 1, Proposition 8.3.2, it suffices to show that the map in question induces an isomorphism at the level of tangent spaces

$$T(\mathcal{X}/\underset{n}{\mathrm{colim}}\, \mathcal{X}^{(n)}) \to T(\mathcal{X}/\mathcal{Y}).$$

By Chapter 5, Corollary 2.3.6, the natural map

$$\underset{n}{\mathrm{colim}}\, T(\mathcal{X}/\mathcal{X}^{(n)}) \to T(\mathcal{X}/\underset{n}{\mathrm{colim}}\, \mathcal{X}^{(n)})$$

is an isomorphism.

Hence, it suffices to show that the colimit

$$\operatorname*{colim}_{n} T(\mathcal{X}^{(n)}/\mathcal{Y})|_{\mathcal{X}}$$

vanishes.

However, this follows from Theorem 5.1.3. Indeed, the above colimit lifts to an object of $\operatorname{IndCoh}(\mathcal{X})^{\operatorname{Fil},\geq 0}$, which belongs to $\operatorname{IndCoh}(\mathcal{X})^{\operatorname{Fil},\geq n}$ for any n.

$\square$

By combining with Chapter 5, Corollary 2.3.7, we obtain:

COROLLARY 5.2.3. *The map*

$$\operatorname*{colim}_{n} \mathcal{X}^{(n)} \to \mathcal{Y}$$

is an isomorphism in $(\operatorname{PreStk}_{\operatorname{laft}})_{\mathcal{X}/}$.

5.2.4. By combining Proposition 5.2.2 with Chapter 7, Corollary 5.3.3(b) (or, alternatively, just using Corollary 5.2.3) above, we obtain:

COROLLARY 5.2.5. *For* $\mathcal{Y} \in \operatorname{FormMod}_{\mathcal{X}/}$, *there is a canonical isomorphism*

$$\operatorname*{colim}_{n} (f_n)_*^{\operatorname{IndCoh}}(\omega_{\mathcal{X}^{(n)}}) \to \omega_{\mathcal{Y}},$$

where f_n *denotes the map* $\mathcal{X}^{(n)} \to \mathcal{Y}$.

5.3. The Hodge filtration (a.k.a., de Rham resolution). Let $\mathfrak{L}$ be a Lie algebroid on $\mathcal{X}$. In the classical setting, the object $\omega_{\mathcal{X}}$, when equipped with the canonical structure of $\mathfrak{L}$-module, admits a canonical 'de Rham' resolution with terms induced from

$$\operatorname{Sym}^n(\mathbf{oblv}_{\operatorname{LieAlgbroid}}(\mathfrak{L})[1])[-n].$$

In this subsection we will carry out the corresponding construction in the derived setting.

The statement will be that the unit object in the category $\mathfrak{L}\text{-mod}(\operatorname{IndCoh})$ has a canonical filtration with subquotients $\mathbf{ind}_{\mathfrak{L}} (\operatorname{Sym}^n(\mathbf{oblv}_{\operatorname{LieAlgbroid}}(\mathfrak{L})[1]))$.

Applying this to $\mathfrak{L} = \mathcal{T}(\mathcal{X})$, we recover the Hodge filtration on

$$\omega_{\mathcal{X}_{\operatorname{dR}}} \in \operatorname{IndCoh}(\mathcal{X}_{\operatorname{dR}}).$$

5.3.1. Let $\mathfrak{L}$ be a Lie algebroid on $\mathcal{X} \in \operatorname{PreStk}_{\operatorname{laft-def}}$, corresponding to an object

$$(f : \mathcal{X} \to \mathcal{Y}) \in \operatorname{FormMod}_{\mathcal{X}/}.$$

Let $\omega_{\mathcal{X},\mathfrak{L}}$ denote the object of $\mathfrak{L}\text{-mod}(\operatorname{IndCoh}(\mathcal{X}))$ corresponding to $\omega_{\mathcal{Y}} \in \operatorname{IndCoh}(\mathcal{Y})$. Tautologically,

$$\mathbf{oblv}_{\mathfrak{L}}(\omega_{\mathcal{X},\mathfrak{L}}) = \omega_{\mathcal{X}}.$$

We will prove the following:

PROPOSITION-CONSTRUCTION 5.3.2. *There exists a canonical lift of* $\omega_{\mathcal{X},\mathfrak{L}}$ *to an object*

$$(\omega_{\mathcal{X},\mathfrak{L}})^{\operatorname{Fil}} \in \mathfrak{L}\text{-mod}(\operatorname{IndCoh}(\mathcal{X}))^{\operatorname{Fil},\geq 0},$$

such that

$$\operatorname{ass-gr}^n(\omega_{\mathcal{X},\mathfrak{L}}) = \mathbf{ind}_{\mathfrak{L}} (\operatorname{Sym}^n(\mathbf{oblv}_{\operatorname{LieAlgbroid}}(\mathfrak{L})[1])).$$

5.3.3. *Proof of Proposition 5.3.2.* Let $\mathcal{X}^{(n)}$ be the n-th infinitesimal neighborhood of $\mathcal{X}$ in $\mathcal{Y}$, see Sect. 5. Let i_n, $i_{n-1,n}$ and f_n denote the maps

$$\mathcal{X} \to \mathcal{X}^{(n)}, \ \mathcal{X}^{(n-1)} \to \mathcal{X}^{(n)} \text{ and } \mathcal{X}^{(n)} \to \mathcal{Y},$$

respectively.

We let

$$(\omega_{\mathcal{X},\mathfrak{L}})^{\leq n} := (f_n)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(n)}}).$$

Recall that by Corollary 5.2.5, the canonical map

$$\operatorname*{colim}_n (f_n)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(n)}}) \to \omega_{\mathcal{Y}}$$

is an isomorphism.

Hence, it remains to construct the isomorphisms

(5.1) $\quad \mathrm{coFib}\left((f_{n-1})_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(n-1)}}) \to (f_n)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(n)}})\right) \simeq$

$$\simeq f_*^{\mathrm{IndCoh}}\left(\mathrm{Sym}^n(\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[1])\right).$$

The left-hand side in (5.1) identifies with

$$(f_n)_*^{\mathrm{IndCoh}}\left(\mathrm{coFib}\left((i_{n-1,n})_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(n-1)}}) \to \omega_{\mathcal{X}^{(n)}}\right)\right).$$

Let us recall that by construction, the map $i_{n-1,n} : \mathcal{X}^{(n-1)} \to \mathcal{X}^{(n)}$ has a structure of square-zero extension corresponding to

$$(i_{n-1})_*^{\mathrm{IndCoh}}\left(\mathrm{Sym}^n\left(T(\mathcal{X}/\mathcal{Y})[1]\right)[-1]\right) \in \mathrm{IndCoh}(\mathcal{X}^{(n-1)}).$$

Hence, by Chapter 8, Proposition 6.4.2,

$$\mathrm{coFib}\left((i_{n-1,n})_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(n-1)}}) \to \omega_{\mathcal{X}^{(n)}}\right) \simeq$$

$$\simeq (i_{n-1,n})_*^{\mathrm{IndCoh}} \circ (i_{n-1})_*^{\mathrm{IndCoh}}\left(\mathrm{Sym}^n\left(T(\mathcal{X}/\mathcal{Y})[1]\right)\right) \simeq$$

$$\simeq (i_n)_*^{\mathrm{IndCoh}}\left(\mathrm{Sym}^n\left(T(\mathcal{X}/\mathcal{Y})[1]\right)\right).$$

And hence, the left-hand side in (5.1) identifies with

$$(f_n)_*^{\mathrm{IndCoh}} \circ (i_n)_*^{\mathrm{IndCoh}}\left(\mathrm{Sym}^n\left(T(\mathcal{X}/\mathcal{Y})[1]\right)\right) \simeq f_*^{\mathrm{IndCoh}}\left(\mathrm{Sym}^n\left(T(\mathcal{X}/\mathcal{Y})[1]\right)\right),$$

where

$$T(\mathcal{X}/\mathcal{Y}) \simeq \mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L}),$$

as desired.

$\square$

5.4. Proof of Theorem 5.1.3: reduction to the case of vector groups. In this subsection we will reduce the assertion of Theorem 5.1.3 to the case when $\mathcal{Y}$ is of the form $\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})$ for $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$.

5.4.1. Since the functor

$$\mathrm{ass\text{-}gr} : \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq 0} \to \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},\geq 0}$$

is conservative, it is enough to prove that in

$$\mathrm{ass\text{-}gr}((T(\mathcal{X}^{(n-1)}/\mathcal{Y})|_{\mathcal{X}})^{\mathrm{Fil}}) \in \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},\geq 0}$$

the lowest graded piece is in degree n, and is canonically isomorphic to $\mathrm{Sym}^n\left(T(\mathcal{X}/\mathcal{Y})[1]\right)[-1]$.

5.4.2.   By Sect. 2.5.2 and the compatibility of the functor RealSqZ with base change (see Chapter 8, Proposition 5.4.3) this reduces the assertion of the proposition to considering the case of

$$\mathcal{Y} := \mathrm{Vect}_{\mathcal{X}}(\mathcal{F}) \in (\mathrm{FormMod}_{/\mathcal{X}})^{\mathbb{A}^1_{\text{left-lax}}},$$

for

$$\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X}) \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},=1} \subset \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},\geq 0} \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathbb{A}^1_{\text{left-lax}}}.$$

5.5. Proof of Theorem 5.1.3: the case of vector groups. When dealing with vector groups we 'know' what the n-infinitesimal neighborhood must be, and this is what we will establish, along with the assertion of Theorem 5.1.3 in this case.

5.5.1.   Consider the symmetric monoidal category $\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},\geq 0}$. Let

$$\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},> 0}) \subset \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},\geq 0})$$

be the full subcategory consisting of objects, for which the augmentation co-ideal belongs to $\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},> 0}$.

Consider also the category

$$\mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},> 0}).$$

We have a pair of adjoint functors
(5.2)
$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},> 0}) \rightleftarrows \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},> 0}) : \mathrm{coChev}^{\mathrm{enh}}.$$

By [**FraG**, Proposition 4.1.2], the adjoint functors in (5.2) are mutually inverse equivalences.

5.5.2.   For

$$\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X}) \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},=1} \subset \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},\geq 0}$$

we consider the objects

$$\mathrm{Sym}(\mathcal{F}) \text{ and } \mathrm{Sym}^{\leq n}(\mathcal{F}) \in \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},> 0}).$$

Note that there is a canonical isomorphism

$$\mathrm{coChev}^{\mathrm{enh}}(\mathrm{Sym}(\mathcal{F})) = \mathbf{triv}_{\mathrm{Lie}}(\mathcal{F}[-1]).$$

(The above isomorphism is a particular case of Chapter 6, Theorem 4.2.4, but is much simpler, since we are in the graded category, and the functors in (5.2) are equivalences.)

Note that

$$B_{\mathcal{X}}(\mathbf{triv}_{\mathrm{Lie}}(\mathcal{F}[-1])) \simeq \mathrm{Vect}_{\mathcal{X}}(\mathcal{F}).$$

Denote

$$\mathfrak{g}^{(n)} := \mathrm{coChev}^{\mathrm{enh}}(\mathrm{Sym}^{\leq n}(\mathcal{F})).$$

For example,

$$\mathfrak{g}^{(1)} = \mathbf{free}_{\mathrm{Lie}}(\mathcal{F}[-1]).$$

(Again, this isomorphism holds because the functors in (5.2) are equivalences.)

Denote

$$\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n)} := B_{\mathcal{X}}(\mathfrak{g}^{(n)}) \in (\mathrm{FormMod}_{\mathcal{X}/})^{\mathbb{A}^1_{\text{left-lax}}}.$$

Let $\widetilde{i}_n$ denote the map $\mathcal{X} \to \mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n)}$.

5.5.3. Consider
$$\mathcal{Y} := \mathrm{Vect}_{\mathcal{X}}(\mathcal{F}) \in (\mathrm{FormMod}_{/\mathcal{X}})^{\mathbb{A}^1_{\text{left-lax}}},$$
and the corresponding object $\mathcal{X}^{(n)} \in (\mathrm{FormMod}_{\mathcal{X}/})^{\mathbb{A}^1_{\text{left-lax}}}$.

We are going to prove that there exists a canonical isomorphism in $(\mathrm{FormMod}_{\mathcal{X}/})^{\mathbb{A}^1_{\text{left-lax}}}$.
$$\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n)} \simeq \mathcal{X}^{(n)},$$
and that the assertion of Theorem 5.1.3 holds for $\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n)}$. More precisely, we will prove the following assertion:

PROPOSITION 5.5.4.

(a) *The lowest graded terms in the objects of* $\mathrm{IndCoh}(\mathcal{X})^{\mathbb{A}^1_{\text{left-lax}}} \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}, \geq 0}$
$$T(\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n-1)}/\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n)})|_{\mathcal{X}} \ and \ T(\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n-1)}/\mathrm{Vect}_{\mathcal{X}}(\mathcal{F}))|_{\mathcal{X}}$$
are in degree n; the map
$$T(\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n-1)}/\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n)})|_{\mathcal{X}} \to T(\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n-1)}/\mathrm{Vect}_{\mathcal{X}}(\mathcal{F}))|_{\mathcal{X}}$$
induces an isomorphism of degree n terms, and both identify canonically with $\mathrm{Sym}^n(\mathcal{F})[-1]$.

(b) *The map*
$$\mathrm{RealSqZ}\left((\widetilde{i}_{n-1})^{\mathrm{IndCoh}}_*(\mathrm{Sym}^n(\mathcal{F})[-1]) \to T(\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n-1)})\right) \to \mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n)},$$
induced by the identification in (a), is an isomorphism.

5.5.5. The assertion of Proposition 5.5.4 implies the required properties of $\mathrm{Vect}_{\mathcal{X}}(\mathcal{F})^{(n)}$ and $\mathcal{X}^{(n)}$ by induction on n.

5.6. Proof of Proposition 5.5.4. The proof of Proposition 5.5.4 will involve some 'cheating': instead of performing the crucial computation, we will reduce it to the case of classical algebraic geometry, namely, the embedding of 0 into a finite-dimensional vector space.

5.6.1. Before we prove Proposition 5.5.4, let us translate its assertion into the language of Lie algebras.

Point (a) says that the objects
$$\mathrm{Fib}\left(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{g}^{(n-1)}) \to \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{g}^{(n)})\right) \ \text{and} \ \mathrm{Fib}\left(\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{g}^{(n-1)}) \to \mathcal{F}[-1]\right)$$
both live in degrees $\geq n$, and their degree n part is isomorphic to $\mathrm{Sym}^n(\mathcal{F})[-2]$.

Point (b) says the following. Let $\mathcal{F}_n$ denote the object
$$\mathrm{coFib}\left(\mathrm{Sym}^n(\mathcal{F})[-2] \to \mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{g}^{(n-1)})\right) \in \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}, \geq 0}.$$
We have a canonical map
$$\mathbf{oblv}_{\mathrm{Lie}}(\mathfrak{g}^{(n-1)}) \to \mathcal{F}_n.$$

Let
$$\mathbf{free}_{\mathrm{LieAlg}_{\mathfrak{g}^{(n-1)}/}}(\mathcal{F}_n)$$
be the corresponding free object in the category of Lie algebras in $\mathrm{IndCoh}(\mathcal{X})$ *under* $\mathfrak{g}^{(n-1)}$.

By point (a) of Proposition 5.5.4, we have a canonical map

(5.3) $$\mathbf{free}_{\mathrm{LieAlg}_{\mathfrak{g}^{(n-1)}/}}(\mathcal{F}_n) \to \mathfrak{g}^{(n)}.$$

Now, point (b) of Proposition 5.5.4 is equivalent to the fact that the map (5.3) is an isomorphism.

5.6.2.   The above reformulation of Proposition 5.5.4 makes sense when $\mathrm{IndCoh}(\mathcal{X})$ is replaced by an arbitrary symmetric monoidal DG category $\mathbf{O}$. Furthermore, the assertion of both points of Proposition 5.5.4 is about the comparison of pairs of functors $\mathbf{O} \to \mathbf{O}$, given by symmetric sequences.

Hence, we can replace $\mathrm{IndCoh}(\mathcal{X})$ by Vect, and we can assume that $\mathcal{F}$ is a finite-dimensional vector space that lives in the cohomological degree 0.

In the latter case, the assertion of Proposition 5.5.4 is manifest. $\square$

6. Filtration on the universal enveloping algebra of a Lie algebroid

Let $\mathfrak{L}$ be a Lie algebroid on $\mathcal{X}$. Recall that in Chapter 8, Sect. 4.2 to $\mathfrak{L}$ we associated its universal enveloping algebra $U(\mathfrak{L})$, which was an algebra object in the monoidal DG category

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})).$$

In this subsection we define a crucial piece of structure that $U(\mathfrak{L})$ possesses, namely, the canonical (a.k.a. PBW) filtration.

6.1. The statement. In this subsection we state the main result of the present section, Theorem 6.1.2.

6.1.1.   Consider the monoidal category

$$\left(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X}))\right)^{\mathrm{Fil}, \geq 0}.$$

We claim:

THEOREM 6.1.2.   *The object*

$$U(\mathfrak{L}) \in \mathrm{AssocAlg}\left(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X}))\right),$$

canonically lifts to an object

$$U(\mathfrak{L})^{\mathrm{Fil}} \in \mathrm{AssocAlg}\left(\left(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X}))\right)^{\mathrm{Fil}, \geq 0}\right).$$

The corresponding associated graded identifies canonically with the monad given by tensor product with $\mathbf{free}_{\mathrm{Com}} \circ \mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})$.

The theorem will be proved in Sects 6.2-6.4.

6.1.3.   The following corollary results from the construction of the filtration and Theorem 4.1.3:

COROLLARY 6.1.4.   *For* $\mathfrak{L} = \mathrm{diag}(\mathfrak{h})$, *the filtration on* $U(\mathfrak{L})$ *defined in Theorem 6.1.2 identifies with the one coming from the canonical filtration on* $U(\mathfrak{h})$.

6.2. Constructing the filtration. As a first step, we will construct $U(\mathfrak{L})^{\mathrm{Fil}}$ as an object of the category

$$\mathrm{AssocAlg}\left(\left(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X}))\right)^{\mathrm{Fil}}\right),$$

where we identify the latter with

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})) \otimes \mathrm{QCoh}(\mathbb{A}^1)^{\mathbb{G}_m}.$$

6.2.1. Consider the forgetful functor

$$\left(\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}\right)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \to \left(\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1//\mathcal{Y}\times\mathbb{A}^1}\right)^{\mathbb{G}_m}.$$

Let us consider the resulting prestacks

$$\mathcal{X}\times\mathbb{A}^1/\mathbb{G}_m \text{ and } \mathcal{Y}_{\mathrm{scaled}}/\mathbb{G}_m$$

over $\mathbb{A}^1/\mathbb{G}_m$.

The map $f_{\mathrm{scaled}} : \mathcal{X}\times\mathbb{A}^1 \to \mathcal{Y}_{\mathrm{scaled}}$ gives rise to a $\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m)$-linear functor

$$(f_{\mathrm{scaled}}/\mathbb{G}_m)^! : \mathrm{IndCoh}(\mathcal{Y}_{\mathrm{scaled}}/\mathbb{G}_m) \to \mathrm{IndCoh}(\mathcal{X}\times\mathbb{A}^1/\mathbb{G}_m).$$

Since the symmetric monoidal category $\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m)$ is rigid, the left adjoint of the functor $(f_{\mathrm{scaled}}/\mathbb{G}_m)^!$, i.e., $(f_{\mathrm{scaled}}/\mathbb{G}_m)_*^{\mathrm{IndCoh}}$, is also $\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m)$-linear.

Hence, by Volume I, Chapter 1, Sect. 8.4.4, the composition

$$(f_{\mathrm{scaled}}/\mathbb{G}_m)^! \circ (f_{\mathrm{scaled}}/\mathbb{G}_m)_*^{\mathrm{IndCoh}}$$

has a natural structure of algebra object on the monoidal category

$$\mathrm{Funct}_{\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m)}(\mathrm{IndCoh}(\mathcal{X}\times\mathbb{A}^1/\mathbb{G}_m), \mathrm{IndCoh}(\mathcal{X}\times\mathbb{A}^1/\mathbb{G}_m)),$$

while the latter identifies with

$$\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})) \otimes \mathrm{QCoh}(\mathbb{A}^1)^{\mathbb{G}_m}.$$

This provides the sought-for lifting.

6.2.2. Our task is now to show that the object

$$U(\mathfrak{L})^{\mathrm{Fil}} \in \mathrm{AssocAlg}\left((\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil}}\right)$$

constructed above, belongs to the essential image of the (fully faithful) functor

$$\mathrm{AssocAlg}\left((\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil},\geq 0}\right) \to$$
$$\to \mathrm{AssocAlg}\left((\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil}}\right).$$

Note, however, that for any monoidal DG category $\mathbf{O}$ the following diagram is a pullback square:

$$\begin{array}{ccc}
\mathrm{AssocAlg}\left(\mathbf{O}^{\mathrm{Fil},\geq 0}\right) & \longrightarrow & \mathrm{AssocAlg}\left(\mathbf{O}^{\mathrm{Fil}}\right) \\
\mathbf{oblv}_{\mathrm{Assoc}}\downarrow & & \downarrow\mathbf{oblv}_{\mathrm{Assoc}} \\
\mathbf{O}^{\mathrm{Fil},\geq 0} & \longrightarrow & \mathbf{O}^{\mathrm{Fil}}.
\end{array}$$

Hence, we obtain that it suffices to show that the object

$$\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L})^{\mathrm{Fil}}) \in (\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil}}$$

in fact belongs to $(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil},\geq 0}$.

6.3. The categorical setting for the non-negative filtration. In this subsection we explain a general categorical paradigm for establishing that the filtration on

$$\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L})^{\mathrm{Fil}})$$

is non-negative.

6.3.1. Consider the following presheaf of categories

(6.1) $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to 1\text{-Cat}, \quad S \mapsto \mathrm{QCoh}(S)\text{-}\mathbf{mod},$

see Example (iii) in Sect. 1.2.5.

We denote the value of this functor on $\mathcal{Z} \in \mathrm{PreStk}$ by $\mathrm{ShvCat}(\mathcal{Z})$.

6.3.2. We regard (6.1) as equipped with the trivial action of a monoid $\mathcal{G}$.

According to Sect. 1.2.3, for a prestack $\mathcal{Z}$, equipped with an action of $\mathcal{G}$ one can talk about the category

$$\mathrm{ShvCat}(\mathcal{Z})^{\mathcal{G}_{\mathrm{left\text{-}lax}}}.$$

6.3.3. Assume now that $\mathcal{Z} = Z \in \mathrm{Sch}^{\mathrm{aff}}$. Let $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{D}'$ be three objects in $\mathrm{ShvCat}(Z)^{\mathcal{G}_{\mathrm{right\text{-}lax}}}$, and let $\mathsf{G} : \mathbf{C} \to \mathbf{D}$ and $\mathsf{F}' : \mathbf{C} \to \mathbf{D}'$ be morphisms.

Applying the forgetful functor

$$\mathrm{ShvCat}(Z)^{\mathcal{G}_{\mathrm{right\text{-}lax}}} \to \mathrm{ShvCat}(Z) \simeq \mathrm{QCoh}(Z)\text{-}\mathbf{mod},$$

the objects $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{D}'$ give rise to $\mathrm{QCoh}(Z)$-module categories, and G and F' to $\mathrm{QCoh}(Z)$-linear functors.

Assume that G, viewed as a functor between $\mathrm{QCoh}(Z)$-linear categories admits a left adjoint, denoted F. Since the monoidal category $\mathrm{QCoh}(Z)$ is rigid, the functor F is also naturally $\mathrm{QCoh}(Z)$-linear.

6.3.4. Assume now that $\mathbf{D}$ and $\mathbf{D}'$ are of the form

$$\mathbf{D}_0 \otimes \mathrm{QCoh}(Z) \text{ and } \mathbf{D}_0' \otimes \mathrm{QCoh}(Z),$$

respectively, where the structure on $\mathbf{D}$ and $\mathbf{D}'$ of objects of $\mathrm{ShvCat}(\mathcal{Z})^{\mathcal{G}_{\mathrm{right\text{-}lax}}}$ is induced by the structure on $\mathrm{QCoh}(Z)$ of an object of $\mathrm{ShvCat}(\mathcal{Z})^{\mathcal{G}_{\mathrm{right\text{-}lax}}}$ (in fact, $\mathrm{ShvCat}(\mathcal{Z})$), arising from the $\mathcal{G}$-action on Z.

We have:

LEMMA 6.3.5. *Under the above circumstances, the object*

$$\mathsf{F}' \circ \mathsf{F} \in \mathrm{Funct}_{\mathrm{QCoh}(Z)}(\mathbf{D}, \mathbf{D}') \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}_0, \mathbf{D}_0') \otimes \mathrm{QCoh}(Z)$$

corresponding to $\mathsf{G} \circ \mathsf{F}$, *admits a canonical lift to an object in the category*

$$\mathrm{Funct}_{\mathrm{cont}}(\mathbf{D}_0, \mathbf{D}_0') \otimes \mathrm{QCoh}(Z)^{\mathcal{G}_{\mathrm{left\text{-}lax}}}.$$

6.4. Implementing the categorical setting. We will now apply the setting of Sect. 6.4 to deduce the filtration on $U(\mathfrak{L})$.

6.4.1. We take the monoid $\mathcal{G}$ to be $\mathbb{A}^1$ and $Z = \mathbb{A}^1$, equipped with an action on itself by multiplication.

We take $\mathbf{D}_0, \mathbf{D}_0' = \mathrm{IndCoh}(\mathcal{X})$.

6.4.2.   We take $\mathbf{C}$ to be $\mathrm{IndCoh}(\mathcal{Y}_{\mathrm{scaled}})$, where the structure on $\mathbf{C}$ of an object of the category $\mathrm{ShvCat}(\mathbb{A}^1)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}$ is given by the lift of $\mathcal{Y}_{\mathrm{scaled}}$ to the object of

$$\mathcal{Y}_{\mathrm{scaled}, \mathbb{A}^1_{\mathrm{left\text{-}lax}}} \in ((\mathrm{PreStk}_{\mathrm{laft}})_{/\mathbb{A}^1})^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}}.$$

We take G and F' to both be the functor of pullback along the map

$$\mathcal{X} \times \mathbb{A}^1 \to \mathcal{Y}_{\mathrm{scaled}}.$$

The functor $\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L}))$ is then one corresponding to $\mathsf{F}' \circ \mathsf{F}$. The lifting of Lemma 6.3.5 defines the sought-for filtered structure.

6.4.3.   The associated graded of $U(\mathfrak{L})^{\mathrm{Fil}}$ has the prescribed shape by Sect. 2.5.2.

6.5. The filtration via infinitesimal neighborhoods. In this subsection we realize the following (intuitively clear) idea: the canonical filtration on $U(\mathfrak{L})$ stated in Theorem 6.1.2, can be realized by considering n-th infinitesimal neighborhoods of the diagonal in the corresponding groupoid.

6.5.1.   For $(f : \mathcal{X} \to \mathcal{Y}) \in \mathrm{FormMod}_{\mathcal{X}/}$, consider the corresponding groupoid

$$\mathcal{R} := \mathcal{X} \underset{\mathcal{Y}}{\times} \mathcal{X},$$

and the algebroid $\mathfrak{L}$.

Let p_s, p_t denote the two projections $\mathcal{R} \rightrightarrows \mathcal{X}$. Let $\Delta_{\mathcal{X}/\mathcal{Y}}$ denote the diagonal (i.e., unit) map $\mathcal{X} \to \mathcal{R}$.

Let $'\mathcal{X}^{(n)}$ denote the n-th infinitesimal neighborhood of $\mathcal{X}$ in $\mathcal{R}$, defined as in Sect. 5.1. Let $p_i^{(n)}$ denote the restriction of p_i to $'\mathcal{X}^{(n)}$, $i = s, t$.

6.5.2.   Consider the object of $(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil}, \geq 0}$ given by

$$(6.2) \qquad n \mapsto (p_t^{(n)})_*^{\mathrm{IndCoh}} \circ (p_s^{(n)})^!, \quad \mathbb{Z}^{\geq 0} \to \mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X}))$$

Let as assume Theorem 6.1.2. From it we will deduce:

THEOREM 6.5.3.   *There exists a canonical isomorphism between the object* (6.2) *and*

$$\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L})^{\mathrm{Fil}})$$

in the category $(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil}, \geq 0}$.

The rest of this subsection is devoted to the proof of this theorem.

6.5.4.   *Proof of Theorem 6.5.3, Step 1.* To prove the proposition, we need to construct a compatible family of maps

$$(p_t^{(n)})_*^{\mathrm{IndCoh}} \circ (p_s^{(n)})^! \to \mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L}))^{\leq n},$$

so that the induced maps

$$\mathrm{coFib}\left((p_t^{(n-1)})_*^{\mathrm{IndCoh}} \circ (p_s^{(n-1)})^! \to (p_t^{(n)})_*^{\mathrm{IndCoh}} \circ (p_s^{(n)})^!\right) \to$$
$$\to \mathrm{coFib}\left(\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L}))^{\leq n-1} \to \mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L}))^{\leq n}\right)$$

are isomorphisms.

First, the base change isomorphism

$$\mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L})) := f^! \circ f_*^{\mathrm{IndCoh}} \simeq (p_t)_*^{\mathrm{IndCoh}} \circ p_s^!$$

of Chapter 3, Proposition 2.1.2 defines a compatible system of maps

$$(6.3) \qquad\qquad (p_t^{(n)})_*^{\mathrm{IndCoh}} \circ (p_s^{(n)})^! \to \mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L})).$$

6.5.5. *Proof of Theorem 6.5.3, Step 2.* As in Sects. 6.3 and 6.4, the prestack

$$'\mathcal{X}^{(n)}_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \in \left(\mathrm{FormMod}_{\mathcal{X}\times\mathbb{A}^1 /\!/ \mathcal{Y}\times\mathbb{A}^1}\right)^{\mathbb{A}^1_{\mathrm{left\text{-}lax}}},$$

and the corresponding maps

$$(p_s^{(n),\mathrm{Fil}}), (p_t^{(n),\mathrm{Fil}}) : \mathcal{X}^{(n)}_{\mathrm{scaled},\mathbb{A}^1_{\mathrm{left\text{-}lax}}} \to \mathcal{X}\times\mathbb{A}^1,$$

lift the system (6.2) to an assignment

$$n \mapsto (p_t^{(n),\mathrm{Fil}})_*^{\mathrm{IndCoh}} \circ (p_s^{(n),\mathrm{Fil}})^!, \quad \mathbb{Z}^{\geq 0} \to \left(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}),\mathrm{IndCoh}(\mathcal{X}))\right)^{\mathrm{Fil},\geq 0}.$$

In addition, the system of maps (6.3) lifts to a system of maps

$$(6.4) \qquad\qquad (p_t^{(n),\mathrm{Fil}})_*^{\mathrm{IndCoh}} \circ (p_s^{(n),\mathrm{Fil}})^! \to \mathbf{oblv}_{\mathrm{Assoc}}(U(\mathfrak{L})^{\mathrm{Fil}}).$$

Hence, taking into account (the filtered version of) Corollary 5.2.5, to prove the proposition, it suffices to show the following:

Lemma 6.5.6.

(a) *For every n, the filtration on $(p_t^{(n),\mathrm{Fil}})_*^{\mathrm{IndCoh}} \circ (p_s^{(n),\mathrm{Fil}})^!$ stabilizes at n, i.e., the maps*

$$\left((p_t^{(n),\mathrm{Fil}})_*^{\mathrm{IndCoh}} \circ (p_s^{(n),\mathrm{Fil}})^!\right)^{\leq m} \to \left((p_t^{(n),\mathrm{Fil}})_*^{\mathrm{IndCoh}} \circ (p_s^{(n),\mathrm{Fil}})^!\right)^{\leq m+1}$$

are isomorphisms for $m \geq n$.

(b) *For every n, the map (6.4) induces an isomorphism of the n-th associated graded quotients.*

6.5.7. *Proof of Theorem 6.5.3, Step 3.* In order to prove Lemma 6.5.6, since the functor ass-gr is conservative on $(\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}),\mathrm{IndCoh}(\mathcal{X})))^{\mathrm{Fil},\geq 0}$, it is enough to prove the corresponding assertion at the associated graded level.

By Sect. 2.5.2, this reduces are to the situation when $\mathcal{Y} = \mathrm{Vect}_{\mathcal{X}}(\mathcal{F})$ for some $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$.

However, in the latter case, the assertion of Lemma 6.5.6 is manifest from Corollary 6.1.4 and Sect. 5.5.3.

7. The case of a regular embedding

Recall that if $f : X \to Y$ is a regular closed embedding of classical schemes, then we have Grothendieck's formula that says that the functors f^* and $f^!$ are related by tensoring by the determinant of the normal bundle.

In this section we will establish an analog of this assertion in the derived setting.

7.1. The notion of regular embedding. In this subsection we will introduce the notion of regular embedding in the context of formal moduli problems.

7.1.1. Let $\mathcal{X}$ be an object of $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$, and let $(f : \mathcal{X} \to \mathcal{Y}) \in \mathrm{FormMod}_{\mathcal{X}/}$.

We shall say that f is a *regular embedding of relative codimension n* if

$$T^*(\mathcal{X}/\mathcal{Y})[-1] \in \mathrm{Pro}(\mathrm{QCoh}(\mathcal{X})^-)$$

belongs to $\mathrm{QCoh}(\mathcal{X})^-$ and is a *vector bundle of rank n* (i.e., its pullback to any affine scheme S is Zariski-locally isomorphic to $\mathcal{O}_S^{\oplus n}$). Throughout this subsection we will assume that f has this property.

7.1.2. Denote

$$\det(T^*(\mathcal{X}/\mathcal{Y})) := \mathrm{Sym}^n(T^*(\mathcal{X}/\mathcal{Y}));$$

this is a cohomologically shifted (by $[n]$) line bundle.

Consider the objects $\mathrm{Sym}^m(T^*(\mathcal{X}/\mathcal{Y})) \in \mathrm{QCoh}(\mathcal{X})$. Note that they all are also vector bundles. Moreover, $\mathrm{Sym}^m(T^*(\mathcal{X}/\mathcal{Y}))$ vanishes for $m > n$.

7.1.3. Note also that

$$T(\mathcal{X}/\mathcal{Y}) \simeq \Upsilon_X((T^*(\mathcal{X}/\mathcal{Y}))^\vee),$$

where $(T^*(\mathcal{X}/\mathcal{Y}))^\vee \in \mathrm{QCoh}(\mathcal{X})$ is the tensor dual of $T^*(\mathcal{X}/\mathcal{Y})$ in $\mathrm{QCoh}(\mathcal{X})$. In particular, $T(\mathcal{X}/\mathcal{Y})$ is dualizable as an object of the symmetric monoidal category $\mathrm{IndCoh}(\mathcal{X})$.

Furthermore, $\mathrm{Sym}^m(T(\mathcal{X}/\mathcal{Y})) \in \mathrm{IndCoh}(\mathcal{X})$ is dualizable for any m, and vanishes for $m > n$.

7.1.4. We now claim:

PROPOSITION 7.1.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a regular embedding. Then the functor*

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{Y})$$

admits a left *adjoint (to be denoted $f^{\mathrm{IndCoh},*}$).*

PROOF. In order to show that the functor f_*^{IndCoh} admits a left adjoint, it suffices to show that it commutes with limits. Since the functor $f^!$ is conservative and commutes with limits (being a right adjoint), it suffices to show that the composition

$$f^! \circ f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X})$$

commutes with limits.

Let $\mathfrak{L}$ denote the Lie algebroid $\mathcal{T}(\mathcal{X}/\mathcal{Y})$. We have to show that $U(\mathfrak{L})$, viewed as an endo-functor of $\mathrm{IndCoh}(\mathcal{X})$, commutes with limits.

Note now that Sect. 7.1.3 implies that the canonical filtration on $U(\mathfrak{L})$ has the property that ass-$\mathrm{gr}^m(U(\mathfrak{L}))$ vanishes for $m > n$. I.e., the filtration is finite. Hence, it is enough to see that each graded term, viewed as endo-functor of $\mathrm{IndCoh}(\mathcal{X})$, commutes with limits.

However,

$$\mathrm{ass\text{-}gr}^m(U(\mathfrak{L})) \simeq \Upsilon_{\mathcal{X}}(\mathrm{Sym}^m(\mathcal{T}(\mathcal{X}/\mathcal{Y}))) \otimes -,$$

and the assertion follows.

$\square$

7.2. Grothendieck's formula. In this subsection we state the main result of this section: Grothendieck's formula that relates $f^{\mathrm{IndCoh},*}$ and $f^!$.

7.2.1.   The goal of this section is to prove the following result:

THEOREM 7.2.2. *Let $\mathcal{X}$ be an object of* $\mathrm{PreStk}_{\mathrm{laft\text{-}def}}$*, and let* $(f : \mathcal{X} \to \mathcal{Y}) \in$
$\mathrm{FormMod}_{\mathcal{X}/}$ *be a regular embedding. Then:*

(a) *The natural transformation*

$$f^{\mathrm{IndCoh},*}(-) \to f^{\mathrm{IndCoh},*}(\omega_{\mathcal{Y}}) \overset{!}{\otimes} f^{!}(-)$$

is an isomorphism.

(b) *There exists a canonical isomorphism*

$$f^{\mathrm{IndCoh},*}(\omega_{\mathcal{X}}) \simeq \Upsilon_{\mathcal{X}}\left(\det(T^{*}(\mathcal{X}/\mathcal{Y}))\right).$$

7.2.3.   Combining points (a) and (b) of the theorem, we obtain:

COROLLARY 7.2.4. *There exists a canonical isomorphism of functors* $\mathrm{IndCoh}(\mathcal{Y}) \to$
$\mathrm{IndCoh}(\mathcal{X})$

$$f^{\mathrm{IndCoh},*}(-) \simeq \det(T^{*}(\mathcal{X}/\mathcal{Y})) \otimes f^{!}(-),$$

where $\otimes$ *is understood in the sense of the action of* $\mathrm{QCoh}(-)$ *on* $\mathrm{IndCoh}(-)$.

7.3. Applications.   In this subsection we give some applications of Theorem 7.2.2.

7.3.1. *Schematic regular embeddings.* Let $f : \mathcal{X} \to \mathcal{Y}$ be a schematic map between
objects of $\mathrm{PreStk}_{\mathrm{laft}}$. Assume that f is a closed embedding, and that the map,
denoted

$$f^{\wedge} : \mathcal{X} \to \mathcal{Y}_{\mathcal{X}}^{\wedge} := \mathcal{X}_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}$$

is a regular embedding of relative codimension n in the sense of Sect. 7.1.1.

From Theorem 7.2.2 we shall now deduce:

COROLLARY 7.3.2. *The functor*

$$f^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{X}),$$

left adjoint to f_{*}^{IndCoh}*, is defined, and we have a canonical isomorphism*

$$f^{\mathrm{IndCoh},*}(-) \simeq \det(T^{*}(\mathcal{X}/\mathcal{Y})) \otimes f^{!}(-).$$

PROOF. By base change, we can assume that $X = \mathcal{X}$ and $Y = \mathcal{Y}$ are schemes.
The condition on f implies that it is *quasi-smooth*, and hence eventually coconnec-
tive (see [**AG**, Corollary 1.2.5]). Hence, existence of the functor $f^{\mathrm{IndCoh},*}$ follows
from [**Ga1**, Proposition 7.1.6].

Let $U \overset{j}{\hookrightarrow} Y$ be the open embedding of the complement of image of f. Let i
denote the map $Y_{X}^{\wedge} \to Y$.

It is easy to see that $f^{\mathrm{IndCoh},*} \circ j_{*} = 0$. Hence, by [**GaRo1**, Proposition
7.4.5], the functor $f^{\mathrm{IndCoh},*}$ factors through the co-localization $i^{!} : \mathrm{IndCoh}(Y) \to$
$\mathrm{IndCoh}(Y_{X}^{\wedge})$, i.e.,

$$f^{\mathrm{IndCoh},*} \simeq (f^{\wedge})^{\mathrm{IndCoh},*} \circ i^{!}.$$

Similarly, $f^{!} \simeq (f^{\wedge})^{!} \circ i^{!}$. Now, the required result follows from the isomorphism
of Corollary 7.2.4 for the morphism $f^{\wedge}$.

$\square$

7.3.3. *Smooth maps.* Let now $g : \mathcal{X} \to \mathcal{Z}$ be a schematic map between objects of $\mathrm{PreStk}_{\mathrm{laft}}$. Assume that g is *smooth* of relative dimension n.

Note that in this case $T^*(\mathcal{X}/\mathcal{Z}) \in \mathrm{QCoh}(\mathcal{X})$ is a vector bundle of rank n. Denote

$$\det(T^*(\mathcal{X}/\mathcal{Z})) := \mathrm{Sym}^n(T^*(\mathcal{X}/\mathcal{Z})[1]).$$

This is a cohomologically shifted (by $[n]$) line bundle.

We claim:

PROPOSITION 7.3.4. *The functor*

$$g^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathcal{Z}) \to \mathrm{IndCoh}(\mathcal{X}),$$

left adjoint to g_*^{IndCoh}, is defined, and we have a canonical isomorphism

$$g^{\mathrm{IndCoh},*}(-) \simeq \det(T^*(\mathcal{X}/\mathcal{Z}))^\vee \otimes g^!(-).$$

7.3.5. *Step 1.* By [**Ga1**, Propositions 7.1.6 and 7.3.8], for any map $f : \mathcal{X} \to \mathcal{X}'$, whose base change by an affine scheme is schematic and Gorenstein, the functor $f^{\mathrm{IndCoh},*}$ exists, and we have:

$$\mathcal{K}_{\mathcal{X}/\mathcal{X}'} \otimes f^{\mathrm{IndCoh},*}(-) \simeq f^!(-)$$

for a canonically defined line bundle on $\mathcal{K}_{\mathcal{X}/\mathcal{X}'}$ on $\mathcal{X}$.

It follows formally that for a Cartesian diagram with vertical arrows Gorenstein

$$
\begin{array}{ccc}
\mathcal{X}_1 & \xrightarrow{\ h\ } & \mathcal{X} \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\
\mathcal{X}_1' & \xrightarrow{\ h'\ } & \mathcal{X}'
\end{array}
$$

we have a canonical isomorphism in $\mathrm{QCoh}(\mathcal{X}_1)$

(7.1) $$h^*(\mathcal{K}_{\mathcal{X}/\mathcal{X}'}) \simeq \mathcal{K}_{\mathcal{X}_1/\mathcal{X}_1'},$$

Furthermore, it follows that for a composition of Gorenstein maps

$$\mathcal{X} \xrightarrow{f} \mathcal{X}' \xrightarrow{h} \mathcal{X}'',$$

we have a canonical isomorphism in $\mathrm{QCoh}(\mathcal{X})$

(7.2) $$f^*(\mathcal{K}_{\mathcal{X}'/\mathcal{X}''}) \otimes \mathcal{K}_{\mathcal{X}/\mathcal{X}'} \simeq \mathcal{K}_{\mathcal{X}/\mathcal{X}''}.$$

Let $g : \mathcal{X} \to \mathcal{Z}$ be a Gorenstein map. Consider the diagram

$$
\begin{array}{ccc}
\mathcal{Y} := \mathcal{X} \underset{\mathcal{Z}}{\times} \mathcal{X} & \xrightarrow{\ p_t\ } & \mathcal{X} \\
{\scriptstyle p_s}\downarrow & & \downarrow{\scriptstyle g} \\
\mathcal{X} & \xrightarrow{\ g\ } & \mathcal{Z}.
\end{array}
$$

By (7.1) and (7.2), we have:

(7.3) $$\mathcal{K}_{\mathcal{Y}/\mathcal{Z}} \simeq (p_s)^*(\mathcal{K}_{\mathcal{X}/\mathcal{Z}}) \otimes (p_t)^*(\mathcal{K}_{\mathcal{X}/\mathcal{Z}}).$$

7.3.6. *Step 2.* Assume now that $g : \mathcal{X} \to \mathcal{Z}$ is smooth. We need to show that

$$\mathcal{K}_{\mathcal{X}/\mathcal{Z}} \simeq \det(T^*(\mathcal{X}/\mathcal{Z})).$$

Let f denote the map

$$\mathcal{X} \to \mathcal{Y} := \mathcal{X} \underset{\mathcal{Z}}{\times} \mathcal{X}.$$

Since g is smooth, the map f is a regular closed embedding, i.e., satisfies the assumptions of Sect. 7.3.1. In particular, it is Gorenstein, and by Corollary 7.3.2

$$\mathcal{K}_{\mathcal{X}/\mathcal{Y}} \simeq \det(T^*(\mathcal{X}/\mathcal{Y}))^{\otimes -1}.$$

Combining (7.3) and (7.2), we obtain:

$$\mathcal{K}_{\mathcal{X}/\mathcal{Z}} \simeq \mathcal{K}_{\mathcal{X}/\mathcal{Z}} \otimes \mathcal{K}_{\mathcal{X}/\mathcal{Z}} \otimes \mathcal{K}_{\mathcal{X}/\mathcal{Y}}.$$

Hence,

$$\mathcal{K}_{\mathcal{X}/\mathcal{Z}} \simeq \mathcal{K}_{\mathcal{X}/\mathcal{Y}}^{\otimes -1} \simeq \det(T^*(\mathcal{X}/\mathcal{Y})).$$

I.e., it remains to show that

$$\det(T^*(\mathcal{X}/\mathcal{Z})) \simeq \det(T^*(\mathcal{X}/\mathcal{Y})).$$

However, this follows from the canonical identification

$$T^*(\mathcal{X}/\mathcal{Y}) \simeq T^*(\mathcal{X}/\mathcal{Z})[1].$$

7.4. Introducing the filtration. The rest of this section is devoted to the proof of Theorem 7.2.2. The idea is to upgrade the required isomorphism to one between filtered objects, using the deformation to the normal cone of Sect. 2.

7.4.1.   Consider again the object

$$\mathcal{Y}_{\text{scaled}, \mathbb{A}^1_{\text{left-lax}}} \in \left(\text{FormMod}_{\mathcal{X} \times \mathbb{A}^1 // \mathcal{Y} \times \mathbb{A}^1} \right)^{\mathbb{A}^1_{\text{left-lax}}},$$

and we will regard it as an object of $\left(\text{FormMod}_{\mathcal{X} \times \mathbb{A}^1 // \mathcal{Y} \times \mathbb{A}^1} \right)^{\mathbb{G}_m}$ via the forgetful functor

$$\left(\text{FormMod}_{\mathcal{X} \times \mathbb{A}^1 // \mathcal{Y} \times \mathbb{A}^1} \right)^{\mathbb{A}^1_{\text{left-lax}}} \to \left(\text{FormMod}_{\mathcal{X} \times \mathbb{A}^1 // \mathcal{Y} \times \mathbb{A}^1} \right)^{\mathbb{G}_m}.$$

7.4.2.   The construction of Sect. 6.2.1 upgrades the endo-functor $f^{\text{IndCoh},*} \circ f_*^{\text{IndCoh}}$ of $\text{IndCoh}(\mathcal{X})$ to an object

$$(7.4) \qquad \left(f^{\text{IndCoh},*} \circ f_*^{\text{IndCoh}}(\mathcal{F}) \right)^{\text{Fil}} \in (\text{Funct}_{\text{cont}}(\text{IndCoh}(\mathcal{X}), \text{IndCoh}(\mathcal{X})))^{\text{Fil}}.$$

By construction, the object (7.4) is the left-dual of $\mathbf{oblv}_{\text{assoc}}(U(\mathfrak{L})^{\text{Fil}})$, when both are viewed as objects in the monoidal category $(\text{Funct}_{\text{cont}}(\text{IndCoh}(\mathcal{X}), \text{IndCoh}(\mathcal{X})))^{\text{Fil}}$.

Since,

$$\mathbf{oblv}_{\text{assoc}}(U(\mathfrak{L})^{\text{Fil}}) \in (\text{Funct}_{\text{cont}}(\text{IndCoh}(\mathcal{X}), \text{IndCoh}(\mathcal{X})))^{\text{Fil},\geq 0,\leq n},$$

we obtain that
(7.5)
$$\left(f^{\text{IndCoh},*} \circ f_*^{\text{IndCoh}}(\mathcal{F}) \right)^{\text{Fil}} \in (\text{Funct}_{\text{cont}}(\text{IndCoh}(\mathcal{X}), \text{IndCoh}(\mathcal{X})))^{\text{Fil},\geq -n,\leq 0}.$$

7.4.3.   Similarly, the object $f^{\mathrm{IndCoh},*}(\omega_{\mathcal{Y}})$ naturally upgrades to an object

$$(f^{\mathrm{IndCoh},*}(\omega_{\mathcal{Y}}))^{\mathrm{Fil}} \in \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil}}.$$

Finally, the natural transformation

$$f^{\mathrm{IndCoh},*}(-) \to f^{\mathrm{IndCoh},*}(\omega_{\mathcal{Y}}) \otimes f^!(-)$$

also lifts to a natural transformation of functors

$$\mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil}}.$$

7.4.4.   We claim:

LEMMA 7.4.5.

$$(f^{\mathrm{IndCoh},*}(\omega_{\mathcal{Y}}))^{\mathrm{Fil}} \in \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq -n}.$$

PROOF.   Recall the construction in Proposition 5.3.2. Note that the assignment

$$k \mapsto f^{\mathrm{IndCoh},*}((f_k)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(k)}})),$$

as well as the maps

$$f^{\mathrm{IndCoh},*} \circ (f_k)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(k)}}) \to f^{\mathrm{IndCoh},*} \circ (f_{k+1})_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(k+1)}})$$

and

$$f^{\mathrm{IndCoh},*} \circ (f_k)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(k)}}) \to f^{\mathrm{IndCoh},*}(\omega_{\mathcal{Y}})$$

all lift to the category $\mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil}}$.

Hence, it is enough to show that for every k, we have

$$\left(f^{\mathrm{IndCoh},*} \circ (f_k)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(k)}})\right)^{\mathrm{Fil}} \in \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq -n}.$$

We note that the identification

$$\mathrm{coFib}(f^{\mathrm{IndCoh},*} \circ (f_{k-1})_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(k-1)}}) \to f^{\mathrm{IndCoh},*} \circ (f_k)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(k)}}) \simeq$$
$$\simeq f^{\mathrm{IndCoh},*} \circ f_*^{\mathrm{IndCoh}}\left(\mathrm{Sym}^k(\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[1])\right)$$

lifts to an isomorphism

$$\mathrm{coFib}\left(\left(f^{\mathrm{IndCoh},*} \circ (f_{k-1})_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(k-1)}})\right)^{\mathrm{Fil}} \to \left(f^{\mathrm{IndCoh},*} \circ (f_k)_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X}^{(k)}})\right)^{\mathrm{Fil}}\right) \simeq$$
$$\simeq (f^{\mathrm{IndCoh},*} \circ f_*^{\mathrm{IndCoh}})^{\mathrm{Fil}}(\omega_{\mathcal{X}}) \otimes \left(\mathrm{Sym}^k(\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[1])\right),$$

where $(f^{\mathrm{IndCoh},*} \circ f_*^{\mathrm{IndCoh}})^{\mathrm{Fil}}$ is as in Sect. 7.4.1, and where $\left(\mathrm{Sym}^k(\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[1])\right)$ is in degree k.

Now,

$$(f^{\mathrm{IndCoh},*} \circ f_*^{\mathrm{IndCoh}})^{\mathrm{Fil}}(\omega_{\mathcal{X}}) \in \mathrm{IndCoh}(\mathcal{X})^{\mathrm{Fil},\geq -n},$$

while $\left(\mathrm{Sym}^k(\mathbf{oblv}_{\mathrm{LieAlgbroid}}(\mathfrak{L})[1])\right)$ is non-negatively filtered.

$\square$

7.5. Reduction to the case of vector groups. In this subsection we will reduce the assertion of Theorem 7.2.2 to the case when $\mathcal{Y} = \mathrm{Vect}_{\mathcal{X}}(\mathcal{F})$ for $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$.

7.5.1.   Since the essential image of $\mathrm{IndCoh}(\mathcal{X})$ under f_*^{IndCoh} generates $\mathrm{IndCoh}(\mathcal{Y})$, in order to prove the isomorphism of Theorem 7.2.2(a), it suffices to show that the natural transformation

$$(7.6) \qquad\qquad f^{\mathrm{IndCoh},*} \circ f_*^{\mathrm{IndCoh}}(\mathcal{F}) \to f^{\mathrm{IndCoh},*}(\omega_{\mathcal{Y}}) \otimes \mathcal{F}$$

is an isomorphism for any $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$.

7.5.2.   Taking into account Lemma 7.4.5 and (7.5), we obtain that in order to show that (7.6) is an isomorphism, it is enough to prove that the isomorphism holds at the associated graded level.

Using Sect. 2.5.2, this reduces point (a) of Theorem 7.2.2 to the verification of the isomorphism (7.6) in the case when

$$\mathcal{Y} := \mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1]).$$

7.5.3.   We will prove the following assertion:

PROPOSITION 7.5.4. *With respect to the canonical filtration on $f^{\mathrm{IndCoh},*}(\omega_{\mathcal{X}})$, we have*

$$\mathrm{ass\text{-}gr}^k(f^{\mathrm{IndCoh},*}(\omega_{\mathcal{X}})) \simeq \left\{ \begin{array}{l} 0 \ \textit{if } k \neq -n \\ \Upsilon_{\mathcal{X}}\left(\det(T^*(\mathcal{X}/\mathcal{Y}))\right) \ \textit{if } k = -n. \end{array} \right.$$

Note that by Lemma 7.4.5, the assertion of Proposition 7.5.4 implies also Theorem 7.2.2(b).

In order to prove Proposition 7.5.4, it is also enough to do so in the case when

$$\mathcal{Y} := \mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1]) \in \left(\mathrm{FormMod}_{\mathcal{X}/}\right)^{\mathbb{G}_m},$$

where $T(\mathcal{X}/\mathcal{Y})$ is given grading 1.

7.6. The case of vector groups. In this subsection we will explicitly perform the calculation stated in Theorem 7.2.2 in the case of vector groups.

7.6.1.   Let

$$\mathcal{Y} = \mathrm{Vect}_{\mathcal{X}}(\mathcal{F}) \in \left(\mathrm{FormMod}_{\mathcal{X}/}\right)^{\mathbb{G}_m},$$

where $\mathcal{F} = \Upsilon_{\mathcal{X}}(\mathcal{E})$, where $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})$ is a vector bundle of rank n, and where we regard $\mathcal{F}$ as an object of

$$\mathrm{IndCoh}(\mathcal{X}) \simeq \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr},=1} \subset \mathrm{IndCoh}(\mathcal{X})^{\mathrm{gr}}.$$

We note that in this case

$$\Upsilon_{\mathcal{X}}\left(\det(T^*(\mathcal{X}/\mathcal{Y}))\right) \simeq \mathrm{Sym}^n(\mathcal{F}^{\vee}[1]),$$

where $\mathcal{F}^{\vee}$ is the monoidal dual of $\mathcal{F}$ in the symmetric monoidal category $\mathrm{IndCoh}(\mathcal{X})$.

7.6.2.   We have:

$$\mathrm{IndCoh}(\mathcal{Y})^{\mathbb{G}_m} \simeq \mathbf{free}_{\mathrm{Com}}(\mathcal{F}[-1])\text{-mod}(\mathrm{IndCoh}(\mathcal{X})^{\mathbb{G}_m}),$$

where $f^!$ is the tautological forgetful functor

$$\mathbf{oblv}_{\mathbf{free}_{\mathrm{Com}}(\mathcal{F}[-1])} : \mathbf{free}_{\mathrm{Com}}(\mathcal{F}[-1])\text{-mod}(\mathrm{IndCoh}(\mathcal{X})^{\mathbb{G}_m}) \to \mathrm{IndCoh}(\mathcal{X})^{\mathbb{G}_m},$$

and f_*^{IndCoh} is the functor

$$\mathbf{ind}_{\mathbf{free}_{\mathrm{Com}}(\mathcal{F}[-1])} := \mathbf{free}_{\mathrm{Com}}(\mathcal{F}[-1]) \otimes -.$$

7.6.3.   Consider the following abstract situation: let $\mathbf{O}$ be a symmetric monoidal DG category, and let $\mathcal{F} \in \mathbf{O}$ be an object *of dimension n*, i.e., $\mathcal{F}$ is dualizable and $\mathrm{Sym}^{n+1}(\mathcal{F}[1]) = 0$. Set

$$\mathfrak{l} := \mathrm{Sym}^n(\mathcal{F}[-1]).$$

Consider the commutative algebra $A := \mathbf{free}_{\mathrm{Com}}(\mathcal{F}[-1])$ and the corresponding adjunction

$$\mathbf{ind}_A : \mathbf{O} \rightleftarrows A\text{-mod} : \mathbf{oblv}_A.$$

The assumption on $\mathcal{F}$ implies that $\mathbf{oblv}_{\mathrm{Com}}(A)$ is dualizable as an object of $\mathbf{O}$. Hence, the functor $\mathbf{ind}_A$ commutes with limits, and thus admits a *left* adjoint.

In this case, it is easy to see that the natural transformation

$$(\mathbf{ind}_A)^L(-) \to (\mathbf{ind}_A)^L(\mathbf{1_O}) \otimes \mathbf{oblv}_A(-)$$

is an isomorphism and that

$$(\mathbf{ind}_A)^L(\mathbf{1_O}) \simeq \mathfrak{l}^{\otimes -1}.$$

$\square$

A. Weil restriction of scalars

In this section we will establish several facts of how the operation of Weil restriction behaves with respect to deformation theory.

In particular, we show that Weil restriction along an affine map of a prestack with deformation theory is a prestack with deformation theory, and we describe the pro-cotangent complex of the Weil restriction. Furthermore, we show that Weil restriction of formal groups can be computed using Lie algebras.

A.1. The operation of Weil restriction of scalars. In this subsection we recall the operation of Weil restriction of scalars of a prestack.

A.1.1.   Let $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ be a map of prestacks. Let $\mathcal{X}_1$ be a prestack over $\mathcal{Z}_1$. Let

$$\mathcal{X}_2 := \mathrm{Weil}^{\mathcal{Z}_1}_{\mathcal{Z}_2}(\mathcal{X}_1) \in \mathrm{PreStk}_{/\mathcal{Z}_2}$$

be the Weil restriction of $\mathcal{X}_1$ along f.

By definition, for $S_2 \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Z}_2}$, we have

$$\mathrm{Maps}_{/\mathcal{Z}_2}(S_2, \mathcal{X}_2) := \mathrm{Maps}_{/\mathcal{Z}_1}(S_1, \mathcal{X}_1), \quad S_1 := \mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} S_2.$$

A.1.2. *Assumption.* From now on we will assume that the morphism f is *affine* (i.e., its base change by an affine scheme yields an affine scheme).

A.2. Weil restriction of scalars and deformation theory. In this subsection we will study the deformation theory of objects obtained as by the operation of Weil restriction of scalars.

A.2.1.   Assume now that $\mathcal{X}_1$ *admits deformation theory relative to* $\mathcal{Z}_1$.

It follows from the definitions that in these circumstance, $\mathcal{X}_2$ will admit deformation theory relative to $\mathcal{Z}_2$. Moreover, its cotangent complex can be described as follows.

For an S_2-point x_2 of $\mathcal{X}_2$, let x_1 be the corresponding S_1-point of $\mathcal{X}_1$, where

$$S_1 := \mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} S_2.$$

Denote by f_S the corresponding map $S_1 \to S_2$.

Let $\mathrm{Pro}((f_S)_*)$ be the corresponding functor

$$\mathrm{Pro}(\mathrm{QCoh}(S_1)^-) \to \mathrm{Pro}(\mathrm{QCoh}(S_2)^-).$$

Then we have:

(A.1) $$T_{x_2}^*(\mathcal{X}_2/\mathcal{Z}_2) \simeq \mathrm{Pro}((f_S)_*)(T_{x_1}^*(\mathcal{X}_1/\mathcal{Z}_1)).$$

A.2.2.   Note that if in the above circumstances, $\mathcal{X}_1$ is itself of the form

$$\mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} \mathcal{X}_2', \quad \mathcal{X}_2' \in \mathrm{PreStk}_{/\mathcal{Z}_2},$$

and the point x_1 comes from an S_2-point x_2' of $\mathcal{X}_2'$, then we have

(A.2) $$T_{x_2}^*(\mathcal{X}_2/\mathcal{Z}_2) \simeq T_{x_2}^*(\mathcal{X}_2'/\mathcal{Z}_2) \underset{\mathcal{O}_{S_2}}{\otimes} (f_S)_*(\mathcal{O}_{S_1}).$$

A.2.3.   Assume that $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{X}_1 \in \mathrm{PreStk}_{\mathrm{laft}}$. It follows from the definitions that in this case $\mathcal{X}_2$ also belongs to $\mathrm{PreStk}_{\mathrm{laft}}$.

In this case we can talk about

$$T(\mathcal{X}_1) \in \mathrm{IndCoh}(\mathcal{X}_1) \text{ and } T(\mathcal{X}_2) \in \mathrm{IndCoh}(\mathcal{X}_2).$$

We have the diagram

$$
\begin{array}{ccc}
\mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} \mathcal{X}_2 & \xrightarrow{\mathrm{ev}} & .\mathcal{X}_1 \\
{\scriptstyle f \times \mathrm{id}} \downarrow & & \\
\mathcal{X}_2. & &
\end{array}
$$

Let $((f \times \mathrm{id})^!)^R : \mathrm{IndCoh}(\mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} \mathcal{X}_2) \to \mathrm{IndCoh}(\mathcal{X}_2)$ be the functor *right adjoint* to

$$(f \times \mathrm{id})^! : \mathrm{IndCoh}(\mathcal{X}_2) \to \mathrm{IndCoh}(\mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} \mathcal{X}_2).$$

Note that $((f \times \mathrm{id})^!)^R$ is continuous because f was assumed is eventually coconnective.

From (A.1) we obtain:

(A.3) $$T(\mathcal{X}_2/\mathcal{Z}_2) \simeq ((f \times \mathrm{id})^!)^R \circ \mathrm{ev}^!(T(\mathcal{X}_1/\mathcal{Z}_1)).$$

Similarly, in the circumstances of Sect. A.2.2, we have

(A.4) $$T_{x_2}(\mathcal{X}_2/\mathcal{Z}_2) \simeq T_{x_2'}(\mathcal{X}_2'/\mathcal{Z}_2) \underset{\mathcal{O}_{\mathcal{X}_2}}{\otimes} f_*(\mathcal{O}_{\mathcal{X}_1}),$$

where $\otimes$ understood in the sense of the action of QCoh on IndCoh.

A.3. Weil restriction of formal groups.

A.3.1. Let $\mathbf{O}$ be a symmetric monoidal DG category and $\mathcal{P}$ an operad (see Chapter 6, Sect. 1.1 for our conventions regarding operads). For a morphism $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ as above, pullback defines a functor

$$f^* : \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathcal{Z}_2)) \to \mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathcal{Z}_1)).$$

This functor admits a right adjoint, denoted also $\mathrm{Weil}^{\mathcal{Z}_1}_{\mathcal{Z}_2}$ that makes the diagram

(A.5)

$$
\begin{array}{ccc}
\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathcal{Z}_2)) & \xrightarrow{\ \mathbf{oblv}_{\mathcal{P}}\ } & \mathrm{QCoh}(\mathcal{Z}_1) \\
{\scriptstyle \mathrm{Weil}^{\mathcal{Z}_1}_{\mathcal{Z}_2}} \downarrow & & \downarrow {\scriptstyle f_*} \\
\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O} \otimes \mathrm{QCoh}(\mathcal{Z}_2)) & \xrightarrow{\ \mathbf{oblv}_{\mathcal{P}}\ } & \mathrm{QCoh}(\mathcal{Z}_2)
\end{array}
$$

commutative.

A.3.2. Assume now that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ belong to $\mathrm{PreStk}_{\mathrm{laft}}$. Then in the above discussion we can replace QCoh and IndCoh, and the diagram (A.5) by

(A.6)

$$
\begin{array}{ccc}
\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O} \otimes \mathrm{IndCoh}(\mathcal{Z}_1)) & \xrightarrow{\ \mathbf{oblv}_{\mathcal{P}}\ } & \mathrm{IndCoh}(\mathcal{Z}_1) \\
{\scriptstyle \mathrm{Weil}^{\mathcal{Z}_1}_{\mathcal{Z}_2}} \downarrow & & \downarrow {\scriptstyle (f^!)^R} \\
\mathcal{P}\text{-}\mathrm{Alg}(\mathbf{O} \otimes \mathrm{IndCoh}(\mathcal{Z}_2)) & \xrightarrow{\ \mathbf{oblv}_{\mathcal{P}}\ } & \mathrm{IndCoh}(\mathcal{Z}_2)
\end{array}
$$

A.3.3. Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ again belong to $\mathrm{PreStk}_{\mathrm{laft}}$. Note that by Chapter 7, Sect. 3.5, we have a commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{Z}_1}) & \xrightarrow{\ \mathrm{Lie}_{\mathcal{Z}_1}\ } & \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{Z}_1)) \\
{\scriptstyle \mathcal{H} \mapsto \mathcal{Z}_1 \underset{\mathcal{Z}_2}{\times} \mathcal{H}} \uparrow & & \uparrow {\scriptstyle f^*} \\
\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{Z}_2}) & \xrightarrow{\ \mathrm{Lie}_{\mathcal{Z}_2}\ } & \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{Z}_2)).
\end{array}
$$

By passing to right adjoints along vertical arrows, we obtain the following commutative diagram:

(A.7)

$$
\begin{array}{ccc}
\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{Z}_1}) & \xrightarrow{\ \mathrm{Lie}_{\mathcal{Z}_1}\ } & \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{Z}_1)) \\
{\scriptstyle \mathrm{Weil}^{\mathcal{Z}_1}_{\mathcal{Z}_2}} \downarrow & & \downarrow {\scriptstyle \mathrm{Weil}^{\mathcal{Z}_1}_{\mathcal{Z}_2}} \\
\mathrm{Grp}(\mathrm{FormMod}_{/\mathcal{Z}_2}) & \xrightarrow{\ \mathrm{Lie}_{\mathcal{Z}_2}\ } & \mathrm{LieAlg}(\mathrm{IndCoh}(\mathcal{Z}_2)).
\end{array}
$$

Bibliography

[AG] D. Arinkin and D. Gaitsgory, *Singular support of coherent sheaves, and the geometric Langlands conjecture*, Selecta Math. N.S. **21** (2015), 1–199.

[BarS] C. Barwick and C. Schommer-Pries, *On the unicity of homotopy theory of higher categories*, arXiv: 1112.0040.

[BB] A. Beilinson and J. Bernstein, *A proof of Jantzen's conjectures*, Advances in Soviet Mathematics **16**, Part I (1993), 1–50.

[BD] A. Beilinson and V. Drinfeld, *Quantization of Hitchin?s integrable system and Hecke eigensheaves*, available at http://www.math.harvard.edu/~gaitsgde/grad_2009/.

[Bez] R. Bezrukavnikov, *On two geometric realizations of the affine Hecke algebra*, arXiv:1209.0403.

[BFN] D. Benzvi, J. Francis and D. Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry*, J. Amer. Math. Soc. **23** (2010), no. 4, 909–966.

[Bezr] R. Bezrukavnikov, *On two geometric realizations of the affine Hecke algebra*, arXiv:1209.0403.

[BoV] J. M. Boardman and J. M. Vogt *Homotopy Invariant Structures on Topological Spaces*, Lecture Notes in Mathematics **347**, Springer-Verlag, Berlin and New York (1973).

[De] P. Deligne, *Catégories tannakiennes*, in: "The Grothendieck Festschrift", Vol. II, 111–195, Progr. Math. **87**, Birkhäuser Boston, Boston, MA, 1990.

[Dr] V. Drinfeld, *DG Quotients of DG Categories*, J. Algebra **272** (2004), no. 2, 643–691.

[DrGa1] V. Drinfeld and D. Gaitsgory, *On some finiteness questions for algebraic stacks*, GAFA **23** (2013), 149–294.

[DrGa2] V. Drinfeld and D. Gaitsgory, *Compact generation of the category of D-modules on the stack of G-bundles on a curve*, joint with V. Drinfeld, arXiv:1112.2402, Cambridge Math Journal, **3** (2015), 19–125.

[Fra] J. Francis, *The tangent complex and Hochschild cohomology of E_n-rings*, Compos. Math. **149** (2013), no. 3, 430–480.

[FraG] J. Francis and D. Gaitsgory, *Chiral Koszul duality*, Selecta Math. (N.S.) **18** (2012), 27–87.

[FG] E. Frenkel and D. Gaitsgory, *D-modules on the affine flag variety and representations of affine Kac-Moody algebras*, Represent. Theory **13** (2009), 470–608.

[Ga1] D. Gaitsgory, *Ind-coherent sheaves*, arXiv:1105.4857.

[Ga2] D. Gaitsgory, *The Atiyah-Bott formula for the cohomology of the moduli space of bundles on a curve*, arXiv: 1505.02331.

[Ga3] D. Gaitsgory, *Sheaves of categories and the notion of 1-affineness*, Contemporary Mathematics **643** (2015), 1–99.

[Ga4] Notes on Geometric Langlands, *Generalities on DG categories*, available at http://www.math.harvard.edu/~gaitsgde/GL/.

[GaRo1] D. Gaitsgory and N. Rozenblyum, *DG indschemes*, Contemporary Mathematics **610** (2014), 139–251.

[GaRo2] D. Gaitsgory and N. Rozenblyum, *D-modules and crystals*, PAMQ **10**, no. 1 (2014), 57–155.

[Jo] A. Joyal, *Quasi-categories and Kan complexes*, (in Special volume celebrating the 70th birthday of Prof. Max Kelly) J. Pure Appl. Algebra **175** (2002), no. 1-3, 207–222.

[JT] A. Joyal, M. Tierney, *Quasi-categories vs Segal spaces*, Categories in algebra, geometry and mathematical physics, Contemp. Math. **431**, Amer. Math. Soc., Providence, RI (2007), 277–326.

[Kr] H. Krause, *The stable derived category of a Noetherian scheme*, Compos. Math., **141**(5) (2005), 1128–1162.

[LM] G. Laumon, L. Morret-Baily, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3 Folge, A Series of Modern Surveys in Mathematics), **39**, Springer-Verlag, Berlin, 2000.

[May] P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics **271**, Springer-Verlag, Berlin and New York (1972).

[LZ1] Y. Liu and W. Zheng, *Enhanced six operations and base change theorem for Artin stacks*, arXiv: 1211.5948.

[LZ2] Y. Liu and W. Zheng, *Gluing restricted nerves of infinity categories*, arXiv: 1211.5294..

[Lu1] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, **170**, Princeton University Press, Princeton, NJ, 2009.

[Lu2] J. Lurie, *Higher Algebra*, available at http://www.math.harvard.edu/~lurie.

[Lu3] J. Lurie, *(∞, 2)-categories and Goodwillie calculus-I*, available at http://www.math.harvard.edu/~lurie.

[Lu4] J. Lurie, *DAG-VII, Spectral schemes*, available at http://www.math.harvard.edu/~lurie.

[Lu5] J. Lurie, *DAG-VIII, Quasi-Coherent Sheaves and Tannaka Duality Theorems*, available at http://www.math.harvard.edu/~lurie.

[Lu6] J. Lurie, *DAG-X, Formal moduli problems*, available at http://www.math.harvard.edu/~lurie.

[Ne] A. Neeman, *The Grothendieck duality theorem via Bousfield techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), no. 1, 205–236.

[Rezk1] C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007.

[Rezk2] C. Rezk, *A Cartesian presentation of weak n-categories*, Geom. Topol. **14** (2010), no. 1, 521–571.

[Seg] G. Segal, *Categories and cohomology theories*, Topology **13**, (1974), 293–312.

[Sim] C. Simpson, *Algebraic (geometric) n-stacks*, arXiv: 9609014.

[To] B. Toen, *Descente fidèlement plate pour les n-champs d'Artin*.

[TV1] B. Toen and G. Vezzosi, *Homotopical Algebraic Geometry-I*.

[TV2] B. Toen and G. Vezzosi, *Homotopical Algebraic Geometry-II*.

[TT] R. Thomason and T. Trobaugh, *Higher algebraic K -theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, 247–435, Progr. Math., **88** (1990).

[YZ] A. Yekutieli and J. Zhang, *Dualizing Complexes and Perverse Sheaves on Noncommutative Ringed Schemes*, Selecta Math. **12** (2006), 137–177.

Index of Notations

Index